

Principles of Mathematical Analysis

THIRD EDITION

WALTER RUDIN



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Principles of Mathematical Analysis

THIRD EDITION



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Principles of Mathematical Analysis

THIRD EDITION

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PREFACE

This book is intended to serve as a text for the course in analysis that is usually taken by advanced undergraduates or by first-year students who study mathematics.

The present edition covers essentially the same topics as the second one, with some additions, a few minor omissions, and considerable rearrangement. I hope that these changes will make the material more accessible and more attractive to the students who take such a course.

Experience has convinced me that it is pedagogically unsound (though logically correct) to start off with the construction of the real numbers from the rational ones. At the beginning, most students simply fail to appreciate the need for doing this. Accordingly, the real number system is introduced as an ordered field with the least-upper-bound property, and a few interesting applications of this property are quickly made. However, Dedekind's construction is not omitted. It is now in an Appendix to Chapter 1, where it may be studied and enjoyed whenever the time seems ripe.

The material on functions of several variables is almost completely rewritten, with many details filled in, and with more examples and more motivation. The proof of the inverse function theorem—the key item in Chapter 9—is

simplified by means of the fixed point theorem about contraction mappings. Differential forms are discussed in much greater detail. Several applications of Stokes' theorem are included.

As regards other changes, the chapter on the Riemann-Stieltjes integral has been trimmed a bit, a short do-it-yourself section on the gamma function has been added to Chapter 8, and there is a large number of new exercises, most of them with fairly detailed hints.

I have also included several references to articles appearing in the *American Mathematical Monthly* and in *Mathematics Magazine*, in the hope that students will develop the habit of looking into the journal literature. Most of these references were kindly supplied by R. B. Burckel.

Over the years, many people, students as well as teachers, have sent me corrections, criticisms, and other comments concerning the previous editions of this book. I have appreciated these, and I take this opportunity to express my sincere thanks to all who have written me.

WALTER RUDIN

THE REAL AND COMPLEX NUMBER SYSTEMS

INTRODUCTION

A satisfactory discussion of the main concepts of analysis (such as convergence, continuity, differentiation, and integration) must be based on an accurately defined number concept. We shall not, however, enter into any discussion of the axioms that govern the arithmetic of the integers, but assume familiarity with the rational numbers (i.e., the numbers of the form m/n , where m and n are integers and $n \neq 0$).

The rational number system is inadequate for many purposes, both as a field and as an ordered set. (These terms will be defined in Secs. 1.6 and 1.12.) For instance, there is no rational p such that $p^2 = 2$. (We shall prove this presently.) This leads to the introduction of so-called “irrational numbers” which are often written as infinite decimal expansions and are considered to be “approximated” by the corresponding finite decimals. Thus the sequence

$$1, 1.4, 1.41, 1.414, 1.4142, \dots$$

“tends to $\sqrt{2}$.” But unless the irrational number $\sqrt{2}$ has been clearly defined, the question must arise: Just what is it that this sequence “tends to”?

This sort of question can be answered as soon as the so-called “real number system” is constructed.

1.1 Example We now show that the equation

$$(1) \quad p^2 = 2$$

is not satisfied by any rational p . If there were such a p , we could write $p = m/n$ where m and n are integers that are not both even. Let us assume this is done. Then (1) implies

$$(2) \quad m^2 = 2n^2,$$

This shows that m^2 is even. Hence m is even (if m were odd, m^2 would be odd), and so m^2 is divisible by 4. It follows that the right side of (2) is divisible by 4, so that n^2 is even, which implies that n is even.

The assumption that (1) holds thus leads to the conclusion that both m and n are even, contrary to our choice of m and n . Hence (1) is impossible for rational p .

We now examine this situation a little more closely. Let A be the set of all positive rationals p such that $p^2 < 2$ and let B consist of all positive rationals p such that $p^2 > 2$. We shall show that A contains no largest number and B contains no smallest.

More explicitly, for every p in A we can find a rational q in A such that $p < q$, and for every p in B we can find a rational q in B such that $q < p$.

To do this, we associate with each rational $p > 0$ the number

$$(3) \quad q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}.$$

Then

$$(4) \quad q^2 - 2 = \frac{2(p^2 - 2)}{(p + 2)^2}.$$

If p is in A then $p^2 - 2 < 0$, (3) shows that $q > p$, and (4) shows that $q^2 < 2$. Thus q is in A .

If p is in B then $p^2 - 2 > 0$, (3) shows that $0 < q < p$, and (4) shows that $q^2 > 2$. Thus q is in B .

1.2 Remark The purpose of the above discussion has been to show that the rational number system has certain gaps, in spite of the fact that between any two rationals there is another: If $r < s$ then $r < (r + s)/2 < s$. The real number system fills these gaps. This is the principal reason for the fundamental role which it plays in analysis.

In order to elucidate its structure, as well as that of the complex numbers, we start with a brief discussion of the general concepts of *ordered set* and *field*.

Here is some of the standard set-theoretic terminology that will be used throughout this book.

1.3 Definitions If A is any set (whose elements may be numbers or any other objects), we write $x \in A$ to indicate that x is a member (or an element) of A .

If x is not a member of A , we write: $x \notin A$.

The set which contains no element will be called the *empty set*. If a set has at least one element, it is called *nonempty*.

If A and B are sets, and if every element of A is an element of B , we say that A is a subset of B , and write $A \subset B$, or $B \supset A$. If, in addition, there is an element of B which is not in A , then A is said to be a *proper* subset of B . Note that $A \subset A$ for every set A .

If $A \subset B$ and $B \subset A$, we write $A = B$. Otherwise $A \neq B$.

1.4 Definition Throughout Chap. 1, the set of all rational numbers will be denoted by Q .

ORDERED SETS

1.5 Definition Let S be a set. An *order* on S is a relation, denoted by $<$, with the following two properties:

(i) If $x \in S$ and $y \in S$ then one and only one of the statements

$$x < y, \quad x = y, \quad y < x$$

is true.

(ii) If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

The statement " $x < y$ " may be read as " x is less than y " or " x is smaller than y " or " x precedes y ".

It is often convenient to write $y > x$ in place of $x < y$.

The notation $x \leq y$ indicates that $x < y$ or $x = y$, without specifying which of these two is to hold. In other words, $x \leq y$ is the negation of $x > y$.

1.6 Definition An *ordered set* is a set S in which an order is defined.

For example, Q is an ordered set if $r < s$ is defined to mean that $s - r$ is a positive rational number.

1.7 Definition Suppose S is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that E is *bounded above*, and call β an *upper bound* of E .

Lower bounds are defined in the same way (with \geq in place of \leq).

1.8 Definition Suppose S is an ordered set, $E \subset S$, and E is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

- (i) α is an upper bound of E .
- (ii) If $\gamma < \alpha$ then γ is not an upper bound of E .

Then α is called the *least upper bound of E* [that there is at most one such α is clear from (ii)] or the *supremum of E* , and we write

$$\alpha = \sup E.$$

The *greatest lower bound*, or *infimum*, of a set E which is bounded below is defined in the same manner: The statement

$$\alpha = \inf E$$

means that α is a lower bound of E and that no β with $\beta > \alpha$ is a lower bound of E .

1.9 Examples

(a) Consider the sets A and B of Example 1.1 as subsets of the ordered set Q . The set A is bounded above. In fact, the upper bounds of A are exactly the members of B . Since B contains no smallest member, A has no least upper bound in Q .

Similarly, B is bounded below: The set of all lower bounds of B consists of A and of all $r \in Q$ with $r \leq 0$. Since A has no largest member, B has no greatest lower bound in Q .

(b) If $\alpha = \sup E$ exists, then α may or may not be a member of E . For instance, let E_1 be the set of all $r \in Q$ with $r < 0$. Let E_2 be the set of all $r \in Q$ with $r \leq 0$. Then

$$\sup E_1 = \sup E_2 = 0,$$

and $0 \notin E_1$, $0 \in E_2$.

(c) Let E consist of all numbers $1/n$, where $n = 1, 2, 3, \dots$. Then $\sup E = 1$, which is in E , and $\inf E = 0$, which is not in E .

1.10 Definition An ordered set S is said to have the *least-upper-bound property* if the following is true:

If $E \subset S$, E is not empty, and E is bounded above, then $\sup E$ exists in S .

Example 1.9(a) shows that Q does not have the least-upper-bound property.

We shall now show that there is a close relation between greatest lower bounds and least upper bounds, and that every ordered set with the least-upper-bound property also has the greatest-lower-bound property.

1.11 Theorem Suppose S is an ordered set with the least-upper-bound property, $B \subset S$, B is not empty, and B is bounded below. Let L be the set of all lower bounds of B . Then

$$\alpha = \sup L$$

exists in S , and $\alpha = \inf B$.

In particular, $\inf B$ exists in S .

Proof Since B is bounded below, L is not empty. Since L consists of exactly those $y \in S$ which satisfy the inequality $y \leq x$ for every $x \in B$, we see that every $x \in B$ is an upper bound of L . Thus L is bounded above. Our hypothesis about S implies therefore that L has a supremum in S ; call it α .

If $\gamma < \alpha$ then (see Definition 1.8) γ is not an upper bound of L , hence $\gamma \notin B$. It follows that $\alpha \leq x$ for every $x \in B$. Thus $\alpha \in L$.

If $\alpha < \beta$ then $\beta \notin L$, since α is an upper bound of L .

We have shown that $\alpha \in L$ but $\beta \notin L$ if $\beta > \alpha$. In other words, α is a lower bound of B , but β is not if $\beta > \alpha$. This means that $\alpha = \inf B$.

FIELDS

1.12 Definition A *field* is a set F with two operations, called *addition* and *multiplication*, which satisfy the following so-called “field axioms” (A), (M), and (D):

(A) Axioms for addition

- (A1) If $x \in F$ and $y \in F$, then their sum $x + y$ is in F .
- (A2) Addition is commutative: $x + y = y + x$ for all $x, y \in F$.
- (A3) Addition is associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.
- (A4) F contains an element 0 such that $0 + x = x$ for every $x \in F$.
- (A5) To every $x \in F$ corresponds an element $-x \in F$ such that

$$x + (-x) = 0.$$

(M) Axioms for multiplication

- (M1) If $x \in F$ and $y \in F$, then their product xy is in F .
- (M2) Multiplication is commutative: $xy = yx$ for all $x, y \in F$.
- (M3) Multiplication is associative: $(xy)z = x(yz)$ for all $x, y, z \in F$.
- (M4) F contains an element $1 \neq 0$ such that $1x = x$ for every $x \in F$.
- (M5) If $x \in F$ and $x \neq 0$ then there exists an element $1/x \in F$ such that

$$x \cdot (1/x) = 1.$$

(D) The distributive law

$$x(y + z) = xy + xz$$

holds for all $x, y, z \in F$.

1.13 Remarks

(a) One usually writes (in any field)

$$x - y, \frac{x}{y}, x + y + z, xyz, x^2, x^3, 2x, 3x, \dots$$

in place of

$$x + (-y), x \cdot \left(\frac{1}{y}\right), (x + y) + z, (xy)z, xx, xxx, x + x, x + x + x, \dots$$

(b) The field axioms clearly hold in Q , the set of all rational numbers, if addition and multiplication have their customary meaning. Thus Q is a field.

(c) Although it is not our purpose to study fields (or any other algebraic structures) in detail, it is worthwhile to prove that some familiar properties of Q are consequences of the field axioms; once we do this, we will not need to do it again for the real numbers and for the complex numbers.

1.14 Proposition *The axioms for addition imply the following statements.*

- (a) *If $x + y = x + z$ then $y = z$.*
- (b) *If $x + y = x$ then $y = 0$.*
- (c) *If $x + y = 0$ then $y = -x$.*
- (d) *$-(-x) = x$.*

Statement (a) is a cancellation law. Note that (b) asserts the uniqueness of the element whose existence is assumed in (A4), and that (c) does the same for (A5).

Proof If $x + y = x + z$, the axioms (A) give

$$\begin{aligned} y &= 0 + y = (-x + x) + y = -x + (x + y) \\ &= -x + (x + z) = (-x + x) + z = 0 + z = z. \end{aligned}$$

This proves (a). Take $z = 0$ in (a) to obtain (b). Take $z = -x$ in (a) to obtain (c).

Since $-x + x = 0$, (c) (with $-x$ in place of x) gives (d).

1.15 Proposition *The axioms for multiplication imply the following statements.*

- (a) *If $x \neq 0$ and $xy = xz$ then $y = z$.*
- (b) *If $x \neq 0$ and $xy = x$ then $y = 1$.*
- (c) *If $x \neq 0$ and $xy = 1$ then $y = 1/x$.*
- (d) *If $x \neq 0$ then $1/(1/x) = x$.*

The proof is so similar to that of Proposition 1.14 that we omit it.

1.16 Proposition *The field axioms imply the following statements, for any $x, y, z \in F$.*

- (a) $0x = 0$.
- (b) *If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.*
- (c) $(-x)y = -(xy) = x(-y)$.
- (d) $(-x)(-y) = xy$.

Proof $0x + 0x = (0 + 0)x = 0x$. Hence 1.14(b) implies that $0x = 0$, and (a) holds.

Next, assume $x \neq 0, y \neq 0$, but $xy = 0$. Then (a) gives

$$1 = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right)xy = \left(\frac{1}{y}\right)\left(\frac{1}{x}\right)0 = 0,$$

a contradiction. Thus (b) holds.

The first equality in (c) comes from

$$(-x)y + xy = (-x + x)y = 0y = 0,$$

combined with 1.14(c); the other half of (c) is proved in the same way. Finally,

$$(-x)(-y) = -[x(-y)] = -[-(xy)] = xy$$

by (c) and 1.14(d).

1.17 Definition *An ordered field is a field F which is also an ordered set, such that*

- (i) $x + y < x + z$ if $x, y, z \in F$ and $y < z$,
- (ii) $xy > 0$ if $x \in F, y \in F, x > 0$, and $y > 0$.

If $x > 0$, we call x *positive*; if $x < 0$, x is *negative*.

For example, \mathbb{Q} is an ordered field.

All the familiar rules for working with inequalities apply in every ordered field: Multiplication by positive [negative] quantities preserves [reverses] inequalities, no square is negative, etc. The following proposition lists some of these.

1.18 Proposition *The following statements are true in every ordered field.*

- (a) *If $x > 0$ then $-x < 0$, and vice versa.*
- (b) *If $x > 0$ and $y < z$ then $xy < xz$.*
- (c) *If $x < 0$ and $y < z$ then $xy > xz$.*
- (d) *If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.*
- (e) *If $0 < x < y$ then $0 < 1/y < 1/x$.*

Proof

(a) If $x > 0$ then $0 = -x + x > -x + 0$, so that $-x < 0$. If $x < 0$ then $0 = -x + x < -x + 0$, so that $-x > 0$. This proves (a).

(b) Since $z > y$, we have $z - y > y - y = 0$, hence $x(z - y) > 0$, and therefore

$$xz = x(z - y) + xy > 0 + xy = xy.$$

(c) By (a), (b), and Proposition 1.16(c),

$$-[x(z - y)] = (-x)(z - y) > 0,$$

so that $x(z - y) < 0$, hence $xz < xy$.

(d) If $x > 0$, part (ii) of Definition 1.17 gives $x^2 > 0$. If $x < 0$, then $-x > 0$, hence $(-x)^2 > 0$. But $x^2 = (-x)^2$, by Proposition 1.16(d). Since $1 = 1^2$, $1 > 0$.

(e) If $y > 0$ and $v \leq 0$, then $yv \leq 0$. But $y \cdot (1/y) = 1 > 0$. Hence $1/y > 0$. Likewise, $1/x > 0$. If we multiply both sides of the inequality $x < y$ by the positive quantity $(1/x)(1/y)$, we obtain $1/y < 1/x$.

THE REAL FIELD

We now state the *existence theorem* which is the core of this chapter.

1.19 Theorem *There exists an ordered field R which has the least-upper-bound property.*

Moreover, R contains \mathbb{Q} as a subfield.

The second statement means that $\mathbb{Q} \subset R$ and that the operations of addition and multiplication in R , when applied to members of \mathbb{Q} , coincide with the usual operations on rational numbers; also, the positive rational numbers are positive elements of R .

The members of R are called *real numbers*.

The proof of Theorem 1.19 is rather long and a bit tedious and is therefore presented in an Appendix to Chap. 1. The proof actually constructs R from \mathbb{Q} .

The next theorem could be extracted from this construction with very little extra effort. However, we prefer to derive it from Theorem 1.19 since this provides a good illustration of what one can do with the least-upper-bound property.

1.20 Theorem

(a) If $x \in R$, $y \in R$, and $x > 0$, then there is a positive integer n such that

$$nx > y.$$

(b) If $x \in R$, $y \in R$, and $x < y$, then there exists a $p \in Q$ such that $x < p < y$.

Part (a) is usually referred to as the *archimedean property* of R . Part (b) may be stated by saying that Q is *dense* in R : Between any two real numbers there is a rational one.

Proof

(a) Let A be the set of all nx , where n runs through the positive integers. If (a) were false, then y would be an upper bound of A . But then A has a *least* upper bound in R . Put $\alpha = \sup A$. Since $x > 0$, $\alpha - x < \alpha$, and $\alpha - x$ is not an upper bound of A . Hence $\alpha - x < mx$ for some positive integer m . But then $\alpha < (m + 1)x \in A$, which is impossible, since α is an upper bound of A .

(b) Since $x < y$, we have $y - x > 0$, and (a) furnishes a positive integer n such that

$$n(y - x) > 1.$$

Apply (a) again, to obtain positive integers m_1 and m_2 such that $m_1 > nx$, $m_2 > -nx$. Then

$$-m_2 < nx < m_1.$$

Hence there is an integer m (with $-m_2 \leq m \leq m_1$) such that

$$m - 1 \leq nx < m.$$

If we combine these inequalities, we obtain

$$nx < m \leq 1 + nx < ny.$$

Since $n > 0$, it follows that

$$x < \frac{m}{n} < y.$$

This proves (b), with $p = m/n$.

We shall now prove the existence of n th roots of positive reals. This proof will show how the difficulty pointed out in the Introduction (irrationality of $\sqrt{2}$) can be handled in R .

1.21 Theorem *For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$.*

This number y is written $\sqrt[n]{x}$ or $x^{1/n}$.

Proof That there is at most one such y is clear, since $0 < y_1 < y_2$ implies $y_1^n < y_2^n$.

Let E be the set consisting of all positive real numbers t such that $t^n < x$.

If $t = x/(1 + x)$ then $0 \leq t < 1$. Hence $t^n \leq t < x$. Thus $t \in E$, and E is not empty.

If $t > 1 + x$ then $t^n \geq t > x$, so that $t \notin E$. Thus $1 + x$ is an upper bound of E .

Hence Theorem 1.19 implies the existence of

$$y = \sup E.$$

To prove that $y^n = x$ we will show that each of the inequalities $y^n < x$ and $y^n > x$ leads to a contradiction.

The identity $b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1})$ yields the inequality

$$b^n - a^n < (b - a)nb^{n-1}$$

when $0 < a < b$.

Assume $y^n < x$. Choose h so that $0 < h < 1$ and

$$h < \frac{x - y^n}{n(y + 1)^{n-1}}.$$

Put $a = y$, $b = y + h$. Then

$$(y + h)^n - y^n < hn(y + h)^{n-1} < hn(y + 1)^{n-1} < x - y^n.$$

Thus $(y + h)^n < x$, and $y + h \in E$. Since $y + h > y$, this contradicts the fact that y is an upper bound of E .

Assume $y^n > x$. Put

$$k = \frac{y^n - x}{ny^{n-1}}.$$

Then $0 < k < y$. If $t \geq y - k$, we conclude that

$$y^n - t^n \leq y^n - (y - k)^n < kny^{n-1} = y^n - x.$$

Thus $t^n > x$, and $t \notin E$. It follows that $y - k$ is an upper bound of E .

But $y - k < y$, which contradicts the fact that y is the *least* upper bound of E .

Hence $y^n = x$, and the proof is complete.

Corollary *If a and b are positive real numbers and n is a positive integer, then*

$$(ab)^{1/n} = a^{1/n}b^{1/n}.$$

Proof Put $\alpha = a^{1/n}$, $\beta = b^{1/n}$. Then

$$ab = \alpha^n \beta^n = (\alpha\beta)^n,$$

since multiplication is commutative. [Axiom (M2) in Definition 1.12.] The uniqueness assertion of Theorem 1.21 shows therefore that

$$(ab)^{1/n} = \alpha\beta = a^{1/n}b^{1/n}.$$

1.22 Decimals We conclude this section by pointing out the relation between real numbers and decimals.

Let $x > 0$ be real. Let n_0 be the largest integer such that $n_0 \leq x$. (Note that the existence of n_0 depends on the archimedean property of R .) Having chosen n_0, n_1, \dots, n_{k-1} , let n_k be the largest integer such that

$$n_0 + \frac{n_1}{10} + \dots + \frac{n_k}{10^k} \leq x.$$

Let E be the set of these numbers

$$(5) \quad n_0 + \frac{n_1}{10} + \dots + \frac{n_k}{10^k} \quad (k = 0, 1, 2, \dots).$$

Then $x = \sup E$. The decimal expansion of x is

$$(6) \quad n_0 \cdot n_1 n_2 n_3 \dots$$

Conversely, for any infinite decimal (6) the set E of numbers (5) is bounded above, and (6) is the decimal expansion of $\sup E$.

Since we shall never use decimals, we do not enter into a detailed discussion.

THE EXTENDED REAL NUMBER SYSTEM

1.23 Definition The extended real number system consists of the real field R and two symbols, $+\infty$ and $-\infty$. We preserve the original order in R , and define

$$-\infty < x < +\infty$$

for every $x \in R$.

It is then clear that $+\infty$ is an upper bound of every subset of the extended real number system, and that every nonempty subset has a least upper bound. If, for example, E is a nonempty set of real numbers which is not bounded above in R , then $\sup E = +\infty$ in the extended real number system.

Exactly the same remarks apply to lower bounds.

The extended real number system does not form a field, but it is customary to make the following conventions:

(a) If x is real then

$$x + \infty = +\infty, \quad x - \infty = -\infty, \quad \frac{x}{+\infty} = \frac{x}{-\infty} = 0.$$

(b) If $x > 0$ then $x \cdot (+\infty) = +\infty$, $x \cdot (-\infty) = -\infty$.

(c) If $x < 0$ then $x \cdot (+\infty) = -\infty$, $x \cdot (-\infty) = +\infty$.

When it is desired to make the distinction between real numbers on the one hand and the symbols $+\infty$ and $-\infty$ on the other quite explicit, the former are called *finite*.

THE COMPLEX FIELD

1.24 Definition A *complex number* is an ordered pair (a, b) of real numbers. "Ordered" means that (a, b) and (b, a) are regarded as distinct if $a \neq b$.

Let $x = (a, b)$, $y = (c, d)$ be two complex numbers. We write $x = y$ if and only if $a = c$ and $b = d$. (Note that this definition is not entirely superfluous; think of equality of rational numbers, represented as quotients of integers.) We define

$$\begin{aligned} x + y &= (a + c, b + d), \\ xy &= (ac - bd, ad + bc). \end{aligned}$$

1.25 Theorem These definitions of addition and multiplication turn the set of all complex numbers into a field, with $(0, 0)$ and $(1, 0)$ in the role of 0 and 1.

Proof We simply verify the field axioms, as listed in Definition 1.12. (Of course, we use the field structure of R .)

Let $x = (a, b)$, $y = (c, d)$, $z = (e, f)$.

(A1) is clear.

(A2) $x + y = (a + c, b + d) = (c + a, d + b) = y + x$.

- (A3) $(x + y) + z = (a + c, b + d) + (e, f)$
 $= (a + c + e, b + d + f)$
 $= (a, b) + (c + e, d + f) = x + (y + z).$
- (A4) $x + 0 = (a, b) + (0, 0) = (a, b) = x.$
- (A5) Put $-x = (-a, -b)$. Then $x + (-x) = (0, 0) = 0.$
- (M1) is clear.
- (M2) $xy = (ac - bd, ad + bc) = (ca - db, da + cb) = yx.$
- (M3) $(xy)z = (ac - bd, ad + bc)(e, f)$
 $= (ace - bde - adf - bcf, acf - bdf + ade + bce)$
 $= (a, b)(ce - df, cf + de) = x(yz).$
- (M4) $1x = (1, 0)(a, b) = (a, b) = x.$
- (M5) If $x \neq 0$ then $(a, b) \neq (0, 0)$, which means that at least one of the real numbers a, b is different from 0. Hence $a^2 + b^2 > 0$, by Proposition 1.18(d), and we can define

$$\frac{1}{x} = \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right).$$

Then

$$x \cdot \frac{1}{x} = (a, b) \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right) = (1, 0) = 1.$$

$$\begin{aligned} \text{(D)} \quad x(y + z) &= (a, b)(c + e, d + f) \\ &= (ac + ae - bd - bf, ad + af + bc + be) \\ &= (ac - bd, ad + bc) + (ae - bf, af + be) \\ &= xy + xz. \end{aligned}$$

1.26 Theorem *For any real numbers a and b we have*

$$(a, 0) + (b, 0) = (a + b, 0), \quad (a, 0)(b, 0) = (ab, 0).$$

The proof is trivial.

Theorem 1.26 shows that the complex numbers of the form $(a, 0)$ have the same arithmetic properties as the corresponding real numbers a . We can therefore identify $(a, 0)$ with a . This identification gives us the real field as a subfield of the complex field.

The reader may have noticed that we have defined the complex numbers without any reference to the mysterious square root of -1 . We now show that the notation (a, b) is equivalent to the more customary $a + bi$.

1.27 Definition $i = (0, 1).$

1.28 Theorem $i^2 = -1$.

Proof $i^2 = (0, 1)(0, 1) = (-1, 0) = -1$.

1.29 Theorem If a and b are real, then $(a, b) = a + bi$.

Proof

$$\begin{aligned} a + bi &= (a, 0) + (b, 0)(0, 1) \\ &= (a, 0) + (0, b) = (a, b). \end{aligned}$$

1.30 Definition If a, b are real and $z = a + bi$, then the complex number $\bar{z} = a - bi$ is called the *conjugate* of z . The numbers a and b are the *real part* and the *imaginary part* of z , respectively.

We shall occasionally write

$$a = \operatorname{Re}(z), \quad b = \operatorname{Im}(z).$$

1.31 Theorem If z and w are complex, then

- (a) $\overline{z + w} = \bar{z} + \bar{w}$,
- (b) $\overline{zw} = \bar{z} \cdot \bar{w}$,
- (c) $z + \bar{z} = 2 \operatorname{Re}(z)$, $z - \bar{z} = 2i \operatorname{Im}(z)$,
- (d) $z\bar{z}$ is real and positive (except when $z = 0$).

Proof (a), (b), and (c) are quite trivial. To prove (d), write $z = a + bi$, and note that $z\bar{z} = a^2 + b^2$.

1.32 Definition If z is a complex number, its *absolute value* $|z|$ is the non-negative square root of $z\bar{z}$; that is, $|z| = (z\bar{z})^{1/2}$.

The existence (and uniqueness) of $|z|$ follows from Theorem 1.21 and part (d) of Theorem 1.31.

Note that when x is real, then $\bar{x} = x$, hence $|x| = \sqrt{x^2}$. Thus $|x| = x$ if $x \geq 0$, $|x| = -x$ if $x < 0$.

1.33 Theorem Let z and w be complex numbers. Then

- (a) $|z| > 0$ unless $z = 0$, $|0| = 0$,
- (b) $|\bar{z}| = |z|$,
- (c) $|zw| = |z| |w|$,
- (d) $|\operatorname{Re} z| \leq |z|$,
- (e) $|z + w| \leq |z| + |w|$.

Proof (a) and (b) are trivial. Put $z = a + bi$, $w = c + di$, with a, b, c, d real. Then

$$|zw|^2 = (ac - bd)^2 + (ad + bc)^2 = (a^2 + b^2)(c^2 + d^2) = |z|^2 |w|^2$$

or $|zw|^2 = (|z| |w|)^2$. Now (c) follows from the uniqueness assertion of Theorem 1.21.

To prove (d), note that $a^2 \leq a^2 + b^2$, hence

$$|a| = \sqrt{a^2} \leq \sqrt{a^2 + b^2}.$$

To prove (e), note that $\bar{z}w$ is the conjugate of $z\bar{w}$, so that $z\bar{w} + \bar{z}w = 2 \operatorname{Re}(z\bar{w})$. Hence

$$\begin{aligned} |z + w|^2 &= (z + w)(\bar{z} + \bar{w}) = z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \\ &= |z|^2 + 2 \operatorname{Re}(z\bar{w}) + |w|^2 \\ &\leq |z|^2 + 2|z\bar{w}| + |w|^2 \\ &= |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2. \end{aligned}$$

Now (e) follows by taking square roots.

1.34 Notation If x_1, \dots, x_n are complex numbers, we write

$$x_1 + x_2 + \cdots + x_n = \sum_{j=1}^n x_j.$$

We conclude this section with an important inequality, usually known as the *Schwarz inequality*.

1.35 Theorem If a_1, \dots, a_n and b_1, \dots, b_n are complex numbers, then

$$\left| \sum_{j=1}^n a_j \bar{b}_j \right|^2 \leq \sum_{j=1}^n |a_j|^2 \sum_{j=1}^n |b_j|^2.$$

Proof Put $A = \sum |a_j|^2$, $B = \sum |b_j|^2$, $C = \sum a_j \bar{b}_j$ (in all sums in this proof, j runs over the values $1, \dots, n$). If $B = 0$, then $b_1 = \cdots = b_n = 0$, and the conclusion is trivial. Assume therefore that $B > 0$. By Theorem 1.31 we have

$$\begin{aligned} \sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\bar{a}_j - \overline{Cb_j}) \\ &= B^2 \sum |a_j|^2 - B\bar{C} \sum a_j \bar{b}_j - BC \sum \bar{a}_j b_j + |C|^2 \sum |b_j|^2 \\ &= B^2 A - B|C|^2 \\ &= B(AB - |C|^2). \end{aligned}$$

Since each term in the first sum is nonnegative, we see that

$$B(AB - |C|^2) \geq 0.$$

Since $B > 0$, it follows that $AB - |C|^2 \geq 0$. This is the desired inequality.

EUCLIDEAN SPACES

1.36 Definitions For each positive integer k , let R^k be the set of all ordered k -tuples

$$\mathbf{x} = (x_1, x_2, \dots, x_k),$$

where x_1, \dots, x_k are real numbers, called the *coordinates* of \mathbf{x} . The elements of R^k are called points, or vectors, especially when $k > 1$. We shall denote vectors by boldfaced letters. If $\mathbf{y} = (y_1, \dots, y_k)$ and if α is a real number, put

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_k + y_k),$$

$$\alpha \mathbf{x} = (\alpha x_1, \dots, \alpha x_k)$$

so that $\mathbf{x} + \mathbf{y} \in R^k$ and $\alpha \mathbf{x} \in R^k$. This defines addition of vectors, as well as multiplication of a vector by a real number (a scalar). These two operations satisfy the commutative, associative, and distributive laws (the proof is trivial, in view of the analogous laws for the real numbers) and make R^k into a *vector space over the real field*. The zero element of R^k (sometimes called the *origin* or the *null vector*) is the point $\mathbf{0}$, all of whose coordinates are 0.

We also define the so-called “inner product” (or scalar product) of \mathbf{x} and \mathbf{y} by

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^k x_i y_i$$

and the *norm* of \mathbf{x} by

$$|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2} = \left(\sum_{i=1}^k x_i^2 \right)^{1/2}.$$

The structure now defined (the vector space R^k with the above inner product and norm) is called *euclidean k -space*.

1.37 Theorem Suppose $\mathbf{x}, \mathbf{y}, \mathbf{z} \in R^k$, and α is real. Then

- (a) $|\mathbf{x}| \geq 0$;
- (b) $|\mathbf{x}| = 0$ if and only if $\mathbf{x} = \mathbf{0}$;
- (c) $|\alpha \mathbf{x}| = |\alpha| |\mathbf{x}|$;
- (d) $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}| |\mathbf{y}|$;
- (e) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$;
- (f) $|\mathbf{x} - \mathbf{z}| \leq |\mathbf{x} - \mathbf{y}| + |\mathbf{y} - \mathbf{z}|$.

Proof (a), (b), and (c) are obvious, and (d) is an immediate consequence of the Schwarz inequality. By (d) we have

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= (\mathbf{x} + \mathbf{y}) \cdot (\mathbf{x} + \mathbf{y}) \\ &= \mathbf{x} \cdot \mathbf{x} + 2\mathbf{x} \cdot \mathbf{y} + \mathbf{y} \cdot \mathbf{y} \\ &\leq |\mathbf{x}|^2 + 2|\mathbf{x}||\mathbf{y}| + |\mathbf{y}|^2 \\ &= (|\mathbf{x}| + |\mathbf{y}|)^2, \end{aligned}$$

so that (e) is proved. Finally, (f) follows from (e) if we replace \mathbf{x} by $\mathbf{x} - \mathbf{y}$ and \mathbf{y} by $\mathbf{y} - \mathbf{z}$.

1.38 Remarks Theorem 1.37 (a), (b), and (f) will allow us (see Chap. 2) to regard R^k as a metric space.

R^1 (the set of all real numbers) is usually called the line, or the real line. Likewise, R^2 is called the plane, or the complex plane (compare Definitions 1.24 and 1.36). In these two cases the norm is just the absolute value of the corresponding real or complex number.

APPENDIX

Theorem 1.19 will be proved in this appendix by constructing R from Q . We shall divide the construction into several steps.

Step 1 The members of R will be certain subsets of Q , called *cuts*. A cut is, by definition, any set $\alpha \subset Q$ with the following three properties.

- (I) α is not empty, and $\alpha \neq Q$.
- (II) If $p \in \alpha$, $q \in Q$, and $q < p$, then $q \in \alpha$.
- (III) If $p \in \alpha$, then $p < r$ for some $r \in \alpha$.

The letters p, q, r, \dots will always denote rational numbers, and $\alpha, \beta, \gamma, \dots$ will denote cuts.

Note that (III) simply says that α has no largest member; (II) implies two facts which will be used freely:

- If $p \in \alpha$ and $q \notin \alpha$ then $p < q$.
- If $r \notin \alpha$ and $r < s$ then $s \notin \alpha$.

Step 2 Define " $\alpha < \beta$ " to mean: α is a proper subset of β .

Let us check that this meets the requirements of Definition 1.5.

If $\alpha < \beta$ and $\beta < \gamma$ it is clear that $\alpha < \gamma$. (A proper subset of a proper subset is a proper subset.) It is also clear that at most one of the three relations

$$\alpha < \beta, \quad \alpha = \beta, \quad \beta < \alpha$$

can hold for any pair α, β . To show that at least one holds, assume that the first two fail. Then α is not a subset of β . Hence there is a $p \in \alpha$ with $p \notin \beta$. If $q \in \beta$, it follows that $q < p$ (since $p \notin \beta$), hence $q \in \alpha$, by (II). Thus $\beta \subset \alpha$. Since $\beta \neq \alpha$, we conclude: $\beta < \alpha$.

Thus R is now an ordered set.

Step 3 *The ordered set R has the least-upper-bound property.*

To prove this, let A be a nonempty subset of R , and assume that $\beta \in R$ is an upper bound of A . Define γ to be the union of all $\alpha \in A$. In other words, $p \in \gamma$ if and only if $p \in \alpha$ for some $\alpha \in A$. We shall prove that $\gamma \in R$ and that $\gamma = \sup A$.

Since A is not empty, there exists an $\alpha_0 \in A$. This α_0 is not empty. Since $\alpha_0 \subset \gamma$, γ is not empty. Next, $\gamma \subset \beta$ (since $\alpha \subset \beta$ for every $\alpha \in A$), and therefore $\gamma \neq Q$. Thus γ satisfies property (I). To prove (II) and (III), pick $p \in \gamma$. Then $p \in \alpha_1$ for some $\alpha_1 \in A$. If $q < p$, then $q \in \alpha_1$, hence $q \in \gamma$; this proves (II). If $r \in \alpha_1$ is so chosen that $r > p$, we see that $r \in \gamma$ (since $\alpha_1 \subset \gamma$), and therefore γ satisfies (III).

Thus $\gamma \in R$.

It is clear that $\alpha \leq \gamma$ for every $\alpha \in A$.

Suppose $\delta < \gamma$. Then there is an $s \in \gamma$ and that $s \notin \delta$. Since $s \in \gamma$, $s \in \alpha$ for some $\alpha \in A$. Hence $\delta < \alpha$, and δ is not an upper bound of A .

This gives the desired result: $\gamma = \sup A$.

Step 4 If $\alpha \in R$ and $\beta \in R$ we define $\alpha + \beta$ to be the set of all sums $r + s$, where $r \in \alpha$ and $s \in \beta$.

We define 0^* to be the set of all negative rational numbers. It is clear that 0^* is a cut. We verify that the axioms for addition (see Definition 1.12) hold in R , with 0^* playing the role of 0.

(A1) We have to show that $\alpha + \beta$ is a cut. It is clear that $\alpha + \beta$ is a nonempty subset of Q . Take $r' \notin \alpha$, $s' \notin \beta$. Then $r' + s' > r + s$ for all choices of $r \in \alpha$, $s \in \beta$. Thus $r' + s' \notin \alpha + \beta$. It follows that $\alpha + \beta$ has property (I).

Pick $p \in \alpha + \beta$. Then $p = r + s$, with $r \in \alpha$, $s \in \beta$. If $q < p$, then $q - s < r$, so $q - s \in \alpha$, and $q = (q - s) + s \in \alpha + \beta$. Thus (II) holds. Choose $t \in \alpha$ so that $t > r$. Then $p < t + s$ and $t + s \in \alpha + \beta$. Thus (III) holds.

(A2) $\alpha + \beta$ is the set of all $r + s$, with $r \in \alpha$, $s \in \beta$. By the same definition, $\beta + \alpha$ is the set of all $s + r$. Since $r + s = s + r$ for all $r \in Q$, $s \in Q$, we have $\alpha + \beta = \beta + \alpha$.

(A3) As above, this follows from the associative law in Q .

(A4) If $r \in \alpha$ and $s \in 0^*$, then $r + s < r$, hence $r + s \in \alpha$. Thus $\alpha + 0^* \subset \alpha$. To obtain the opposite inclusion, pick $p \in \alpha$, and pick $r \in \alpha$, $r > p$. Then

$p - r \in 0^*$, and $p = r + (p - r) \in \alpha + 0^*$. Thus $\alpha \subset \alpha + 0^*$. We conclude that $\alpha + 0^* = \alpha$.

(A5) Fix $\alpha \in R$. Let β be the set of all p with the following property:

There exists $r > 0$ such that $-p - r \notin \alpha$.

In other words, some rational number smaller than $-p$ fails to be in α .

We show that $\beta \in R$ and that $\alpha + \beta = 0^$.*

If $s \notin \alpha$ and $p = -s - 1$, then $-p - 1 \notin \alpha$, hence $p \in \beta$. So β is not empty. If $q \in \alpha$, then $-q \notin \beta$. So $\beta \neq Q$. Hence β satisfies (I).

Pick $p \in \beta$, and pick $r > 0$, so that $-p - r \notin \alpha$. If $q < p$, then $-q - r > -p - r$, hence $-q - r \notin \alpha$. Thus $q \in \beta$, and (II) holds. Put $t = p + (r/2)$. Then $t > p$, and $-t - (r/2) = -p - r \notin \alpha$, so that $t \in \beta$. Hence β satisfies (III).

We have proved that $\beta \in R$.

If $r \in \alpha$ and $s \in \beta$, then $-s \notin \alpha$, hence $r < -s$, $r + s < 0$. Thus $\alpha + \beta \subset 0^*$.

To prove the opposite inclusion, pick $v \in 0^*$, put $w = -v/2$. Then $w > 0$, and there is an integer n such that $nw \in \alpha$ but $(n+1)w \notin \alpha$. (Note that this depends on the fact that Q has the archimedean property!) Put $p = -(n+2)w$. Then $p \in \beta$, since $-p - w \notin \alpha$, and

$$v = nw + p \in \alpha + \beta.$$

Thus $0^* \subset \alpha + \beta$.

We conclude that $\alpha + \beta = 0^*$.

This β will of course be denoted by $-\alpha$.

Step 5 Having proved that the addition defined in Step 4 satisfies Axioms (A) of Definition 1.12, it follows that Proposition 1.14 is valid in R , and we can prove one of the requirements of Definition 1.17:

If $\alpha, \beta, \gamma \in R$ and $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.

Indeed, it is obvious from the definition of $+$ in R that $\alpha + \beta \subset \alpha + \gamma$; if we had $\alpha + \beta = \alpha + \gamma$, the cancellation law (Proposition 1.14) would imply $\beta = \gamma$.

It also follows that $\alpha > 0^*$ if and only if $-\alpha < 0^*$.

Step 6 Multiplication is a little more bothersome than addition in the present context, since products of negative rationals are positive. For this reason we confine ourselves first to R^+ , the set of all $\alpha \in R$ with $\alpha > 0^*$.

If $\alpha \in R^+$ and $\beta \in R^+$, we define $\alpha\beta$ to be the set of all p such that $p \leq rs$ for some choice of $r \in \alpha$, $s \in \beta$, $r > 0$, $s > 0$.

We define 1^* to be the set of all $q < 1$.

Then the axioms (M) and (D) of Definition 1.12 hold, with R^+ in place of F , and with 1^* in the role of 1.

The proofs are so similar to the ones given in detail in Step 4 that we omit them.

Note, in particular, that the second requirement of Definition 1.17 holds: If $\alpha > 0^*$ and $\beta > 0^*$ then $\alpha\beta > 0^*$.

Step 7 We complete the definition of multiplication by setting $\alpha 0^* = 0^* \alpha = 0^*$, and by setting

$$\alpha\beta = \begin{cases} (-\alpha)(-\beta) & \text{if } \alpha < 0^*, \beta < 0^*, \\ -[(-\alpha)\beta] & \text{if } \alpha < 0^*, \beta > 0^*, \\ -[\alpha \cdot (-\beta)] & \text{if } \alpha > 0^*, \beta < 0^*. \end{cases}$$

The products on the right were defined in Step 6.

Having proved (in Step 6) that the axioms (M) hold in R^+ , it is now perfectly simple to prove them in R , by repeated application of the identity $\gamma = -(-\gamma)$ which is part of Proposition 1.14. (See Step 5.)

The proof of the distributive law

$$\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$$

breaks into cases. For instance, suppose $\alpha > 0^*$, $\beta < 0^*$, $\beta + \gamma > 0^*$. Then $\gamma = (\beta + \gamma) + (-\beta)$, and (since we already know that the distributive law holds in R^+)

$$\alpha\gamma = \alpha(\beta + \gamma) + \alpha \cdot (-\beta).$$

But $\alpha \cdot (-\beta) = -(\alpha\beta)$. Thus

$$\alpha\beta + \alpha\gamma = \alpha(\beta + \gamma).$$

The other cases are handled in the same way.

We have now completed the proof that R is an ordered field with the least-upper-bound property.

Step 8 We associate with each $r \in Q$ the set r^* which consists of all $p \in Q$ such that $p < r$. It is clear that each r^* is a cut; that is, $r^* \in R$. These cuts satisfy the following relations:

- (a) $r^* + s^* = (r + s)^*$,
- (b) $r^* s^* = (rs)^*$,
- (c) $r^* < s^*$ if and only if $r < s$.

To prove (a), choose $p \in r^* + s^*$. Then $p = u + v$, where $u < r$, $v < s$. Hence $p < r + s$, which says that $p \in (r + s)^*$.

Conversely, suppose $p \in (r + s)^*$. Then $p < r + s$. Choose t so that $2t = r + s - p$, put

$$r' = r - t, s' = s - t.$$

Then $r' \in r^*$, $s' \in s^*$, and $p = r' + s'$, so that $p \in r^* + s^*$.

This proves (a). The proof of (b) is similar.

If $r < s$ then $r \in s^*$, but $r \notin r^*$; hence $r^* < s^*$.

If $r^* < s^*$, then there is a $p \in s^*$ such that $p \notin r^*$. Hence $r \leq p < s$, so that $r < s$.

This proves (c).

Step 9 We saw in Step 8 that the replacement of the rational numbers r by the corresponding “rational cuts” $r^* \in R$ preserves sums, products, and order. This fact may be expressed by saying that the ordered field Q is *isomorphic* to the ordered field Q^* whose elements are the rational cuts. Of course, r^* is by no means the same as r , but the properties we are concerned with (arithmetic and order) are the same in the two fields.

It is this identification of Q with Q^ which allows us to regard Q as a subfield of R .*

The second part of Theorem 1.19 is to be understood in terms of this identification. Note that the same phenomenon occurs when the real numbers are regarded as a subfield of the complex field, and it also occurs at a much more elementary level, when the integers are identified with a certain subset of Q .

It is a fact, which we will not prove here, that *any two ordered fields with the least-upper-bound property are isomorphic*. The first part of Theorem 1.19 therefore characterizes the real field R completely.

The books by Landau and Thurston cited in the Bibliography are entirely devoted to number systems. Chapter 1 of Knopp’s book contains a more leisurely description of how R can be obtained from Q . Another construction, in which each real number is defined to be an equivalence class of Cauchy sequences of rational numbers (see Chap. 3), is carried out in Sec. 5 of the book by Hewitt and Stromberg.

The cuts in Q which we used here were invented by Dedekind. The construction of R from Q by means of Cauchy sequences is due to Cantor. Both Cantor and Dedekind published their constructions in 1872.

EXERCISES

Unless the contrary is explicitly stated, all numbers that are mentioned in these exercises are understood to be real.

1. If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

2. Prove that there is no rational number whose square is 12.
3. Prove Proposition 1.15.
4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E . Prove that $\alpha \leq \beta$.
5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

6. Fix $b > 1$.

(a) If m, n, p, q are integers, $n > 0, q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

(b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(c) If x is real, define $B(x)$ to be the set of all numbers b^t , where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x .

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y .

7. Fix $b > 1, y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the *logarithm of y to the base b* .)

(a) For any positive integer n , $b^n - 1 \geq n(b - 1)$.

(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

(c) If $t > 1$ and $n > (b - 1)/(t - 1)$, then $b^{1/n} < t$.

(d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n ; to see this, apply part (c) with $t = y \cdot b^{-w}$.

(e) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n .

(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

(g) Prove that this x is unique.

8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.
9. Suppose $z = a + bi, w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?
10. Suppose $z = a + bi, w = u + iv$, and

$$a = \left(\frac{|w| + u}{2} \right)^{1/2}, \quad b = \left(\frac{|w| - u}{2} \right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception!) has two complex square roots.

11. If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z ?

12. If z_1, \dots, z_n are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

13. If x, y are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

14. If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2.$$

15. Under what conditions does equality hold in the Schwarz inequality?

16. Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in R^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:

(a) If $2r > d$, there are infinitely many $\mathbf{z} \in R^k$ such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

(b) If $2r = d$, there is exactly one such \mathbf{z} .

(c) If $2r < d$, there is no such \mathbf{z} .

How must these statements be modified if k is 2 or 1?

17. Prove that

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

if $\mathbf{x} \in R^k$ and $\mathbf{y} \in R^k$. Interpret this geometrically, as a statement about parallelograms.

18. If $k \geq 2$ and $\mathbf{x} \in R^k$, prove that there exists $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

19. Suppose $\mathbf{a} \in R^k$, $\mathbf{b} \in R^k$. Find $\mathbf{c} \in R^k$ and $r > 0$ such that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

if and only if $|\mathbf{x} - \mathbf{c}| = r$.

(Solution: $3\mathbf{c} = 4\mathbf{b} - \mathbf{a}$, $3r = 2|\mathbf{b} - \mathbf{a}|$.)

20. With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero-element!) but that (A5) fails.

2

BASIC TOPOLOGY

FINITE, COUNTABLE, AND UNCOUNTABLE SETS

We begin this section with a definition of the function concept.

2.1 Definition Consider two sets A and B , whose elements may be any objects whatsoever, and suppose that with each element x of A there is associated, in some manner, an element of B , which we denote by $f(x)$. Then f is said to be a *function* from A to B (or a *mapping* of A into B). The set A is called the *domain* of f (we also say f is defined on A), and the elements $f(x)$ are called the *values* of f . The set of all values of f is called the *range* of f .

2.2 Definition Let A and B be two sets and let f be a mapping of A into B . If $E \subset A$, $f(E)$ is defined to be the set of all elements $f(x)$, for $x \in E$. We call $f(E)$ the *image* of E under f . In this notation, $f(A)$ is the range of f . It is clear that $f(A) \subset B$. If $f(A) = B$, we say that f maps A *onto* B . (Note that, according to this usage, *onto* is more specific than *into*.)

If $E \subset B$, $f^{-1}(E)$ denotes the set of all $x \in A$ such that $f(x) \in E$. We call $f^{-1}(E)$ the *inverse image* of E under f . If $y \in B$, $f^{-1}(y)$ is the set of all $x \in A$

such that $f(x) = y$. If, for each $y \in B$, $f^{-1}(y)$ consists of at most one element of A , then f is said to be a 1-1 (*one-to-one*) mapping of A into B . This may also be expressed as follows: f is a 1-1 mapping of A into B provided that $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$, $x_1 \in A$, $x_2 \in A$.

(The notation $x_1 \neq x_2$ means that x_1 and x_2 are distinct elements; otherwise we write $x_1 = x_2$.)

2.3 Definition If there exists a 1-1 mapping of A onto B , we say that A and B can be put in 1-1 *correspondence*, or that A and B have the same *cardinal number*, or, briefly, that A and B are *equivalent*, and we write $A \sim B$. This relation clearly has the following properties:

It is reflexive: $A \sim A$.

It is symmetric: If $A \sim B$, then $B \sim A$.

It is transitive: If $A \sim B$ and $B \sim C$, then $A \sim C$.

Any relation with these three properties is called an *equivalence relation*.

2.4 Definition For any positive integer n , let J_n be the set whose elements are the integers $1, 2, \dots, n$; let J be the set consisting of all positive integers. For any set A , we say:

- (a) A is *finite* if $A \sim J_n$ for some n (the empty set is also considered to be finite).
- (b) A is *infinite* if A is not finite.
- (c) A is *countable* if $A \sim J$.
- (d) A is *uncountable* if A is neither finite nor countable.
- (e) A is *at most countable* if A is finite or countable.

Countable sets are sometimes called *enumerable*, or *denumerable*.

For two finite sets A and B , we evidently have $A \sim B$ if and only if A and B contain the same number of elements. For infinite sets, however, the idea of "having the same number of elements" becomes quite vague, whereas the notion of 1-1 correspondence retains its clarity.

2.5 Example Let A be the set of all integers. Then A is countable. For, consider the following arrangement of the sets A and J :

$$\begin{array}{ll} A: & 0, 1, -1, 2, -2, 3, -3, \dots \\ J: & 1, 2, 3, 4, 5, 6, 7, \dots \end{array}$$

We can, in this example, even give an explicit formula for a function f from J to A which sets up a 1-1 correspondence:

$$f(n) = \begin{cases} \frac{n}{2} & (n \text{ even}), \\ -\frac{n-1}{2} & (n \text{ odd}). \end{cases}$$

2.6 Remark A finite set cannot be equivalent to one of its proper subsets. That this is, however, possible for infinite sets, is shown by Example 2.5, in which J is a proper subset of A .

In fact, we could replace Definition 2.4(b) by the statement: A is infinite if A is equivalent to one of its proper subsets.

2.7 Definition By a *sequence*, we mean a function f defined on the set J of all positive integers. If $f(n) = x_n$, for $n \in J$, it is customary to denote the sequence f by the symbol $\{x_n\}$, or sometimes by x_1, x_2, x_3, \dots . The values of f , that is, the elements x_n , are called the *terms* of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a *sequence in A* , or a *sequence of elements of A* .

Note that the terms x_1, x_2, x_3, \dots of a sequence need not be distinct.

Since every countable set is the range of a 1-1 function defined on J , we may regard every countable set as the range of a sequence of distinct terms. Speaking more loosely, we may say that the elements of any countable set can be "arranged in a sequence."

Sometimes it is convenient to replace J in this definition by the set of all nonnegative integers, i.e., to start with 0 rather than with 1.

2.8 Theorem Every infinite subset of a countable set A is countable.

Proof Suppose $E \subset A$, and E is infinite. Arrange the elements x of A in a sequence $\{x_n\}$ of distinct elements. Construct a sequence $\{n_k\}$ as follows:

Let n_1 be the smallest positive integer such that $x_{n_1} \in E$. Having chosen n_1, \dots, n_{k-1} ($k = 2, 3, 4, \dots$), let n_k be the smallest integer greater than n_{k-1} such that $x_{n_k} \in E$.

Putting $f(k) = x_{n_k}$ ($k = 1, 2, 3, \dots$), we obtain a 1-1 correspondence between E and J .

The theorem shows that, roughly speaking, countable sets represent the "smallest" infinity: No uncountable set can be a subset of a countable set.

2.9 Definition Let A and Ω be sets, and suppose that with each element α of A there is associated a subset of Ω which we denote by E_α .

The set whose elements are the sets E_α will be denoted by $\{E_\alpha\}$. Instead of speaking of sets of sets, we shall sometimes speak of a collection of sets, or a family of sets.

The *union* of the sets E_α is defined to be the set S such that $x \in S$ if and only if $x \in E_\alpha$ for at least one $\alpha \in A$. We use the notation

$$(1) \quad S = \bigcup_{\alpha \in A} E_\alpha.$$

If A consists of the integers $1, 2, \dots, n$, one usually writes

$$(2) \quad S = \bigcup_{m=1}^n E_m$$

or

$$(3) \quad S = E_1 \cup E_2 \cup \cdots \cup E_n.$$

If A is the set of all positive integers, the usual notation is

$$(4) \quad S = \bigcup_{m=1}^{\infty} E_m.$$

The symbol ∞ in (4) merely indicates that the union of a *countable* collection of sets is taken, and should not be confused with the symbols $+\infty$, $-\infty$, introduced in Definition 1.23.

The *intersection* of the sets E_α is defined to be the set P such that $x \in P$ if and only if $x \in E_\alpha$ for every $\alpha \in A$. We use the notation

$$(5) \quad P = \bigcap_{\alpha \in A} E_\alpha,$$

or

$$(6) \quad P = \bigcap_{m=1}^n E_m = E_1 \cap E_2 \cap \cdots \cap E_n,$$

or

$$(7) \quad P = \bigcap_{m=1}^{\infty} E_m,$$

as for unions. If $A \cap B$ is not empty, we say that A and B *intersect*; otherwise they are *disjoint*.

2.10 Examples

(a) Suppose E_1 consists of 1, 2, 3 and E_2 consists of 2, 3, 4. Then $E_1 \cup E_2$ consists of 1, 2, 3, 4, whereas $E_1 \cap E_2$ consists of 2, 3.

(b) Let A be the set of real numbers x such that $0 < x \leq 1$. For every $x \in A$, let E_x be the set of real numbers y such that $0 < y < x$. Then

- (i) $E_x \subset E_z$ if and only if $0 < x \leq z \leq 1$;
- (ii) $\bigcup_{x \in A} E_x = E_1$;
- (iii) $\bigcap_{x \in A} E_x$ is empty;

(i) and (ii) are clear. To prove (iii), we note that for every $y > 0$, $y \notin E_x$ if $x < y$. Hence $y \notin \bigcap_{x \in A} E_x$.

2.11 Remarks Many properties of unions and intersections are quite similar to those of sums and products; in fact, the words sum and product were sometimes used in this connection, and the symbols Σ and Π were written in place of \bigcup and \bigcap .

The commutative and associative laws are trivial:

$$(8) \quad A \cup B = B \cup A; \quad A \cap B = B \cap A.$$

$$(9) \quad (A \cup B) \cup C = A \cup (B \cup C); \quad (A \cap B) \cap C = A \cap (B \cap C).$$

Thus the omission of parentheses in (3) and (6) is justified.

The distributive law also holds:

$$(10) \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

To prove this, let the left and right members of (10) be denoted by E and F , respectively.

Suppose $x \in E$. Then $x \in A$ and $x \in B \cup C$, that is, $x \in B$ or $x \in C$ (possibly both). Hence $x \in A \cap B$ or $x \in A \cap C$, so that $x \in F$. Thus $E \subset F$.

Next, suppose $x \in F$. Then $x \in A \cap B$ or $x \in A \cap C$. That is, $x \in A$, and $x \in B \cup C$. Hence $x \in A \cap (B \cup C)$, so that $F \subset E$.

It follows that $E = F$.

We list a few more relations which are easily verified:

$$(11) \quad A \subset A \cup B,$$

$$(12) \quad A \cap B \subset A.$$

If 0 denotes the empty set, then

$$(13) \quad A \cup 0 = A, \quad A \cap 0 = 0.$$

If $A \subset B$, then

$$(14) \quad A \cup B = B, \quad A \cap B = A.$$

2.12 Theorem Let $\{E_n\}$, $n = 1, 2, 3, \dots$, be a sequence of countable sets, and put

$$(15) \quad S = \bigcup_{n=1}^{\infty} E_n.$$

Then S is countable.

Proof Let every set E_n be arranged in a sequence $\{x_{nk}\}$, $k = 1, 2, 3, \dots$, and consider the infinite array

$$(16) \quad \begin{array}{ccccccc} x_{11} & x_{12} & x_{13} & x_{14} & \dots & & \\ x_{21} & x_{22} & x_{23} & x_{24} & \dots & & \\ x_{31} & x_{32} & x_{33} & x_{34} & \dots & & \\ x_{41} & x_{42} & x_{43} & x_{44} & \dots & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \end{array}$$

in which the elements of E_n form the n th row. The array contains all elements of S . As indicated by the arrows, these elements can be arranged in a sequence

$$(17) \quad x_{11}; x_{21}, x_{12}; x_{31}, x_{22}, x_{13}; x_{41}, x_{32}, x_{23}, x_{14}; \dots$$

If any two of the sets E_n have elements in common, these will appear more than once in (17). Hence there is a subset T of the set of all positive integers such that $S \sim T$, which shows that S is at most countable (Theorem 2.8). Since $E_1 \subset S$, and E_1 is infinite, S is infinite, and thus countable.

Corollary Suppose A is at most countable, and, for every $\alpha \in A$, B_α is at most countable. Put

$$T = \bigcup_{\alpha \in A} B_\alpha.$$

Then T is at most countable.

For T is equivalent to a subset of (15).

2.13 Theorem Let A be a countable set, and let B_n be the set of all n -tuples (a_1, \dots, a_n) , where $a_k \in A$ ($k = 1, \dots, n$), and the elements a_1, \dots, a_n need not be distinct. Then B_n is countable.

Proof That B_1 is countable is evident, since $B_1 = A$. Suppose B_{n-1} is countable ($n = 2, 3, 4, \dots$). The elements of B_n are of the form

$$(18) \quad (b, a) \quad (b \in B_{n-1}, a \in A).$$

For every fixed b , the set of pairs (b, a) is equivalent to A , and hence countable. Thus B_n is the union of a countable set of countable sets. By Theorem 2.12, B_n is countable.

The theorem follows by induction.

Corollary *The set of all rational numbers is countable.*

Proof We apply Theorem 2.13, with $n = 2$, noting that every rational r is of the form b/a , where a and b are integers. The set of pairs (a, b) , and therefore the set of fractions b/a , is countable.

In fact, even the set of all algebraic numbers is countable (see Exercise 2).

That not all infinite sets are, however, countable, is shown by the next theorem.

2.14 Theorem *Let A be the set of all sequences whose elements are the digits 0 and 1. This set A is uncountable.*

The elements of A are sequences like 1, 0, 0, 1, 0, 1, 1, 1,

Proof Let E be a countable subset of A , and let E consist of the sequences s_1, s_2, s_3, \dots . We construct a sequence s as follows. If the n th digit in s_n is 1, we let the n th digit of s be 0, and vice versa. Then the sequence s differs from every member of E in at least one place; hence $s \notin E$. But clearly $s \in A$, so that E is a proper subset of A .

We have shown that every countable subset of A is a proper subset of A . It follows that A is uncountable (for otherwise A would be a proper subset of A , which is absurd).

The idea of the above proof was first used by Cantor, and is called Cantor's diagonal process; for, if the sequences s_1, s_2, s_3, \dots are placed in an array like (16), it is the elements on the diagonal which are involved in the construction of the new sequence.

Readers who are familiar with the binary representation of the real numbers (base 2 instead of 10) will notice that Theorem 2.14 implies that the set of all real numbers is uncountable. We shall give a second proof of this fact in Theorem 2.43.

METRIC SPACES

2.15 Definition A set X , whose elements we shall call *points*, is said to be a *metric space* if with any two points p and q of X there is associated a real number $d(p, q)$, called the *distance* from p to q , such that

- (a) $d(p, q) > 0$ if $p \neq q$; $d(p, p) = 0$;
- (b) $d(p, q) = d(q, p)$;
- (c) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$.

Any function with these three properties is called a *distance function*, or a *metric*.

2.16 Examples The most important examples of metric spaces, from our standpoint, are the euclidean spaces R^k , especially R^1 (the real line) and R^2 (the complex plane); the distance in R^k is defined by

$$(19) \quad d(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}| \quad (\mathbf{x}, \mathbf{y} \in R^k).$$

By Theorem 1.37, the conditions of Definition 2.15 are satisfied by (19).

It is important to observe that every subset Y of a metric space X is a metric space in its own right, with the same distance function. For it is clear that if conditions (a) to (c) of Definition 2.15 hold for $p, q, r \in X$, they also hold if we restrict p, q, r to lie in Y .

Thus every subset of a euclidean space is a metric space. Other examples are the spaces $\mathcal{C}(K)$ and $\mathcal{L}^2(\mu)$, which are discussed in Chaps. 7 and 11, respectively.

2.17 Definition By the *segment* (a, b) we mean the set of all real numbers x such that $a < x < b$.

By the *interval* $[a, b]$ we mean the set of all real numbers x such that $a \leq x \leq b$.

Occasionally we shall also encounter "half-open intervals" $[a, b)$ and $(a, b]$; the first consists of all x such that $a \leq x < b$, the second of all x such that $a < x \leq b$.

If $a_i < b_i$ for $i = 1, \dots, k$, the set of all points $\mathbf{x} = (x_1, \dots, x_k)$ in R^k whose coordinates satisfy the inequalities $a_i \leq x_i \leq b_i$ ($1 \leq i \leq k$) is called a *k-cell*. Thus a 1-cell is an interval, a 2-cell is a rectangle, etc.

If $\mathbf{x} \in R^k$ and $r > 0$, the *open* (or *closed*) *ball* B with center at \mathbf{x} and radius r is defined to be the set of all $\mathbf{y} \in R^k$ such that $|\mathbf{y} - \mathbf{x}| < r$ (or $|\mathbf{y} - \mathbf{x}| \leq r$).

We call a set $E \subset R^k$ *convex* if

$$\lambda \mathbf{x} + (1 - \lambda) \mathbf{y} \in E$$

whenever $\mathbf{x} \in E$, $\mathbf{y} \in E$, and $0 < \lambda < 1$.

For example, *balls are convex*. For if $|\mathbf{y} - \mathbf{x}| < r$, $|\mathbf{z} - \mathbf{x}| < r$, and $0 < \lambda < 1$, we have

$$\begin{aligned} |\lambda \mathbf{y} + (1 - \lambda) \mathbf{z} - \mathbf{x}| &= |\lambda(\mathbf{y} - \mathbf{x}) + (1 - \lambda)(\mathbf{z} - \mathbf{x})| \\ &\leq \lambda |\mathbf{y} - \mathbf{x}| + (1 - \lambda) |\mathbf{z} - \mathbf{x}| < \lambda r + (1 - \lambda)r \\ &= r. \end{aligned}$$

The same proof applies to closed balls. It is also easy to see that *k-cells are convex*.

2.18 Definition Let X be a metric space. All points and sets mentioned below are understood to be elements and subsets of X .

- (a) A *neighborhood* of p is a set $N_r(p)$ consisting of all q such that $d(p, q) < r$, for some $r > 0$. The number r is called the *radius* of $N_r(p)$.
- (b) A point p is a *limit point* of the set E if *every* neighborhood of p contains a point $q \neq p$ such that $q \in E$.
- (c) If $p \in E$ and p is not a limit point of E , then p is called an *isolated point* of E .
- (d) E is *closed* if every limit point of E is a point of E .
- (e) A point p is an *interior point* of E if there is a neighborhood N of p such that $N \subset E$.
- (f) E is *open* if every point of E is an interior point of E .
- (g) The *complement* of E (denoted by E^c) is the set of all points $p \in X$ such that $p \notin E$.
- (h) E is *perfect* if E is closed and if every point of E is a limit point of E .
- (i) E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.
- (j) E is *dense in* X if every point of X is a limit point of E , or a point of E (or both).

Let us note that in R^1 neighborhoods are segments, whereas in R^2 neighborhoods are interiors of circles.

2.19 Theorem *Every neighborhood is an open set.*

Proof Consider a neighborhood $E = N_r(p)$, and let q be any point of E . Then there is a positive real number h such that

$$d(p, q) = r - h.$$

For all points s such that $d(q, s) < h$, we have then

$$d(p, s) \leq d(p, q) + d(q, s) < r - h + h = r,$$

so that $s \in E$. Thus q is an interior point of E .

2.20 Theorem *If p is a limit point of a set E , then every neighborhood of p contains infinitely many points of E .*

Proof Suppose there is a neighborhood N of p which contains only a finite number of points of E . Let q_1, \dots, q_n be those points of $N \cap E$, which are distinct from p , and put

$$r = \min_{1 \leq m \leq n} d(p, q_m)$$

[we use this notation to denote the smallest of the numbers $d(p, q_1), \dots, d(p, q_n)$]. The minimum of a finite set of positive numbers is clearly positive, so that $r > 0$.

The neighborhood $N_r(p)$ contains no point q of E such that $q \neq p$, so that p is not a limit point of E . This contradiction establishes the theorem.

Corollary *A finite point set has no limit points.*

2.21 Examples Let us consider the following subsets of R^2 :

- (a) The set of all complex z such that $|z| < 1$.
- (b) The set of all complex z such that $|z| \leq 1$.
- (c) A nonempty finite set.
- (d) The set of all integers.
- (e) The set consisting of the numbers $1/n$ ($n = 1, 2, 3, \dots$). Let us note that this set E has a limit point (namely, $z = 0$) but that no point of E is a limit point of E ; we wish to stress the difference between having a limit point and containing one.
- (f) The set of all complex numbers (that is, R^2).
- (g) The segment (a, b) .

Let us note that (d), (e), (g) can be regarded also as subsets of R^1 .

Some properties of these sets are tabulated below:

	<i>Closed</i>	<i>Open</i>	<i>Perfect</i>	<i>Bounded</i>
(a)	No	Yes	No	Yes
(b)	Yes	No	Yes	Yes
(c)	Yes	No	No	Yes
(d)	Yes	No	No	No
(e)	No	No	No	Yes
(f)	Yes	Yes	Yes	No
(g)	No		No	Yes

In (g), we left the second entry blank. The reason is that the segment (a, b) is not open if we regard it as a subset of R^2 , but it is an open subset of R^1 .

2.22 Theorem *Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α . Then*

$$(20) \quad \left(\bigcup_{\alpha} E_{\alpha} \right)^c = \bigcap_{\alpha} (E_{\alpha}^c).$$

Proof Let A and B be the left and right members of (20). If $x \in A$, then $x \notin \bigcup_{\alpha} E_{\alpha}$, hence $x \notin E_{\alpha}$ for any α , hence $x \in E_{\alpha}^c$ for every α , so that $x \in \bigcap_{\alpha} E_{\alpha}^c$. Thus $A \subset B$.

Conversely, if $x \in B$, then $x \in E_\alpha^c$ for every α , hence $x \notin E_\alpha$ for any α , hence $x \notin \bigcup_\alpha E_\alpha$, so that $x \in (\bigcup_\alpha E_\alpha)^c$. Thus $B \subset A$.

It follows that $A = B$.

2.23 Theorem *A set E is open if and only if its complement is closed.*

Proof First, suppose E^c is closed. Choose $x \in E$. Then $x \notin E^c$, and x is not a limit point of E^c . Hence there exists a neighborhood N of x such that $E^c \cap N$ is empty, that is, $N \subset E$. Thus x is an interior point of E , and E is open.

Next, suppose E is open. Let x be a limit point of E^c . Then every neighborhood of x contains a point of E^c , so that x is not an interior point of E . Since E is open, this means that $x \in E^c$. It follows that E^c is closed.

Corollary *A set F is closed if and only if its complement is open.*

2.24 Theorem

- (a) *For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.*
- (b) *For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.*
- (c) *For any finite collection G_1, \dots, G_n of open sets, $\bigcap_{i=1}^n G_i$ is open.*
- (d) *For any finite collection F_1, \dots, F_n of closed sets, $\bigcup_{i=1}^n F_i$ is closed.*

Proof Put $G = \bigcup_\alpha G_\alpha$. If $x \in G$, then $x \in G_\alpha$ for some α . Since x is an interior point of G_α , x is also an interior point of G , and G is open. This proves (a).

By Theorem 2.22,

$$(21) \quad \left(\bigcap_\alpha F_\alpha \right)^c = \bigcup_\alpha (F_\alpha^c),$$

and F_α^c is open, by Theorem 2.23. Hence (a) implies that (21) is open so that $\bigcap_\alpha F_\alpha$ is closed.

Next, put $H = \bigcap_{i=1}^n G_i$. For any $x \in H$, there exist neighborhoods N_i of x , with radii r_i , such that $N_i \subset G_i$ ($i = 1, \dots, n$). Put

$$r = \min(r_1, \dots, r_n),$$

and let N be the neighborhood of x of radius r . Then $N \subset G_i$ for $i = 1, \dots, n$, so that $N \subset H$, and H is open.

By taking complements, (d) follows from (c):

$$\left(\bigcup_{i=1}^n F_i \right)^c = \bigcap_{i=1}^n (F_i^c).$$

2.25 Examples In parts (c) and (d) of the preceding theorem, the finiteness of the collections is essential. For let G_n be the segment $\left(-\frac{1}{n}, \frac{1}{n}\right)$ ($n = 1, 2, 3, \dots$). Then G_n is an open subset of R^1 . Put $G = \bigcap_{n=1}^{\infty} G_n$. Then G consists of a single point (namely, $x = 0$) and is therefore not an open subset of R^1 .

Thus the intersection of an infinite collection of open sets need not be open. Similarly, the union of an infinite collection of closed sets need not be closed.

2.26 Definition If X is a metric space, if $E \subset X$, and if E' denotes the set of all limit points of E in X , then the *closure* of E is the set $\bar{E} = E \cup E'$.

2.27 Theorem If X is a metric space and $E \subset X$, then

- (a) \bar{E} is closed,
- (b) $E = \bar{E}$ if and only if E is closed,
- (c) $\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

By (a) and (c), \bar{E} is the *smallest* closed subset of X that contains E .

Proof

- (a) If $p \in X$ and $p \notin \bar{E}$ then p is neither a point of E nor a limit point of E . Hence p has a neighborhood which does not intersect E . The complement of \bar{E} is therefore open. Hence \bar{E} is closed.
- (b) If $E = \bar{E}$, (a) implies that E is closed. If E is closed, then $E' \subset E$ [by Definitions 2.18(d) and 2.26], hence $\bar{E} = E$.
- (c) If F is closed and $F \supset E$, then $F \supset F'$, hence $F \supset E'$. Thus $F \supset \bar{E}$.

2.28 Theorem Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \bar{E}$. Hence $y \in E$ if E is closed.

Compare this with the examples in Sec. 1.9.

Proof If $y \in E$ then $y \in \bar{E}$. Assume $y \notin E$. For every $h > 0$ there exists then a point $x \in E$ such that $y - h < x < y$, for otherwise $y - h$ would be an upper bound of E . Thus y is a limit point of E . Hence $y \in \bar{E}$.

2.29 Remark Suppose $E \subset Y \subset X$, where X is a metric space. To say that E is an open subset of X means that to each point $p \in E$ there is associated a positive number r such that the conditions $d(p, q) < r$, $q \in X$ imply that $q \in E$. But we have already observed (Sec. 2.16) that Y is also a metric space, so that our definitions may equally well be made within Y . To be quite explicit, let us say that E is *open relative to Y* if to each $p \in E$ there is associated an $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$. Example 2.21(g) showed that a set

may be open relative to Y without being an open subset of X . However, there is a simple relation between these concepts, which we now state.

2.30 Theorem *Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open subset G of X .*

Proof Suppose E is open relative to Y . To each $p \in E$ there is a positive number r_p such that the conditions $d(p, q) < r_p$, $q \in Y$ imply that $q \in E$. Let V_p be the set of all $q \in X$ such that $d(p, q) < r_p$, and define

$$G = \bigcup_{p \in E} V_p.$$

Then G is an open subset of X , by Theorems 2.19 and 2.24.

Since $p \in V_p$ for all $p \in E$, it is clear that $E \subset G \cap Y$.

By our choice of V_p , we have $V_p \cap Y \subset E$ for every $p \in E$, so that $G \cap Y \subset E$. Thus $E = G \cap Y$, and one half of the theorem is proved.

Conversely, if G is open in X and $E = G \cap Y$, every $p \in E$ has a neighborhood $V_p \subset G$. Then $V_p \cap Y \subset E$, so that E is open relative to Y .

COMPACT SETS

2.31 Definition By an *open cover* of a set E in a metric space X we mean a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \bigcup_\alpha G_\alpha$.

2.32 Definition A subset K of a metric space X is said to be *compact* if every open cover of K contains a *finite* subcover.

More explicitly, the requirement is that if $\{G_\alpha\}$ is an open cover of K , then there are finitely many indices $\alpha_1, \dots, \alpha_n$ such that

$$K \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}.$$

The notion of compactness is of great importance in analysis, especially in connection with continuity (Chap. 4).

It is clear that every finite set is compact. The existence of a large class of infinite compact sets in R^k will follow from Theorem 2.41.

We observed earlier (in Sec. 2.29) that if $E \subset Y \subset X$, then E may be open relative to Y without being open relative to X . The property of being open thus depends on the space in which E is embedded. The same is true of the property of being closed.

Compactness, however, behaves better, as we shall now see. To formulate the next theorem, let us say, temporarily, that K is compact relative to X if the requirements of Definition 2.32 are met.

2.33 Theorem *Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y .*

By virtue of this theorem we are able, in many situations, to regard compact sets as metric spaces in their own right, without paying any attention to any embedding space. In particular, although it makes little sense to talk of *open* spaces, or of *closed* spaces (every metric space X is an open subset of itself, and is a closed subset of itself), it does make sense to talk of *compact* metric spaces.

Proof Suppose K is compact relative to X , and let $\{V_\alpha\}$ be a collection of sets, open relative to Y , such that $K \subset \bigcup_\alpha V_\alpha$. By theorem 2.30, there are sets G_α , open relative to X , such that $V_\alpha = Y \cap G_\alpha$, for all α ; and since K is compact relative to X , we have

$$(22) \quad K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$$

for some choice of finitely many indices $\alpha_1, \dots, \alpha_n$. Since $K \subset Y$, (22) implies

$$(23) \quad K \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}.$$

This proves that K is compact relative to Y .

Conversely, suppose K is compact relative to Y , let $\{G_\alpha\}$ be a collection of open subsets of X which covers K , and put $V_\alpha = Y \cap G_\alpha$. Then (23) will hold for some choice of $\alpha_1, \dots, \alpha_n$; and since $V_\alpha \subset G_\alpha$, (23) implies (22).

This completes the proof.

2.34 Theorem *Compact subsets of metric spaces are closed.*

Proof Let K be a compact subset of a metric space X . We shall prove that the complement of K is an open subset of X .

Suppose $p \in X, p \notin K$. If $q \in K$, let V_q and W_q be neighborhoods of p and q , respectively, of radius less than $\frac{1}{2}d(p, q)$ [see Definition 2.18(a)]. Since K is compact, there are finitely many points q_1, \dots, q_n in K such that

$$K \subset W_{q_1} \cup \cdots \cup W_{q_n} = W.$$

If $V = V_{q_1} \cap \cdots \cap V_{q_n}$, then V is a neighborhood of p which does not intersect W . Hence $V \subset K^c$, so that p is an interior point of K^c . The theorem follows.

2.35 Theorem *Closed subsets of compact sets are compact.*

Proof Suppose $F \subset K \subset X$, F is closed (relative to X), and K is compact. Let $\{V_\alpha\}$ be an open cover of F . If F^c is adjoined to $\{V_\alpha\}$, we obtain an

open cover Ω of K . Since K is compact, there is a finite subcollection Φ of Ω which covers K , and hence F . If F^c is a member of Φ , we may remove it from Φ and still retain an open cover of F . We have thus shown that a finite subcollection of $\{V_\alpha\}$ covers F .

Corollary *If F is closed and K is compact, then $F \cap K$ is compact.*

Proof Theorems 2.24(b) and 2.34 show that $F \cap K$ is closed; since $F \cap K \subset K$, Theorem 2.35 shows that $F \cap K$ is compact.

2.36 Theorem *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.*

Proof Fix a member K_1 of $\{K_\alpha\}$ and put $G_\alpha = K_\alpha^c$. Assume that no point of K_1 belongs to every K_α . Then the sets G_α form an open cover of K_1 ; and since K_1 is compact, there are finitely many indices $\alpha_1, \dots, \alpha_n$ such that $K_1 \subset G_{\alpha_1} \cup \dots \cup G_{\alpha_n}$. But this means that

$$K_1 \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_n}$$

is empty, in contradiction to our hypothesis.

Corollary *If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$ ($n = 1, 2, 3, \dots$), then $\bigcap_1^\infty K_n$ is not empty.*

2.37 Theorem *If E is an infinite subset of a compact set K , then E has a limit point in K .*

Proof If no point of K were a limit point of E , then each $q \in K$ would have a neighborhood V_q which contains at most one point of E (namely, q , if $q \in E$). It is clear that no finite subcollection of $\{V_q\}$ can cover E ; and the same is true of K , since $E \subset K$. This contradicts the compactness of K .

2.38 Theorem *If $\{I_n\}$ is a sequence of intervals in R^1 , such that $I_n \supset I_{n+1}$ ($n = 1, 2, 3, \dots$), then $\bigcap_1^\infty I_n$ is not empty.*

Proof If $I_n = [a_n, b_n]$, let E be the set of all a_n . Then E is nonempty and bounded above (by b_1). Let x be the sup of E . If m and n are positive integers, then

$$a_n \leq a_{m+n} \leq b_{m+n} \leq b_m,$$

so that $x \leq b_m$ for each m . Since it is obvious that $a_m \leq x$, we see that $x \in I_m$ for $m = 1, 2, 3, \dots$

2.39 Theorem *Let k be a positive integer. If $\{I_n\}$ is a sequence of k -cells such that $I_n \supset I_{n+1}$ ($n = 1, 2, 3, \dots$), then $\bigcap_1^\infty I_n$ is not empty.*

Proof Let I_n consist of all points $\mathbf{x} = (x_1, \dots, x_k)$ such that

$$a_{n,j} \leq x_j \leq b_{n,j} \quad (1 \leq j \leq k; n = 1, 2, 3, \dots),$$

and put $I_{n,j} = [a_{n,j}, b_{n,j}]$. For each j , the sequence $\{I_{n,j}\}$ satisfies the hypotheses of Theorem 2.38. Hence there are real numbers x_j^* ($1 \leq j \leq k$) such that

$$a_{n,j} \leq x_j^* \leq b_{n,j} \quad (1 \leq j \leq k; n = 1, 2, 3, \dots).$$

Setting $\mathbf{x}^* = (x_1^*, \dots, x_k^*)$, we see that $\mathbf{x}^* \in I_n$ for $n = 1, 2, 3, \dots$. The theorem follows.

2.40 Theorem *Every k -cell is compact.*

Proof Let I be a k -cell, consisting of all points $\mathbf{x} = (x_1, \dots, x_k)$ such that $a_j \leq x_j \leq b_j$ ($1 \leq j \leq k$). Put

$$\delta = \left\{ \sum_1^k (b_j - a_j)^2 \right\}^{1/2}.$$

Then $|\mathbf{x} - \mathbf{y}| \leq \delta$, if $\mathbf{x} \in I, \mathbf{y} \in I$.

Suppose, to get a contradiction, that there exists an open cover $\{G_\alpha\}$ of I which contains no finite subcover of I . Put $c_j = (a_j + b_j)/2$. The intervals $[a_j, c_j]$ and $[c_j, b_j]$ then determine 2^k k -cells Q_i whose union is I . At least one of these sets Q_i , call it I_1 , cannot be covered by any finite subcollection of $\{G_\alpha\}$ (otherwise I could be so covered). We next subdivide I_1 and continue the process. We obtain a sequence $\{I_n\}$ with the following properties:

- (a) $I \supset I_1 \supset I_2 \supset I_3 \supset \dots$;
- (b) I_n is not covered by any finite subcollection of $\{G_\alpha\}$;
- (c) if $\mathbf{x} \in I_n$ and $\mathbf{y} \in I_n$, then $|\mathbf{x} - \mathbf{y}| \leq 2^{-n} \delta$.

By (a) and Theorem 2.39, there is a point \mathbf{x}^* which lies in every I_n . For some α , $\mathbf{x}^* \in G_\alpha$. Since G_α is open, there exists $r > 0$ such that $|\mathbf{y} - \mathbf{x}^*| < r$ implies that $\mathbf{y} \in G_\alpha$. If n is so large that $2^{-n} \delta < r$ (there is such an n , for otherwise $2^n \leq \delta/r$ for all positive integers n , which is absurd since R is archimedean), then (c) implies that $I_n \subset G_\alpha$, which contradicts (b).

This completes the proof.

The equivalence of (a) and (b) in the next theorem is known as the Heine-Borel theorem.

2.41 Theorem *If a set E in R^k has one of the following three properties, then it has the other two:*

- (a) E is closed and bounded.
- (b) E is compact.
- (c) Every infinite subset of E has a limit point in E .

Proof If (a) holds, then $E \subset I$ for some k -cell I , and (b) follows from Theorems 2.40 and 2.35. Theorem 2.37 shows that (b) implies (c). It remains to be shown that (c) implies (a).

If E is not bounded, then E contains points \mathbf{x}_n with

$$|\mathbf{x}_n| > n \quad (n = 1, 2, 3, \dots).$$

The set S consisting of these points \mathbf{x}_n is infinite and clearly has no limit point in R^k , hence has none in E . Thus (c) implies that E is bounded.

If E is not closed, then there is a point $\mathbf{x}_0 \in R^k$ which is a limit point of E but not a point of E . For $n = 1, 2, 3, \dots$, there are points $\mathbf{x}_n \in E$ such that $|\mathbf{x}_n - \mathbf{x}_0| < 1/n$. Let S be the set of these points \mathbf{x}_n . Then S is infinite (otherwise $|\mathbf{x}_n - \mathbf{x}_0|$ would have a constant positive value, for infinitely many n), S has \mathbf{x}_0 as a limit point, and S has no other limit point in R^k . For if $\mathbf{y} \in R^k$, $\mathbf{y} \neq \mathbf{x}_0$, then

$$\begin{aligned} |\mathbf{x}_n - \mathbf{y}| &\geq |\mathbf{x}_0 - \mathbf{y}| - |\mathbf{x}_n - \mathbf{x}_0| \\ &\geq |\mathbf{x}_0 - \mathbf{y}| - \frac{1}{n} \geq \frac{1}{2} |\mathbf{x}_0 - \mathbf{y}| \end{aligned}$$

for all but finitely many n ; this shows that \mathbf{y} is not a limit point of S (Theorem 2.20).

Thus S has no limit point in E ; hence E must be closed if (c) holds.

We should remark, at this point, that (b) and (c) are equivalent in any metric space (Exercise 26) but that (a) does not, in general, imply (b) and (c). Examples are furnished by Exercise 16 and by the space \mathcal{L}^2 , which is discussed in Chap. 11.

2.42 Theorem (Weierstrass) *Every bounded infinite subset of R^k has a limit point in R^k .*

Proof Being bounded, the set E in question is a subset of a k -cell $I \subset R^k$. By Theorem 2.40, I is compact, and so E has a limit point in I , by Theorem 2.37.

PERFECT SETS

2.43 Theorem *Let P be a nonempty perfect set in R^k . Then P is uncountable.*

Proof Since P has limit points, P must be infinite. Suppose P is countable, and denote the points of P by x_1, x_2, x_3, \dots . We shall construct a sequence $\{V_n\}$ of neighborhoods, as follows.

Let V_1 be any neighborhood of x_1 . If V_1 consists of all $y \in R^k$ such that $|y - x_1| < r$, the closure \bar{V}_1 of V_1 is the set of all $y \in R^k$ such that $|y - x_1| \leq r$.

Suppose V_n has been constructed, so that $V_n \cap P$ is not empty. Since every point of P is a limit point of P , there is a neighborhood V_{n+1} such that (i) $\bar{V}_{n+1} \subset V_n$, (ii) $x_n \notin \bar{V}_{n+1}$, (iii) $V_{n+1} \cap P$ is not empty. By (iii), V_{n+1} satisfies our induction hypothesis, and the construction can proceed.

Put $K_n = \bar{V}_n \cap P$. Since \bar{V}_n is closed and bounded, \bar{V}_n is compact. Since $x_n \notin K_{n+1}$, no point of P lies in $\bigcap_1^\infty K_n$. Since $K_n \subset P$, this implies that $\bigcap_1^\infty K_n$ is empty. But each K_n is nonempty, by (iii), and $K_n \supset K_{n+1}$, by (i); this contradicts the Corollary to Theorem 2.36.

Corollary *Every interval $[a, b]$ ($a < b$) is uncountable. In particular, the set of all real numbers is uncountable.*

2.44 The Cantor set The set which we are now going to construct shows that there exist perfect sets in R^1 which contain no segment.

Let E_0 be the interval $[0, 1]$. Remove the segment $(\frac{1}{3}, \frac{2}{3})$, and let E_1 be the union of the intervals

$$[0, \frac{1}{3}] \text{ and } [\frac{2}{3}, 1].$$

Remove the middle thirds of these intervals, and let E_2 be the union of the intervals

$$[0, \frac{1}{9}], [\frac{2}{9}, \frac{3}{9}], [\frac{6}{9}, \frac{7}{9}], [\frac{8}{9}, 1].$$

Continuing in this way, we obtain a sequence of compact sets E_n , such that

- (a) $E_1 \supset E_2 \supset E_3 \supset \dots$;
- (b) E_n is the union of 2^n intervals, each of length 3^{-n} .

The set

$$P = \bigcap_{n=1}^{\infty} E_n$$

is called the *Cantor set*. P is clearly compact, and Theorem 2.36 shows that P is not empty.

No segment of the form

$$(24) \quad \left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right),$$

where k and m are positive integers, has a point in common with P . Since every segment (α, β) contains a segment of the form (24), if

$$3^{-m} < \frac{\beta - \alpha}{6},$$

P contains no segment.

To show that P is perfect, it is enough to show that P contains no isolated point. Let $x \in P$, and let S be any segment containing x . Let I_n be that interval of E_n which contains x . Choose n large enough, so that $I_n \subset S$. Let x_n be an endpoint of I_n , such that $x_n \neq x$.

It follows from the construction of P that $x_n \in P$. Hence x is a limit point of P , and P is perfect.

One of the most interesting properties of the Cantor set is that it provides us with an example of an uncountable set of measure zero (the concept of measure will be discussed in Chap. 11).

CONNECTED SETS

2.45 Definition Two subsets A and B of a metric space X are said to be *separated* if both $A \cap \bar{B}$ and $\bar{A} \cap B$ are empty, i.e., if no point of A lies in the closure of B and no point of B lies in the closure of A .

A set $E \subset X$ is said to be *connected* if E is *not* a union of two nonempty separated sets.

2.46 Remark Separated sets are of course disjoint, but disjoint sets need not be separated. For example, the interval $[0, 1]$ and the segment $(1, 2)$ are *not* separated, since 1 is a limit point of $(1, 2)$. However, the segments $(0, 1)$ and $(1, 2)$ are separated.

The connected subsets of the line have a particularly simple structure:

2.47 Theorem A subset E of the real line R^1 is connected if and only if it has the following property: If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.

Proof If there exist $x \in E$, $y \in E$, and some $z \in (x, y)$ such that $z \notin E$, then $E = A_z \cup B_z$ where

$$A_z = E \cap (-\infty, z), \quad B_z = E \cap (z, \infty).$$

Since $x \in A_z$ and $y \in B_z$, A and B are nonempty. Since $A_z \subset (-\infty, z)$ and $B_z \subset (z, \infty)$, they are separated. Hence E is not connected.

To prove the converse, suppose E is not connected. Then there are nonempty separated sets A and B such that $A \cup B = E$. Pick $x \in A$, $y \in B$, and assume (without loss of generality) that $x < y$. Define

$$z = \sup (A \cap [x, y]).$$

By Theorem 2.28, $z \in \bar{A}$; hence $z \notin B$. In particular, $x \leq z < y$.

If $z \notin A$, it follows that $x < z < y$ and $z \notin E$.

If $z \in A$, then $z \notin \bar{B}$, hence there exists z_1 such that $z < z_1 < y$ and $z_1 \notin B$. Then $x < z_1 < y$ and $z_1 \notin E$.

EXERCISES

1. Prove that the empty set is a subset of every set.
2. A complex number z is said to be *algebraic* if there are integers a_0, \dots, a_n , not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \dots + |a_n| = N.$$

3. Prove that there exist real numbers which are not algebraic.
4. Is the set of all irrational real numbers countable?
5. Construct a bounded set of real numbers with exactly three limit points.
6. Let E' be the set of all limit points of a set E . Prove that E' is closed. Prove that E and \bar{E} have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?
7. Let A_1, A_2, A_3, \dots be subsets of a metric space.
 - (a) If $B_n = \bigcup_{i=1}^n A_i$, prove that $\bar{B}_n = \bigcup_{i=1}^n \bar{A}_i$, for $n = 1, 2, 3, \dots$.
 - (b) If $B = \bigcup_{i=1}^{\infty} A_i$, prove that $\bar{B} \supset \bigcup_{i=1}^{\infty} \bar{A}_i$.
 Show, by an example, that this inclusion can be proper.
8. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E ? Answer the same question for closed sets in \mathbb{R}^2 .
9. Let E° denote the set of all interior points of a set E . [See Definition 2.18(e); E° is called the *interior* of E .]
 - (a) Prove that E° is always open.
 - (b) Prove that E is open if and only if $E^\circ = E$.
 - (c) If $G \subset E$ and G is open, prove that $G \subset E^\circ$.
 - (d) Prove that the complement of E° is the closure of the complement of E .
 - (e) Do E and \bar{E} always have the same interiors?
 - (f) Do E and E° always have the same closures?

10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p, q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

11. For $x \in R^1$ and $y \in R^1$, define

$$\begin{aligned} d_1(x, y) &= (x - y)^2, \\ d_2(x, y) &= \sqrt{|x - y|}, \\ d_3(x, y) &= |x^2 - y^2|, \\ d_4(x, y) &= |x - 2y|, \\ d_5(x, y) &= \frac{|x - y|}{1 + |x - y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not.

12. Let $K \subset R^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \dots$. Prove that K is compact directly from the definition (without using the Heine-Borel theorem).
13. Construct a compact set of real numbers whose limit points form a countable set.
14. Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.
15. Show that Theorem 2.36 and its Corollary become false (in R^1 , for example) if the word “compact” is replaced by “closed” or by “bounded.”
16. Regard Q , the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in Q$ such that $2 < p^2 < 3$. Show that E is closed and bounded in Q , but that E is not compact. Is E open in Q ?
17. Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?
18. Is there a nonempty perfect set in R^1 which contains no rational number?
19. (a) If A and B are disjoint closed sets in some metric space X , prove that they are separated.
 (b) Prove the same for disjoint open sets.
 (c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.
 (d) Prove that every connected metric space with at least two points is uncountable. *Hint:* Use (c).
20. Are closures and interiors of connected sets always connected? (Look at subsets of R^2 .)
21. Let A and B be separated subsets of some R^k , suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

for $t \in R^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]

- (a) Prove that A_0 and B_0 are separated subsets of R^1 .
- (b) Prove that there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.
- (c) Prove that every convex subset of R^k is connected.
22. A metric space is called *separable* if it contains a countable dense subset. Show that R^k is separable. *Hint*: Consider the set of points which have only rational coordinates.
23. A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α . In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.
- Prove that every separable metric space has a *countable* base. *Hint*: Take all neighborhoods with rational radius and center in some countable dense subset of X .
24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint*: Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \dots, x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, \dots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ . Take $\delta = 1/n$ ($n = 1, 2, 3, \dots$), and consider the centers of the corresponding neighborhoods.
25. Prove that every compact metric space K has a countable base, and that K is therefore separable. *Hint*: For every positive integer n , there are finitely many neighborhoods of radius $1/n$ whose union covers K .
26. Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. *Hint*: By Exercises 23 and 24, X has a countable base. It follows that every open cover of X has a *countable* subcover $\{G_n\}$, $n = 1, 2, 3, \dots$. If no finite subcollection of $\{G_n\}$ covers X , then the complement F_n of $G_1 \cup \dots \cup G_n$ is nonempty for each n , but $\bigcap F_n$ is empty. If E is a set which contains a point from each F_n , consider a limit point of E , and obtain a contradiction.
27. Define a point p in a metric space X to be a *condensation point* of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E .
- Suppose $E \subset R^k$, E is uncountable, and let P be the set of all condensation points of E . Prove that P is perfect and that at most countably many points of E are not in P . In other words, show that $P^c \cap E$ is at most countable. *Hint*: Let $\{V_n\}$ be a countable base of R^k , let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.
28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (*Corollary*: Every countable closed set in R^k has isolated points.) *Hint*: Use Exercise 27.
29. Prove that every open set in R^1 is the union of an at most countable collection of disjoint segments. *Hint*: Use Exercise 22.

30. Imitate the proof of Theorem 2.43 to obtain the following result:

If $R^k = \bigcup_1^\infty F_n$, where each F_n is a closed subset of R^k , then at least one F_n has a nonempty interior.

Equivalent statement: If G_n is a dense open subset of R^k , for $n = 1, 2, 3, \dots$, then $\bigcap_1^\infty G_n$ is not empty (in fact, it is dense in R^k).

(This is a special case of Baire's theorem; see Exercise 22, Chap. 3, for the general case.)

NUMERICAL SEQUENCES AND SERIES

As the title indicates, this chapter will deal primarily with sequences and series of complex numbers. The basic facts about convergence, however, are just as easily explained in a more general setting. The first three sections will therefore be concerned with sequences in euclidean spaces, or even in metric spaces.

CONVERGENT SEQUENCES

3.1 Definition A sequence $\{p_n\}$ in a metric space X is said to *converge* if there is a point $p \in X$ with the following property: For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \varepsilon$. (Here d denotes the distance in X .)

In this case we also say that $\{p_n\}$ converges to p , or that p is the limit of $\{p_n\}$ [see Theorem 3.2(b)], and we write $p_n \rightarrow p$, or

$$\lim_{n \rightarrow \infty} p_n = p.$$

If $\{p_n\}$ does not converge, it is said to *diverge*.

It might be well to point out that our definition of “convergent sequence” depends not only on $\{p_n\}$ but also on X ; for instance, the sequence $\{1/n\}$ converges in R^1 (to 0), but fails to converge in the set of all positive real numbers [with $d(x, y) = |x - y|$]. In cases of possible ambiguity, we can be more precise and specify “convergent in X ” rather than “convergent.”

We recall that the set of all points p_n ($n = 1, 2, 3, \dots$) is the *range* of $\{p_n\}$. The range of a sequence may be a finite set, or it may be infinite. The sequence $\{p_n\}$ is said to be *bounded* if its range is bounded.

As examples, consider the following sequences of complex numbers (that is, $X = R^2$):

- (a) If $s_n = 1/n$, then $\lim_{n \rightarrow \infty} s_n = 0$; the range is infinite, and the sequence is bounded.
- (b) If $s_n = n^2$, the sequence $\{s_n\}$ is unbounded, is divergent, and has infinite range.
- (c) If $s_n = 1 + [(-1)^n/n]$, the sequence $\{s_n\}$ converges to 1, is bounded, and has infinite range.
- (d) If $s_n = i^n$, the sequence $\{s_n\}$ is divergent, is bounded, and has finite range.
- (e) If $s_n = 1$ ($n = 1, 2, 3, \dots$), then $\{s_n\}$ converges to 1, is bounded, and has finite range.

We now summarize some important properties of convergent sequences in metric spaces.

3.2 Theorem Let $\{p_n\}$ be a sequence in a metric space X .

- (a) $\{p_n\}$ converges to $p \in X$ if and only if every neighborhood of p contains p_n for all but finitely many n .
- (b) If $p \in X$, $p' \in X$, and if $\{p_n\}$ converges to p and to p' , then $p' = p$.
- (c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.
- (d) If $E \subset X$ and if p is a limit point of E , then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n \rightarrow \infty} p_n$.

Proof (a) Suppose $p_n \rightarrow p$ and let V be a neighborhood of p . For some $\varepsilon > 0$, the conditions $d(q, p) < \varepsilon$, $q \in X$ imply $q \in V$. Corresponding to this ε , there exists N such that $n \geq N$ implies $d(p_n, p) < \varepsilon$. Thus $n \geq N$ implies $p_n \in V$.

Conversely, suppose every neighborhood of p contains all but finitely many of the p_n . Fix $\varepsilon > 0$, and let V be the set of all $q \in X$ such that $d(p, q) < \varepsilon$. By assumption, there exists N (corresponding to this V) such that $p_n \in V$ if $n \geq N$. Thus $d(p_n, p) < \varepsilon$ if $n \geq N$; hence $p_n \rightarrow p$.

(b) Let $\varepsilon > 0$ be given. There exist integers N, N' such that

$$n \geq N \quad \text{implies} \quad d(p_n, p) < \frac{\varepsilon}{2},$$

$$n \geq N' \quad \text{implies} \quad d(p_n, p') < \frac{\varepsilon}{2}.$$

Hence if $n \geq \max(N, N')$, we have

$$d(p, p') \leq d(p, p_n) + d(p_n, p') < \varepsilon.$$

Since ε was arbitrary, we conclude that $d(p, p') = 0$.

(c) Suppose $p_n \rightarrow p$. There is an integer N such that $n > N$ implies $d(p_n, p) < 1$. Put

$$r = \max \{1, d(p_1, p), \dots, d(p_N, p)\}.$$

Then $d(p_n, p) \leq r$ for $n = 1, 2, 3, \dots$.

(d) For each positive integer n , there is a point $p_n \in E$ such that $d(p_n, p) < 1/n$. Given $\varepsilon > 0$, choose N so that $N\varepsilon > 1$. If $n > N$, it follows that $d(p_n, p) < \varepsilon$. Hence $p_n \rightarrow p$.

This completes the proof.

For sequences in R^k we can study the relation between convergence, on the one hand, and the algebraic operations on the other. We first consider sequences of complex numbers.

3.3 Theorem Suppose $\{s_n\}, \{t_n\}$ are complex sequences, and $\lim_{n \rightarrow \infty} s_n = s$, $\lim_{n \rightarrow \infty} t_n = t$. Then

$$(a) \quad \lim_{n \rightarrow \infty} (s_n + t_n) = s + t;$$

$$(b) \quad \lim_{n \rightarrow \infty} cs_n = cs, \quad \lim_{n \rightarrow \infty} (c + s_n) = c + s, \text{ for any number } c;$$

$$(c) \quad \lim_{n \rightarrow \infty} s_n t_n = st;$$

$$(d) \quad \lim_{n \rightarrow \infty} \frac{1}{s_n} = \frac{1}{s}, \text{ provided } s_n \neq 0 \ (n = 1, 2, 3, \dots), \text{ and } s \neq 0.$$

Proof

(a) Given $\varepsilon > 0$, there exist integers N_1, N_2 such that

$$n \geq N_1 \quad \text{implies} \quad |s_n - s| < \frac{\varepsilon}{2},$$

$$n \geq N_2 \quad \text{implies} \quad |t_n - t| < \frac{\varepsilon}{2}.$$

If $N = \max (N_1, N_2)$, then $n \geq N$ implies

$$|(s_n + t_n) - (s + t)| \leq |s_n - s| + |t_n - t| < \varepsilon.$$

This proves (a). The proof of (b) is trivial.

(c) We use the identity

$$(1) \quad s_n t_n - st = (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s).$$

Given $\varepsilon > 0$, there are integers N_1, N_2 such that

$$n \geq N_1 \quad \text{implies} \quad |s_n - s| < \sqrt{\varepsilon},$$

$$n \geq N_2 \quad \text{implies} \quad |t_n - t| < \sqrt{\varepsilon}.$$

If we take $N = \max (N_1, N_2)$, $n \geq N$ implies

$$|(s_n - s)(t_n - t)| < \varepsilon,$$

so that

$$\lim_{n \rightarrow \infty} (s_n - s)(t_n - t) = 0.$$

We now apply (a) and (b) to (1), and conclude that

$$\lim_{n \rightarrow \infty} (s_n t_n - st) = 0.$$

(d) Choosing m such that $|s_n - s| < \frac{1}{2}|s|$ if $n \geq m$, we see that

$$|s_n| > \frac{1}{2}|s| \quad (n \geq m).$$

Given $\varepsilon > 0$, there is an integer $N > m$ such that $n \geq N$ implies

$$|s_n - s| < \frac{1}{2}|s|^2 \varepsilon.$$

Hence, for $n \geq N$,

$$\left| \frac{1}{s_n} - \frac{1}{s} \right| = \left| \frac{s_n - s}{s_n s} \right| < \frac{2}{|s|^2} |s_n - s| < \varepsilon.$$

3.4 Theorem

(a) Suppose $\mathbf{x}_n \in R^k$ ($n = 1, 2, 3, \dots$) and

$$\mathbf{x}_n = (\alpha_{1,n}, \dots, \alpha_{k,n}).$$

Then $\{\mathbf{x}_n\}$ converges to $\mathbf{x} = (\alpha_1, \dots, \alpha_k)$ if and only if

$$(2) \quad \lim_{n \rightarrow \infty} \alpha_{j,n} = \alpha_j \quad (1 \leq j \leq k).$$

(b) Suppose $\{\mathbf{x}_n\}, \{\mathbf{y}_n\}$ are sequences in R^k , $\{\beta_n\}$ is a sequence of real numbers, and $\mathbf{x}_n \rightarrow \mathbf{x}$, $\mathbf{y}_n \rightarrow \mathbf{y}$, $\beta_n \rightarrow \beta$. Then

$$\lim_{n \rightarrow \infty} (\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y}, \quad \lim_{n \rightarrow \infty} \mathbf{x}_n \cdot \mathbf{y}_n = \mathbf{x} \cdot \mathbf{y}, \quad \lim_{n \rightarrow \infty} \beta_n \mathbf{x}_n = \beta \mathbf{x}.$$

Proof

(a) If $\mathbf{x}_n \rightarrow \mathbf{x}$, the inequalities

$$|\alpha_{j,n} - \alpha_j| \leq |\mathbf{x}_n - \mathbf{x}|,$$

which follow immediately from the definition of the norm in R^k , show that (2) holds.

Conversely, if (2) holds, then to each $\varepsilon > 0$ there corresponds an integer N such that $n \geq N$ implies

$$|\alpha_{j,n} - \alpha_j| < \frac{\varepsilon}{\sqrt{k}} \quad (1 \leq j \leq k).$$

Hence $n \geq N$ implies

$$|\mathbf{x}_n - \mathbf{x}| = \left\{ \sum_{j=1}^k |\alpha_{j,n} - \alpha_j|^2 \right\}^{1/2} < \varepsilon,$$

so that $\mathbf{x}_n \rightarrow \mathbf{x}$. This proves (a).

Part (b) follows from (a) and Theorem 3.3.

SUBSEQUENCES

3.5 Definition Given a sequence $\{p_n\}$, consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \dots$. Then the sequence $\{p_{n_k}\}$ is called a *subsequence* of $\{p_n\}$. If $\{p_{n_k}\}$ converges, its limit is called a *subsequential limit* of $\{p_n\}$.

It is clear that $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p . We leave the details of the proof to the reader.

3.6 Theorem

- (a) If $\{p_n\}$ is a sequence in a compact metric space X , then some subsequence of $\{p_n\}$ converges to a point of X .
- (b) Every bounded sequence in R^k contains a convergent subsequence.

Proof

(a) Let E be the range of $\{p_n\}$. If E is finite then there is a $p \in E$ and a sequence $\{n_i\}$ with $n_1 < n_2 < n_3 < \cdots$, such that

$$p_{n_1} = p_{n_2} = \cdots = p.$$

The subsequence $\{p_{n_i}\}$ so obtained converges evidently to p .

If E is infinite, Theorem 2.37 shows that E has a limit point $p \in X$. Choose n_1 so that $d(p, p_{n_1}) < 1$. Having chosen n_1, \dots, n_{i-1} , we see from Theorem 2.20 that there is an integer $n_i > n_{i-1}$ such that $d(p, p_{n_i}) < 1/i$. Then $\{p_{n_i}\}$ converges to p .

(b) This follows from (a), since Theorem 2.41 implies that every bounded subset of R^k lies in a compact subset of R^k .

3.7 Theorem *The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X .*

Proof Let E^* be the set of all subsequential limits of $\{p_n\}$ and let q be a limit point of E^* . We have to show that $q \in E^*$.

Choose n_1 so that $p_{n_1} \neq q$. (If no such n_1 exists, then E^* has only one point, and there is nothing to prove.) Put $\delta = d(q, p_{n_1})$. Suppose n_1, \dots, n_{i-1} are chosen. Since q is a limit point of E^* , there is an $x \in E^*$ with $d(x, q) < 2^{-i}\delta$. Since $x \in E^*$, there is an $n_i > n_{i-1}$ such that $d(x, p_{n_i}) < 2^{-i}\delta$. Thus

$$d(q, p_{n_i}) \leq 2^{1-i}\delta$$

for $i = 1, 2, 3, \dots$. This says that $\{p_{n_i}\}$ converges to q . Hence $q \in E^*$.

CAUCHY SEQUENCES

3.8 Definition A sequence $\{p_n\}$ in a metric space X is said to be a *Cauchy sequence* if for every $\varepsilon > 0$ there is an integer N such that $d(p_n, p_m) < \varepsilon$ if $n \geq N$ and $m \geq N$.

In our discussion of Cauchy sequences, as well as in other situations which will arise later, the following geometric concept will be useful.

3.9 Definition Let E be a nonempty subset of a metric space X , and let S be the set of all real numbers of the form $d(p, q)$, with $p \in E$ and $q \in E$. The sup of S is called the *diameter* of E .

If $\{p_n\}$ is a sequence in X and if E_N consists of the points $p_N, p_{N+1}, p_{N+2}, \dots$, it is clear from the two preceding definitions that $\{p_n\}$ is a Cauchy sequence if and only if

$$\lim_{N \rightarrow \infty} \text{diam } E_N = 0.$$

3.10 Theorem

(a) If \bar{E} is the closure of a set E in a metric space X , then

$$\text{diam } \bar{E} = \text{diam } E.$$

(b) If K_n is a sequence of compact sets in X such that $K_n \supset K_{n+1}$ ($n = 1, 2, 3, \dots$) and if

$$\lim_{n \rightarrow \infty} \text{diam } K_n = 0,$$

then $\bigcap_1^\infty K_n$ consists of exactly one point.

Proof

(a) Since $E \subset \bar{E}$, it is clear that

$$\text{diam } E \leq \text{diam } \bar{E}.$$

Fix $\varepsilon > 0$, and choose $p \in \bar{E}, q \in \bar{E}$. By the definition of \bar{E} , there are points p', q' , in E such that $d(p, p') < \varepsilon, d(q, q') < \varepsilon$. Hence

$$\begin{aligned} d(p, q) &\leq d(p, p') + d(p', q') + d(q', q) \\ &< 2\varepsilon + d(p', q') \leq 2\varepsilon + \text{diam } E. \end{aligned}$$

It follows that

$$\text{diam } \bar{E} \leq 2\varepsilon + \text{diam } E,$$

and since ε was arbitrary, (a) is proved.

(b) Put $K = \bigcap_1^\infty K_n$. By Theorem 2.36, K is not empty. If K contains more than one point, then $\text{diam } K > 0$. But for each n , $K_n \supset K$, so that $\text{diam } K_n \geq \text{diam } K$. This contradicts the assumption that $\text{diam } K_n \rightarrow 0$.

3.11 Theorem

- (a) In any metric space X , every convergent sequence is a Cauchy sequence.
- (b) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X , then $\{p_n\}$ converges to some point of X .
- (c) In R^k , every Cauchy sequence converges.

Note: The difference between the definition of convergence and the definition of a Cauchy sequence is that the limit is explicitly involved in the former, but not in the latter. Thus Theorem 3.11(b) may enable us

to decide whether or not a given sequence converges without knowledge of the limit to which it may converge.

The fact (contained in Theorem 3.11) that a sequence converges in R^k if and only if it is a Cauchy sequence is usually called the *Cauchy criterion* for convergence.

Proof

(a) If $p_n \rightarrow p$ and if $\varepsilon > 0$, there is an integer N such that $d(p, p_n) < \varepsilon$ for all $n \geq N$. Hence

$$d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < 2\varepsilon$$

as soon as $n \geq N$ and $m \geq N$. Thus $\{p_n\}$ is a Cauchy sequence.

(b) Let $\{p_n\}$ be a Cauchy sequence in the compact space X . For $N = 1, 2, 3, \dots$, let E_N be the set consisting of $p_N, p_{N+1}, p_{N+2}, \dots$. Then

$$(3) \quad \lim_{N \rightarrow \infty} \text{diam } \bar{E}_N = 0,$$

by Definition 3.9 and Theorem 3.10(a). Being a closed subset of the compact space X , each \bar{E}_N is compact (Theorem 2.35). Also $E_N \supset E_{N+1}$, so that $\bar{E}_N \supset \bar{E}_{N+1}$.

Theorem 3.10(b) shows now that there is a unique $p \in X$ which lies in every \bar{E}_N .

Let $\varepsilon > 0$ be given. By (3) there is an integer N_0 such that $\text{diam } \bar{E}_N < \varepsilon$ if $N \geq N_0$. Since $p \in \bar{E}_N$, it follows that $d(p, q) < \varepsilon$ for every $q \in \bar{E}_N$, hence for every $q \in E_N$. In other words, $d(p, p_n) < \varepsilon$ if $n \geq N_0$. This says precisely that $p_n \rightarrow p$.

(c) Let $\{\mathbf{x}_n\}$ be a Cauchy sequence in R^k . Define E_N as in (b), with \mathbf{x}_i in place of p_i . For some N , $\text{diam } E_N < 1$. The range of $\{\mathbf{x}_n\}$ is the union of E_N and the finite set $\{\mathbf{x}_1, \dots, \mathbf{x}_{N-1}\}$. Hence $\{\mathbf{x}_n\}$ is bounded. Since every bounded subset of R^k has compact closure in R^k (Theorem 2.41), (c) follows from (b).

3.12 Definition A metric space in which every Cauchy sequence converges is said to be *complete*.

Thus Theorem 3.11 says that *all compact metric spaces and all Euclidean spaces are complete*. Theorem 3.11 implies also that *every closed subset E of a complete metric space X is complete*. (Every Cauchy sequence in E is a Cauchy sequence in X , hence it converges to some $p \in X$, and actually $p \in E$ since E is closed.) An example of a metric space which is not complete is the space of all rational numbers, with $d(x, y) = |x - y|$.

Theorem 3.2(c) and example (d) of Definition 3.1 show that convergent sequences are bounded, but that bounded sequences in R^k need not converge. However, there is one important case in which convergence is equivalent to boundedness; this happens for monotonic sequences in R^1 .

3.13 Definition A sequence $\{s_n\}$ of real numbers is said to be

- (a) *monotonically increasing* if $s_n \leq s_{n+1}$ ($n = 1, 2, 3, \dots$);
- (b) *monotonically decreasing* if $s_n \geq s_{n+1}$ ($n = 1, 2, 3, \dots$).

The class of monotonic sequences consists of the increasing and the decreasing sequences.

3.14 Theorem Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.

Proof Suppose $s_n \leq s_{n+1}$ (the proof is analogous in the other case). Let E be the range of $\{s_n\}$. If $\{s_n\}$ is bounded, let s be the least upper bound of E . Then

$$s_n \leq s \quad (n = 1, 2, 3, \dots).$$

For every $\varepsilon > 0$, there is an integer N such that

$$s - \varepsilon < s_N \leq s,$$

for otherwise $s - \varepsilon$ would be an upper bound of E . Since $\{s_n\}$ increases, $n \geq N$ therefore implies

$$s - \varepsilon < s_n \leq s,$$

which shows that $\{s_n\}$ converges (to s).

The converse follows from Theorem 3.2(c).

UPPER AND LOWER LIMITS

3.15 Definition Let $\{s_n\}$ be a sequence of real numbers with the following property: For every real M there is an integer N such that $n \geq N$ implies $s_n \geq M$. We then write

$$s_n \rightarrow +\infty.$$

Similarly, if for every real M there is an integer N such that $n \geq N$ implies $s_n \leq M$, we write

$$s_n \rightarrow -\infty.$$

It should be noted that we now use the symbol \rightarrow (introduced in Definition 3.1) for certain types of divergent sequences, as well as for convergent sequences, but that the definitions of convergence and of limit, given in Definition 3.1, are in no way changed.

3.16 Definition Let $\{s_n\}$ be a sequence of real numbers. Let E be the set of numbers x (in the extended real number system) such that $s_{n_k} \rightarrow x$ for some subsequence $\{s_{n_k}\}$. This set E contains all subsequential limits as defined in Definition 3.5, plus possibly the numbers $+\infty$, $-\infty$.

We now recall Definitions 1.8 and 1.23 and put

$$s^* = \sup E,$$

$$s_* = \inf E.$$

The numbers s^* , s_* are called the *upper* and *lower limits* of $\{s_n\}$; we use the notation

$$\limsup_{n \rightarrow \infty} s_n = s^*, \quad \liminf_{n \rightarrow \infty} s_n = s_*.$$

3.17 Theorem Let $\{s_n\}$ be a sequence of real numbers. Let E and s^* have the same meaning as in Definition 3.16. Then s^* has the following two properties:

- (a) $s^* \in E$.
- (b) If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$.

Moreover, s^* is the only number with the properties (a) and (b).

Of course, an analogous result is true for s_* .

Proof

(a) If $s^* = +\infty$, then E is not bounded above; hence $\{s_n\}$ is not bounded above, and there is a subsequence $\{s_{n_k}\}$ such that $s_{n_k} \rightarrow +\infty$.

If s^* is real, then E is bounded above, and at least one subsequential limit exists, so that (a) follows from Theorems 3.7 and 2.28.

If $s^* = -\infty$, then E contains only one element, namely $-\infty$, and there is no subsequential limit. Hence, for any real M , $s_n > M$ for at most a finite number of values of n , so that $s_n \rightarrow -\infty$.

This establishes (a) in all cases.

(b) Suppose there is a number $x > s^*$ such that $s_n \geq x$ for infinitely many values of n . In that case, there is a number $y \in E$ such that $y \geq x > s^*$, contradicting the definition of s^* .

Thus s^* satisfies (a) and (b).

To show the uniqueness, suppose there are two numbers, p and q , which satisfy (a) and (b), and suppose $p < q$. Choose x such that $p < x < q$. Since p satisfies (b), we have $s_n < x$ for $n \geq N$. But then q cannot satisfy (a).

3.18 Examples

(a) Let $\{s_n\}$ be a sequence containing all rationals. Then every real number is a subsequential limit, and

$$\limsup_{n \rightarrow \infty} s_n = +\infty, \quad \liminf_{n \rightarrow \infty} s_n = -\infty.$$

(b) Let $s_n = (-1^n)/[1 + (1/n)]$. Then

$$\limsup_{n \rightarrow \infty} s_n = 1, \quad \liminf_{n \rightarrow \infty} s_n = -1.$$

(c) For a real-valued sequence $\{s_n\}$, $\lim_{n \rightarrow \infty} s_n = s$ if and only if

$$\limsup_{n \rightarrow \infty} s_n = \liminf_{n \rightarrow \infty} s_n = s.$$

We close this section with a theorem which is useful, and whose proof is quite trivial:

3.19 Theorem *If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then*

$$\liminf_{n \rightarrow \infty} s_n \leq \liminf_{n \rightarrow \infty} t_n,$$

$$\limsup_{n \rightarrow \infty} s_n \leq \limsup_{n \rightarrow \infty} t_n.$$

SOME SPECIAL SEQUENCES

We shall now compute the limits of some sequences which occur frequently. The proofs will all be based on the following remark: If $0 \leq x_n \leq s_n$ for $n \geq N$, where N is some fixed number, and if $s_n \rightarrow 0$, then $x_n \rightarrow 0$.

3.20 Theorem

(a) *If $p > 0$, then $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$.*

(b) *If $p > 0$, then $\lim_{n \rightarrow \infty} \sqrt[n]{p} = 1$.*

(c) $\lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$.

(d) *If $p > 0$ and α is real, then $\lim_{n \rightarrow \infty} \frac{n^\alpha}{(1+p)^n} = 0$.*

(e) *If $|x| < 1$, then $\lim_{n \rightarrow \infty} x^n = 0$.*

Proof

(a) Take $n > (1/\varepsilon)^{1/p}$. (Note that the archimedean property of the real number system is used here.)

(b) If $p > 1$, put $x_n = \sqrt[n]{p} - 1$. Then $x_n > 0$, and, by the binomial theorem,

$$1 + nx_n \leq (1 + x_n)^n = p,$$

so that

$$0 < x_n \leq \frac{p-1}{n}.$$

Hence $x_n \rightarrow 0$. If $p = 1$, (b) is trivial, and if $0 < p < 1$, the result is obtained by taking reciprocals.

(c) Put $x_n = \sqrt[n]{n} - 1$. Then $x_n \geq 0$, and, by the binomial theorem,

$$n = (1 + x_n)^n \geq \frac{n(n-1)}{2} x_n^2.$$

Hence

$$0 \leq x_n \leq \sqrt{\frac{2}{n-1}} \quad (n \geq 2).$$

(d) Let k be an integer such that $k > \alpha$, $k > 0$. For $n > 2k$,

$$(1+p)^n > \binom{n}{k} p^k = \frac{n(n-1) \cdots (n-k+1)}{k!} p^k > \frac{n^k p^k}{2^k k!}.$$

Hence

$$0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{\alpha-k} \quad (n > 2k).$$

Since $\alpha - k < 0$, $n^{\alpha-k} \rightarrow 0$, by (a).

(e) Take $\alpha = 0$ in (d).

SERIES

In the remainder of this chapter, all sequences and series under consideration will be complex-valued, unless the contrary is explicitly stated. Extensions of some of the theorems which follow, to series with terms in R^k , are mentioned in Exercise 15.

3.21 Definition Given a sequence $\{a_n\}$, we use the notation

$$\sum_{n=p}^q a_n \quad (p \leq q)$$

to denote the sum $a_p + a_{p+1} + \cdots + a_q$. With $\{a_n\}$ we associate a sequence $\{s_n\}$, where

$$s_n = \sum_{k=1}^n a_k.$$

For $\{s_n\}$ we also use the symbolic expression

$$a_1 + a_2 + a_3 + \cdots$$

or, more concisely,

$$(4) \quad \sum_{n=1}^{\infty} a_n.$$

The symbol (4) we call an *infinite series*, or just a *series*. The numbers s_n are called the *partial sums* of the series. If $\{s_n\}$ converges to s , we say that the series *converges*, and write

$$\sum_{n=1}^{\infty} a_n = s.$$

The number s is called the sum of the series; but it should be clearly understood that s is the limit of a sequence of sums, and is not obtained simply by addition.

If $\{s_n\}$ diverges, the series is said to diverge.

Sometimes, for convenience of notation, we shall consider series of the form

$$(5) \quad \sum_{n=0}^{\infty} a_n.$$

And frequently, when there is no possible ambiguity, or when the distinction is immaterial, we shall simply write Σa_n in place of (4) or (5).

It is clear that every theorem about sequences can be stated in terms of series (putting $a_1 = s_1$, and $a_n = s_n - s_{n-1}$ for $n > 1$), and vice versa. But it is nevertheless useful to consider both concepts.

The Cauchy criterion (Theorem 3.11) can be restated in the following form:

3.22 Theorem Σa_n converges if and only if for every $\varepsilon > 0$ there is an integer N such that

$$(6) \quad \left| \sum_{k=n}^m a_k \right| \leq \varepsilon$$

if $m \geq n \geq N$.

In particular, by taking $m = n$, (6) becomes

$$|a_n| \leq \varepsilon \quad (n \geq N).$$

In other words:

3.23 Theorem *If Σa_n converges, then $\lim_{n \rightarrow \infty} a_n = 0$.*

The condition $a_n \rightarrow 0$ is not, however, sufficient to ensure convergence of Σa_n . For instance, the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges; for the proof we refer to Theorem 3.28.

Theorem 3.14, concerning monotonic sequences, also has an immediate counterpart for series.

3.24 Theorem *A series of nonnegative¹ terms converges if and only if its partial sums form a bounded sequence.*

We now turn to a convergence test of a different nature, the so-called “comparison test.”

3.25 Theorem

- (a) *If $|a_n| \leq c_n$ for $n \geq N_0$, where N_0 is some fixed integer, and if Σc_n converges, then Σa_n converges.*
- (b) *If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if Σd_n diverges, then Σa_n diverges.*

Note that (b) applies only to series of nonnegative terms a_n .

Proof Given $\varepsilon > 0$, there exists $N \geq N_0$ such that $m \geq n \geq N$ implies

$$\sum_{k=n}^m c_k \leq \varepsilon,$$

by the Cauchy criterion. Hence

$$\left| \sum_{k=n}^m a_k \right| \leq \sum_{k=n}^m |a_k| \leq \sum_{k=n}^m c_k \leq \varepsilon,$$

and (a) follows.

Next, (b) follows from (a), for if Σa_n converges, so must Σd_n [note that (b) also follows from Theorem 3.24].

¹ The expression “nonnegative” always refers to *real* numbers.

The comparison test is a very useful one; to use it efficiently, we have to become familiar with a number of series of nonnegative terms whose convergence or divergence is known.

SERIES OF NONNEGATIVE TERMS

The simplest of all is perhaps the geometric series.

3.26 Theorem *If $0 \leq x < 1$, then*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

If $x \geq 1$, the series diverges.

Proof If $x \neq 1$,

$$s_n = \sum_{k=0}^n x^k = \frac{1 - x^{n+1}}{1 - x}.$$

The result follows if we let $n \rightarrow \infty$. For $x = 1$, we get

$$1 + 1 + 1 + \cdots,$$

which evidently diverges.

In many cases which occur in applications, the terms of the series decrease monotonically. The following theorem of Cauchy is therefore of particular interest. The striking feature of the theorem is that a rather “thin” subsequence of $\{a_n\}$ determines the convergence or divergence of Σa_n .

3.27 Theorem *Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series*

$$(7) \quad \sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \cdots$$

converges.

Proof By Theorem 3.24, it suffices to consider boundedness of the partial sums. Let

$$\begin{aligned} s_n &= a_1 + a_2 + \cdots + a_n, \\ t_k &= a_1 + 2a_2 + \cdots + 2^k a_{2^k}. \end{aligned}$$

For $n < 2^k$,

$$\begin{aligned} s_n &\leq a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + \cdots + 2^k a_{2^k} \\ &= t_k, \end{aligned}$$

so that

$$(8) \quad s_n \leq t_k.$$

On the other hand, if $n > 2^k$,

$$\begin{aligned} s_n &\geq a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\ &\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{k-1}a_{2^k} \\ &= \frac{1}{2}t_k, \end{aligned}$$

so that

$$(9) \quad 2s_n \geq t_k.$$

By (8) and (9), the sequences $\{s_n\}$ and $\{t_k\}$ are either both bounded or both unbounded. This completes the proof.

3.28 Theorem $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Proof If $p \leq 0$, divergence follows from Theorem 3.23. If $p > 0$, Theorem 3.27 is applicable, and we are led to the series

$$\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k}.$$

Now, $2^{1-p} < 1$ if and only if $1 - p < 0$, and the result follows by comparison with the geometric series (take $x = 2^{1-p}$ in Theorem 3.26).

As a further application of Theorem 3.27, we prove:

3.29 Theorem If $p > 1$,

$$(10) \quad \sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges; if $p \leq 1$, the series diverges.

Remark “ $\log n$ ” denotes the logarithm of n to the base e (compare Exercise 7, Chap. 1); the number e will be defined in a moment (see Definition 3.30). We let the series start with $n = 2$, since $\log 1 = 0$.

Proof The monotonicity of the logarithmic function (which will be discussed in more detail in Chap. 8) implies that $\{\log n\}$ increases. Hence $\{1/n \log n\}$ decreases, and we can apply Theorem 3.27 to (10); this leads us to the series

$$(11) \quad \sum_{k=1}^{\infty} 2^k \cdot \frac{1}{2^k (\log 2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{(k \log 2)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p},$$

and Theorem 3.29 follows from Theorem 3.28.

This procedure may evidently be continued. For instance,

$$(12) \quad \sum_{n=3}^{\infty} \frac{1}{n \log n \log \log n}$$

diverges, whereas

$$(13) \quad \sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^2}$$

converges.

We may now observe that the terms of the series (12) differ very little from those of (13). Still, one diverges, the other converges. If we continue the process which led us from Theorem 3.28 to Theorem 3.29, and then to (12) and (13), we get pairs of convergent and divergent series whose terms differ even less than those of (12) and (13). One might thus be led to the conjecture that there is a limiting situation of some sort, a “boundary” with all convergent series on one side, all divergent series on the other side—at least as far as series with monotonic coefficients are concerned. This notion of “boundary” is of course quite vague. The point we wish to make is this: No matter how we make this notion precise, the conjecture is false. Exercises 11(b) and 12(b) may serve as illustrations.

We do not wish to go any deeper into this aspect of convergence theory, and refer the reader to Knopp’s “Theory and Application of Infinite Series,” Chap. IX, particularly Sec. 41.

THE NUMBER e

3.30 Definition $e = \sum_{n=0}^{\infty} \frac{1}{n!}.$

Here $n! = 1 \cdot 2 \cdot 3 \cdots n$ if $n \geq 1$, and $0! = 1$.

Since

$$\begin{aligned} s_n &= 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots + \frac{1}{1 \cdot 2 \cdots n} \\ &< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 3, \end{aligned}$$

the series converges, and the definition makes sense. In fact, the series converges very rapidly and allows us to compute e with great accuracy.

It is of interest to note that e can also be defined by means of another limit process; the proof provides a good illustration of operations with limits:

3.31 Theorem $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e.$

Proof Let

$$s_n = \sum_{k=0}^n \frac{1}{k!}, \quad t_n = \left(1 + \frac{1}{n}\right)^n.$$

By the binomial theorem,

$$\begin{aligned} t_n &= 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \cdots \\ &\quad + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \cdots \left(1 - \frac{n-1}{n}\right). \end{aligned}$$

Hence $t_n \leq s_n$, so that

$$(14) \quad \limsup_{n \rightarrow \infty} t_n \leq e,$$

by Theorem 3.19. Next, if $n \geq m$,

$$t_n \geq 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{m!} \left(1 - \frac{1}{n}\right) \cdots \left(1 - \frac{m-1}{n}\right).$$

Let $n \rightarrow \infty$, keeping m fixed. We get

$$\liminf_{n \rightarrow \infty} t_n \geq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{m!},$$

so that

$$s_m \leq \liminf_{n \rightarrow \infty} t_n.$$

Letting $m \rightarrow \infty$, we finally get

$$(15) \quad e \leq \liminf_{n \rightarrow \infty} t_n.$$

The theorem follows from (14) and (15).

The rapidity with which the series $\sum \frac{1}{n!}$ converges can be estimated as follows: If s_n has the same meaning as above, we have

$$\begin{aligned} e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \frac{1}{(n+3)!} + \cdots \\ &< \frac{1}{(n+1)!} \left\{ 1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots \right\} = \frac{1}{n!n} \end{aligned}$$

so that

$$(16) \quad 0 < e - s_n < \frac{1}{n!n}.$$

Thus s_{10} , for instance, approximates e with an error less than 10^{-7} . The inequality (16) is of theoretical interest as well, since it enables us to prove the irrationality of e very easily.

3.32 Theorem e is irrational.

Proof Suppose e is rational. Then $e = p/q$, where p and q are positive integers. By (16),

$$(17) \quad 0 < q!(e - s_q) < \frac{1}{q}.$$

By our assumption, $q!e$ is an integer. Since

$$q!s_q = q! \left(1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{q!} \right)$$

is an integer, we see that $q!(e - s_q)$ is an integer.

Since $q \geq 1$, (17) implies the existence of an integer between 0 and 1. We have thus reached a contradiction.

Actually, e is not even an algebraic number. For a simple proof of this, see page 25 of Niven's book, or page 176 of Herstein's, cited in the Bibliography.

THE ROOT AND RATIO TESTS

3.33 Theorem (Root Test) Given Σa_n , put $\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|}$.

Then

- (a) if $\alpha < 1$, Σa_n converges;
- (b) if $\alpha > 1$, Σa_n diverges;
- (c) if $\alpha = 1$, the test gives no information.

Proof If $\alpha < 1$, we can choose β so that $\alpha < \beta < 1$, and an integer N such that

$$\sqrt[n]{|a_n|} < \beta$$

for $n \geq N$ [by Theorem 3.17(b)]. That is, $n \geq N$ implies

$$|a_n| < \beta^n.$$

Since $0 < \beta < 1$, $\Sigma \beta^n$ converges. Convergence of Σa_n follows now from the comparison test.

If $\alpha > 1$, then, again by Theorem 3.17, there is a sequence $\{n_k\}$ such that

$$\sqrt[n_k]{|a_{n_k}|} \rightarrow \alpha.$$

Hence $|a_n| > 1$ for infinitely many values of n , so that the condition $a_n \rightarrow 0$, necessary for convergence of Σa_n , does not hold (Theorem 3.23).

To prove (c), we consider the series

$$\Sigma \frac{1}{n}, \quad \Sigma \frac{1}{n^2}.$$

For each of these series $\alpha = 1$, but the first diverges, the second converges.

3.34 Theorem (Ratio Test) *The series Σa_n*

(a) *converges if $\limsup_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$,*

(b) *diverges if $\left| \frac{a_{n+1}}{a_n} \right| \geq 1$ for all $n \geq n_0$, where n_0 is some fixed integer.*

Proof If condition (a) holds, we can find $\beta < 1$, and an integer N , such that

$$\left| \frac{a_{n+1}}{a_n} \right| < \beta$$

for $n \geq N$. In particular,

$$\begin{aligned} |a_{N+1}| &< \beta |a_N|, \\ |a_{N+2}| &< \beta |a_{N+1}| < \beta^2 |a_N|, \\ &\dots\dots\dots \\ |a_{N+p}| &< \beta^p |a_N|. \end{aligned}$$

That is,

$$|a_n| < |a_N| \beta^{-N} \cdot \beta^n$$

for $n \geq N$, and (a) follows from the comparison test, since $\Sigma \beta^n$ converges.

If $|a_{n+1}| \geq |a_n|$ for $n \geq n_0$, it is easily seen that the condition $a_n \rightarrow 0$ does not hold, and (b) follows.

Note: The knowledge that $\lim a_{n+1}/a_n = 1$ implies nothing about the convergence of Σa_n . The series $\Sigma 1/n$ and $\Sigma 1/n^2$ demonstrate this.

3.35 Examples

(a) Consider the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots,$$

for which

$$\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^n = 0,$$

$$\liminf_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[2n]{\frac{1}{3^n}} = \frac{1}{\sqrt{3}},$$

$$\limsup_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \sqrt[2n]{\frac{1}{2^n}} = \frac{1}{\sqrt{2}},$$

$$\limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{3}{2}\right)^n = +\infty.$$

The root test indicates convergence; the ratio test does not apply.

(b) The same is true for the series

$$\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \cdots,$$

where

$$\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \frac{1}{8},$$

$$\limsup_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 2,$$

but

$$\lim \sqrt[n]{a_n} = \frac{1}{2}.$$

3.36 Remarks The ratio test is frequently easier to apply than the root test, since it is usually easier to compute ratios than n th roots. However, the root test has wider scope. More precisely: Whenever the ratio test shows convergence, the root test does too; whenever the root test is inconclusive, the ratio test is too. This is a consequence of Theorem 3.37, and is illustrated by the above examples.

Neither of the two tests is subtle with regard to divergence. Both deduce divergence from the fact that a_n does not tend to zero as $n \rightarrow \infty$.

3.37 Theorem *For any sequence $\{c_n\}$ of positive numbers,*

$$\liminf_{n \rightarrow \infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n \rightarrow \infty} \sqrt[n]{c_n},$$

$$\limsup_{n \rightarrow \infty} \sqrt[n]{c_n} \leq \limsup_{n \rightarrow \infty} \frac{c_{n+1}}{c_n}.$$

Proof We shall prove the second inequality; the proof of the first is quite similar. Put

$$\alpha = \limsup_{n \rightarrow \infty} \frac{c_{n+1}}{c_n}.$$

If $\alpha = +\infty$, there is nothing to prove. If α is finite, choose $\beta > \alpha$. There is an integer N such that

$$\frac{c_{n+1}}{c_n} \leq \beta$$

for $n \geq N$. In particular, for any $p > 0$,

$$c_{N+k+1} \leq \beta c_{N+k} \quad (k = 0, 1, \dots, p-1).$$

Multiplying these inequalities, we obtain

$$c_{N+p} \leq \beta^p c_N,$$

or

$$c_n \leq c_N \beta^{-N} \cdot \beta^n \quad (n \geq N).$$

Hence

$$\sqrt[n]{c_n} \leq \sqrt[n]{c_N \beta^{-N}} \cdot \beta,$$

so that

$$(18) \quad \limsup_{n \rightarrow \infty} \sqrt[n]{c_n} \leq \beta,$$

by Theorem 3.20(b). Since (18) is true for every $\beta > \alpha$, we have

$$\limsup_{n \rightarrow \infty} \sqrt[n]{c_n} \leq \alpha.$$

POWER SERIES

3.38 Definition Given a sequence $\{c_n\}$ of complex numbers, the series

$$(19) \quad \sum_{n=0}^{\infty} c_n z^n$$

is called a *power series*. The numbers c_n are called the *coefficients* of the series; z is a complex number.

In general, the series will converge or diverge, depending on the choice of z . More specifically, with every power series there is associated a circle, the circle of convergence, such that (19) converges if z is in the interior of the circle and diverges if z is in the exterior (to cover all cases, we have to consider the plane as the interior of a circle of infinite radius, and a point as a circle of radius zero). The behavior on the circle of convergence is much more varied and cannot be described so simply.

3.39 Theorem Given the power series $\sum c_n z^n$, put

$$\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|}, \quad R = \frac{1}{\alpha}.$$

(If $\alpha = 0$, $R = +\infty$; if $\alpha = +\infty$, $R = 0$.) Then $\sum c_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$.

Proof Put $a_n = c_n z^n$, and apply the root test:

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n|} = |z| \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|} = \frac{|z|}{R}.$$

Note: R is called the radius of convergence of $\sum c_n z^n$.

3.40 Examples

(a) The series $\sum n^n z^n$ has $R = 0$.

(b) The series $\sum \frac{z^n}{n!}$ has $R = +\infty$. (In this case the ratio test is easier to apply than the root test.)

- (c) The series $\sum z^n$ has $R = 1$. If $|z| = 1$, the series diverges, since $\{z^n\}$ does not tend to 0 as $n \rightarrow \infty$.
- (d) The series $\sum \frac{z^n}{n}$ has $R = 1$. It diverges if $z = 1$. It converges for all other z with $|z| = 1$. (The last assertion will be proved in Theorem 3.44.)
- (e) The series $\sum \frac{z^n}{n^2}$ has $R = 1$. It converges for all z with $|z| = 1$, by the comparison test, since $|z^n/n^2| = 1/n^2$.

SUMMATION BY PARTS

3.41 Theorem *Given two sequences $\{a_n\}$, $\{b_n\}$, put*

$$A_n = \sum_{k=0}^n a_k$$

if $n \geq 0$; put $A_{-1} = 0$. Then, if $0 \leq p \leq q$, we have

$$(20) \quad \sum_{n=p}^q a_n b_n = \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

Proof

$$\sum_{n=p}^q a_n b_n = \sum_{n=p}^q (A_n - A_{n-1}) b_n = \sum_{n=p}^q A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1},$$

and the last expression on the right is clearly equal to the right side of (20).

Formula (20), the so-called “partial summation formula,” is useful in the investigation of series of the form $\sum a_n b_n$, particularly when $\{b_n\}$ is monotonic. We shall now give applications.

3.42 Theorem *Suppose*

- (a) *the partial sums A_n of $\sum a_n$ form a bounded sequence;*
- (b) *$b_0 \geq b_1 \geq b_2 \geq \cdots$;*
- (c) *$\lim_{n \rightarrow \infty} b_n = 0$.*

Then $\sum a_n b_n$ converges.

Proof Choose M such that $|A_n| \leq M$ for all n . Given $\varepsilon > 0$, there is an integer N such that $b_N \leq (\varepsilon/2M)$. For $N \leq p \leq q$, we have

$$\begin{aligned} \left| \sum_{n=p}^q a_n b_n \right| &= \left| \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \right| \\ &\leq M \left| \sum_{n=p}^{q-1} (b_n - b_{n+1}) + b_q + b_p \right| \\ &= 2M b_p \leq 2M b_N \leq \varepsilon. \end{aligned}$$

Convergence now follows from the Cauchy criterion. We note that the first inequality in the above chain depends of course on the fact that $b_n - b_{n+1} \geq 0$.

3.43 Theorem Suppose

- (a) $|c_1| \geq |c_2| \geq |c_3| \geq \cdots$;
- (b) $c_{2m-1} \geq 0, c_{2m} \leq 0 \quad (m = 1, 2, 3, \dots)$;
- (c) $\lim_{n \rightarrow \infty} c_n = 0$.

Then Σc_n converges.

Series for which (b) holds are called “alternating series”; the theorem was known to Leibnitz.

Proof Apply Theorem 3.42, with $a_n = (-1)^{n+1}, b_n = |c_n|$.

3.44 Theorem Suppose the radius of convergence of $\Sigma c_n z^n$ is 1, and suppose $c_0 \geq c_1 \geq c_2 \geq \cdots, \lim_{n \rightarrow \infty} c_n = 0$. Then $\Sigma c_n z^n$ converges at every point on the circle $|z| = 1$, except possibly at $z = 1$.

Proof Put $a_n = z^n, b_n = c_n$. The hypotheses of Theorem 3.42 are then satisfied, since

$$|A_n| = \left| \sum_{m=0}^n z^m \right| = \left| \frac{1 - z^{n+1}}{1 - z} \right| \leq \frac{2}{|1 - z|},$$

if $|z| = 1, z \neq 1$.

ABSOLUTE CONVERGENCE

The series Σa_n is said to *converge absolutely* if the series $\Sigma |a_n|$ converges.

3.45 Theorem If Σa_n converges absolutely, then Σa_n converges.

Proof The assertion follows from the inequality

$$\left| \sum_{k=n}^m a_k \right| \leq \sum_{k=n}^m |a_k|,$$

plus the Cauchy criterion.

3.46 Remarks For series of positive terms, absolute convergence is the same as convergence.

If $\sum a_n$ converges, but $\sum |a_n|$ diverges, we say that $\sum a_n$ converges *non-absolutely*. For instance, the series

$$\sum \frac{(-1)^n}{n}$$

converges nonabsolutely (Theorem 3.43).

The comparison test, as well as the root and ratio tests, is really a test for absolute convergence, and therefore cannot give any information about non-absolutely convergent series. Summation by parts can sometimes be used to handle the latter. In particular, power series converge absolutely in the interior of the circle of convergence.

We shall see that we may operate with absolutely convergent series very much as with finite sums. We may multiply them term by term and we may change the order in which the additions are carried out, without affecting the sum of the series. But for nonabsolutely convergent series this is no longer true, and more care has to be taken when dealing with them.

ADDITION AND MULTIPLICATION OF SERIES

3.47 Theorem If $\sum a_n = A$, and $\sum b_n = B$, then $\sum(a_n + b_n) = A + B$, and $\sum ca_n = cA$, for any fixed c .

Proof Let

$$A_n = \sum_{k=0}^n a_k, \quad B_n = \sum_{k=0}^n b_k.$$

Then

$$A_n + B_n = \sum_{k=0}^n (a_k + b_k).$$

Since $\lim_{n \rightarrow \infty} A_n = A$ and $\lim_{n \rightarrow \infty} B_n = B$, we see that

$$\lim_{n \rightarrow \infty} (A_n + B_n) = A + B.$$

The proof of the second assertion is even simpler.

Thus two convergent series may be added term by term, and the resulting series converges to the sum of the two series. The situation becomes more complicated when we consider multiplication of two series. To begin with, we have to define the product. This can be done in several ways; we shall consider the so-called "Cauchy product."

3.48 Definition Given Σa_n and Σb_n , we put

$$c_n = \sum_{k=0}^n a_k b_{n-k} \quad (n = 0, 1, 2, \dots)$$

and call Σc_n the *product* of the two given series.

This definition may be motivated as follows. If we take two power series $\Sigma a_n z^n$ and $\Sigma b_n z^n$, multiply them term by term, and collect terms containing the same power of z , we get

$$\begin{aligned} \sum_{n=0}^{\infty} a_n z^n \cdot \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \cdots \\ &= c_0 + c_1 z + c_2 z^2 + \cdots. \end{aligned}$$

Setting $z = 1$, we arrive at the above definition.

3.49 Example If

$$A_n = \sum_{k=0}^n a_k, \quad B_n = \sum_{k=0}^n b_k, \quad C_n = \sum_{k=0}^n c_k,$$

and $A_n \rightarrow A$, $B_n \rightarrow B$, then it is not at all clear that $\{C_n\}$ will converge to AB , since we do not have $C_n = A_n B_n$. The dependence of $\{C_n\}$ on $\{A_n\}$ and $\{B_n\}$ is quite a complicated one (see the proof of Theorem 3.50). We shall now show that the product of two convergent series may actually diverge.

The series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} = 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$$

converges (Theorem 3.43). We form the product of this series with itself and obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_n &= 1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}\sqrt{2}} + \frac{1}{\sqrt{3}} \right) \\ &\quad - \left(\frac{1}{\sqrt{4}} + \frac{1}{\sqrt{3}\sqrt{2}} + \frac{1}{\sqrt{2}\sqrt{3}} + \frac{1}{\sqrt{4}} \right) + \cdots, \end{aligned}$$

so that

$$c_n = (-1)^n \sum_{k=0}^n \frac{1}{\sqrt{(n-k+1)(k+1)}}.$$

Since

$$(n-k+1)(k+1) = \left(\frac{n}{2} + 1\right)^2 - \left(\frac{n}{2} - k\right)^2 \leq \left(\frac{n}{2} + 1\right)^2.$$

we have

$$|c_n| \geq \sum_{k=0}^n \frac{2}{n+2} = \frac{2(n+1)}{n+2},$$

so that the condition $c_n \rightarrow 0$, which is necessary for the convergence of $\sum c_n$, is not satisfied.

In view of the next theorem, due to Mertens, we note that we have here considered the product of two nonabsolutely convergent series.

3.50 Theorem *Suppose*

- (a) $\sum_{n=0}^{\infty} a_n$ *converges absolutely*,
- (b) $\sum_{n=0}^{\infty} a_n = A$,
- (c) $\sum_{n=0}^{\infty} b_n = B$,
- (d) $c_n = \sum_{k=0}^n a_k b_{n-k} \quad (n = 0, 1, 2, \dots).$

Then

$$\sum_{n=0}^{\infty} c_n = AB.$$

That is, the product of two convergent series converges, and to the right value, if at least one of the two series converges absolutely.

Proof Put

$$A_n = \sum_{k=0}^n a_k, \quad B_n = \sum_{k=0}^n b_k, \quad C_n = \sum_{k=0}^n c_k, \quad \beta_n = B_n - B.$$

Then

$$\begin{aligned} C_n &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \\ &= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0 \\ &= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + \cdots + a_n(B + \beta_0) \\ &= A_n B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0 \end{aligned}$$

Put

$$\gamma_n = a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0.$$

We wish to show that $C_n \rightarrow AB$. Since $A_n B \rightarrow AB$, it suffices to show that

$$(21) \quad \lim_{n \rightarrow \infty} \gamma_n = 0.$$

Put

$$\alpha = \sum_{n=0}^{\infty} |a_n|.$$

[It is here that we use (a).] Let $\varepsilon > 0$ be given. By (c), $\beta_n \rightarrow 0$. Hence we can choose N such that $|\beta_n| \leq \varepsilon$ for $n \geq N$, in which case

$$\begin{aligned} |\gamma_n| &\leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + |\beta_{N+1} a_{n-N-1} + \cdots + \beta_n a_0| \\ &\leq |\beta_0 a_n + \cdots + \beta_N a_{n-N}| + \varepsilon \alpha. \end{aligned}$$

Keeping N fixed, and letting $n \rightarrow \infty$, we get

$$\limsup_{n \rightarrow \infty} |\gamma_n| \leq \varepsilon \alpha,$$

since $a_k \rightarrow 0$ as $k \rightarrow \infty$. Since ε is arbitrary, (21) follows.

Another question which may be asked is whether the series Σc_n , if convergent, must have the sum AB . Abel showed that the answer is in the affirmative.

3.51 Theorem *If the series Σa_n , Σb_n , Σc_n converge to A , B , C , and $c_n = a_0 b_n + \cdots + a_n b_0$, then $C = AB$.*

Here no assumption is made concerning absolute convergence. We shall give a simple proof (which depends on the continuity of power series) after Theorem 8.2.

REARRANGEMENTS

3.52 Definition Let $\{k_n\}$, $n = 1, 2, 3, \dots$, be a sequence in which every positive integer appears once and only once (that is, $\{k_n\}$ is a 1-1 function from J onto J , in the notation of Definition 2.2). Putting

$$a'_n = a_{k_n} \quad (n = 1, 2, 3, \dots),$$

we say that $\Sigma a'_n$ is a *rearrangement* of Σa_n .

If $\{s_n\}, \{s'_n\}$ are the sequences of partial sums of $\Sigma a_n, \Sigma a'_n$, it is easily seen that, in general, these two sequences consist of entirely different numbers. We are thus led to the problem of determining under what conditions all rearrangements of a convergent series will converge and whether the sums are necessarily the same.

3.53 Example Consider the convergent series

$$(22) \quad 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

and one of its rearrangements

$$(23) \quad 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$$

in which two positive terms are always followed by one negative. If s is the sum of (22), then

$$s < 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

Since

$$\frac{1}{4k-3} + \frac{1}{4k-1} - \frac{1}{2k} > 0$$

for $k \geq 1$, we see that $s'_3 < s'_6 < s'_9 < \cdots$, where s'_n is n th partial sum of (23). Hence

$$\limsup_{n \rightarrow \infty} s'_n > s'_3 = \frac{5}{6},$$

so that (23) certainly does not converge to s [we leave it to the reader to verify that (23) does, however, converge].

This example illustrates the following theorem, due to Riemann.

3.54 Theorem Let Σa_n be a series of real numbers which converges, but not absolutely. Suppose

$$-\infty \leq \alpha \leq \beta \leq \infty.$$

Then there exists a rearrangement $\Sigma a'_n$ with partial sums s'_n such that

$$(24) \quad \liminf_{n \rightarrow \infty} s'_n = \alpha, \quad \limsup_{n \rightarrow \infty} s'_n = \beta.$$

Proof Let

$$p_n = \frac{|a_n| + a_n}{2}, \quad q_n = \frac{|a_n| - a_n}{2} \quad (n = 1, 2, 3, \dots).$$

Then $p_n - q_n = a_n$, $p_n + q_n = |a_n|$, $p_n \geq 0$, $q_n \geq 0$. The series Σp_n , Σq_n must both diverge.

For if both were convergent, then

$$\Sigma(p_n + q_n) = \Sigma|a_n|$$

would converge, contrary to hypothesis. Since

$$\sum_{n=1}^N a_n = \sum_{n=1}^N (p_n - q_n) = \sum_{n=1}^N p_n - \sum_{n=1}^N q_n,$$

divergence of Σp_n and convergence of Σq_n (or vice versa) implies divergence of Σa_n , again contrary to hypothesis.

Now let P_1, P_2, P_3, \dots denote the nonnegative terms of Σa_n , in the order in which they occur, and let Q_1, Q_2, Q_3, \dots be the absolute values of the negative terms of Σa_n , also in their original order.

The series $\Sigma P_n, \Sigma Q_n$ differ from $\Sigma p_n, \Sigma q_n$ only by zero terms, and are therefore divergent.

We shall construct sequences $\{m_n\}, \{k_n\}$, such that the series

$$(25) \quad P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} + P_{m_1+1} + \dots \\ + P_{m_2} - Q_{k_1+1} - \dots - Q_{k_2} + \dots,$$

which clearly is a rearrangement of Σa_n , satisfies (24).

Choose real-valued sequences $\{\alpha_n\}, \{\beta_n\}$ such that $\alpha_n \rightarrow \alpha$, $\beta_n \rightarrow \beta$, $\alpha_n < \beta_n$, $\beta_1 > 0$.

Let m_1, k_1 be the smallest integers such that

$$P_1 + \dots + P_{m_1} > \beta_1,$$

$$P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} < \alpha_1;$$

let m_2, k_2 be the smallest integers such that

$$P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} + P_{m_1+1} + \dots + P_{m_2} > \beta_2,$$

$$P_1 + \dots + P_{m_1} - Q_1 - \dots - Q_{k_1} + P_{m_1+1} + \dots + P_{m_2} - Q_{k_1+1} \\ - \dots - Q_{k_2} < \alpha_2;$$

and continue in this way. This is possible since ΣP_n and ΣQ_n diverge.

If x_n, y_n denote the partial sums of (25) whose last terms are P_{m_n} , $-Q_{k_n}$, then

$$|x_n - \beta_n| \leq P_{m_n}, \quad |y_n - \alpha_n| \leq Q_{k_n}.$$

Since $P_n \rightarrow 0$ and $Q_n \rightarrow 0$ as $n \rightarrow \infty$, we see that $x_n \rightarrow \beta$, $y_n \rightarrow \alpha$.

Finally, it is clear that no number less than α or greater than β can be a subsequential limit of the partial sums of (25).

3.55 Theorem *If Σa_n is a series of complex numbers which converges absolutely, then every rearrangement of Σa_n converges, and they all converge to the same sum.*

Proof Let $\Sigma a'_n$ be a rearrangement, with partial sums s'_n . Given $\varepsilon > 0$, there exists an integer N such that $m \geq n \geq N$ implies

$$(26) \quad \sum_{i=n}^m |a_i| \leq \varepsilon.$$

Now choose p such that the integers $1, 2, \dots, N$ are all contained in the set k_1, k_2, \dots, k_p (we use the notation of Definition 3.52). Then if $n > p$, the numbers a_1, \dots, a_N will cancel in the difference $s_n - s'_n$, so that $|s_n - s'_n| \leq \varepsilon$, by (26). Hence $\{s'_n\}$ converges to the same sum as $\{s_n\}$.

EXERCISES

1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?
2. Calculate $\lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n)$.
3. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \quad (n = 1, 2, 3, \dots),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \dots$.

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}; \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

5. For any two real sequences $\{a_n\}, \{b_n\}$, prove that

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

6. Investigate the behavior (convergence or divergence) of Σa_n if

$$(a) \quad a_n = \sqrt{n+1} - \sqrt{n};$$

$$(b) \quad a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n};$$

$$(c) \quad a_n = (\sqrt[n]{n} - 1)^n;$$

$$(d) \quad a_n = \frac{1}{1 + z^n}, \quad \text{for complex values of } z.$$

7. Prove that the convergence of Σa_n implies the convergence of

$$\Sigma \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

8. If Σa_n converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\Sigma a_n b_n$ converges.
9. Find the radius of convergence of each of the following power series:
- (a) $\Sigma n^3 z^n$, (b) $\Sigma \frac{2^n}{n!} z^n$,
- (c) $\Sigma \frac{2^n}{n^2} z^n$, (d) $\Sigma \frac{n^3}{3^n} z^n$.
10. Suppose that the coefficients of the power series $\Sigma a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.
11. Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and Σa_n diverges.

(a) Prove that $\Sigma \frac{a_n}{1 + a_n}$ diverges.

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\Sigma \frac{a_n}{s_n}$ diverges.

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\Sigma \frac{a_n}{s_n^2}$ converges.

(d) What can be said about

$$\Sigma \frac{a_n}{1 + na_n} \quad \text{and} \quad \Sigma \frac{a_n}{1 + n^2 a_n}?$$

12. Suppose $a_n > 0$ and Σa_n converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) Prove that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\Sigma \frac{a_n}{r_n}$ diverges.

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

13. Prove that the Cauchy product of two absolutely convergent series converges absolutely.

14. If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \quad (n = 0, 1, 2, \dots).$$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.

(c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?

(d) Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^n k a_k.$$

Assume that $\lim (n a_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges.

[This gives a converse of (a), but under the additional assumption that $n a_n \rightarrow 0$.]

(e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|n a_n| \leq M$ for all n , and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$, by completing the following outline:

If $m < n$, then

$$s_n - \sigma_n = \frac{m+1}{n-m} (\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{i=m+1}^n (s_n - s_i).$$

For these i ,

$$|s_n - s_i| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

Fix $\varepsilon > 0$ and associate with each n the integer m that satisfies

$$m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1.$$

Then $(m+1)/(n-m) \leq 1/\varepsilon$ and $|s_n - s_i| < M\varepsilon$. Hence

$$\limsup_{n \rightarrow \infty} |s_n - \sigma| \leq M\varepsilon.$$

Since ε was arbitrary, $\lim s_n = \sigma$.

15. Definition 3.21 can be extended to the case in which the a_n lie in some fixed R^k . Absolute convergence is defined as convergence of $\sum |a_n|$. Show that Theorems 3.22, 3.23, 3.25(a), 3.33, 3.34, 3.42, 3.45, 3.47, and 3.55 are true in this more general setting. (Only slight modifications are required in any of the proofs.)
16. Fix a positive number α . Choose $x_1 > \sqrt{\alpha}$, and define x_2, x_3, x_4, \dots , by the recursion formula

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{\alpha}{x_n} \right).$$

- (a) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.
 (b) Put $\varepsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta \left(\frac{\varepsilon_1}{\beta} \right)^{2^n} \quad (n = 1, 2, 3, \dots).$$

- (c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\varepsilon_1/\beta < \frac{1}{10}$ and that therefore

$$\varepsilon_5 < 4 \cdot 10^{-16}, \quad \varepsilon_6 < 4 \cdot 10^{-32}.$$

17. Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}.$$

- (a) Prove that $x_1 > x_3 > x_5 > \dots$.
 (b) Prove that $x_2 < x_4 < x_6 < \dots$.
 (c) Prove that $\lim x_n = \sqrt{\alpha}$.
 (d) Compare the rapidity of convergence of this process with the one described in Exercise 16.
18. Replace the recursion formula of Exercise 16 by

$$x_{n+1} = \frac{p-1}{p} x_n + \frac{\alpha}{p} x_n^{-p+1}$$

where p is a fixed positive integer, and describe the behavior of the resulting sequences $\{x_n\}$.

19. Associate to each sequence $a = \{\alpha_n\}$, in which α_n is 0 or 2, the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove that the set of all $x(a)$ is precisely the Cantor set described in Sec. 2.44.

20. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X , and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p .
21. Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a *complete* metric space X , if $E_n \supset E_{n+1}$, and if

$$\lim_{n \rightarrow \infty} \text{diam } E_n = 0,$$

then $\bigcap_{i=1}^{\infty} E_n$ consists of exactly one point.

22. Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X . Prove Baire's theorem, namely, that $\bigcap_{i=1}^{\infty} G_n$ is not empty. (In fact, it is dense in X .) *Hint*: Find a shrinking sequence of neighborhoods E_n such that $E_n \subset G_n$, and apply Exercise 21.
23. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X . Show that the sequence $\{d(p_n, q_n)\}$ converges. *Hint*: For any m, n ,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if m and n are large.

24. Let X be a metric space.

(a) Call two Cauchy sequences $\{p_n\}, \{q_n\}$ in X *equivalent* if

$$\lim_{n \rightarrow \infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

(b) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*, Q \in X^*, \{p_n\} \in P, \{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n \rightarrow \infty} d(p_n, q_n);$$

by Exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^* .

(c) Prove that the resulting metric space X^* is complete.

(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p ; let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry (i.e., a distance-preserving mapping) of X into X^* .

(e) Prove that $\varphi(X)$ is dense in X^* , and that $\varphi(X) = X^*$ if X is complete. By (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^* . We call X^* the *completion* of X .

25. Let X be the metric space whose points are the rational numbers, with the metric $d(x, y) = |x - y|$. What is the completion of this space? (Compare Exercise 24.)

The function concept and some of the related terminology were introduced in Definitions 2.1 and 2.2. Although we shall (in later chapters) be mainly interested in real and complex functions (i.e., in functions whose values are real or complex numbers) we shall also discuss vector-valued functions (i.e., functions with values in R^k) and functions with values in an arbitrary metric space. The theorems we shall discuss in this general setting would not become any easier if we restricted ourselves to real functions, for instance, and it actually simplifies and clarifies the picture to discard unnecessary hypotheses and to state and prove theorems in an appropriately general context.

The domains of definition of our functions will also be metric spaces, suitably specialized in various instances.

LIMITS OF FUNCTIONS

4.1 Definition Let X and Y be metric spaces; suppose $E \subset X$, f maps E into Y , and p is a limit point of E . We write $f(x) \rightarrow q$ as $x \rightarrow p$, or

$$(1) \quad \lim_{x \rightarrow p} f(x) = q$$

if there is a point $q \in Y$ with the following property: For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$(2) \quad d_Y(f(x), q) < \varepsilon$$

for all points $x \in E$ for which

$$(3) \quad 0 < d_X(x, p) < \delta.$$

The symbols d_X and d_Y refer to the distances in X and Y , respectively.

If X and/or Y are replaced by the real line, the complex plane, or by some euclidean space R^k , the distances d_X, d_Y are of course replaced by absolute values, or by norms of differences (see Sec. 2.16).

It should be noted that $p \in X$, but that p need not be a point of E in the above definition. Moreover, even if $p \in E$, we may very well have $f(p) \neq \lim_{x \rightarrow p} f(x)$.

We can recast this definition in terms of limits of sequences:

4.2 Theorem Let X, Y, E, f , and p be as in Definition 4.1. Then

$$(4) \quad \lim_{x \rightarrow p} f(x) = q$$

if and only if

$$(5) \quad \lim_{n \rightarrow \infty} f(p_n) = q$$

for every sequence $\{p_n\}$ in E such that

$$(6) \quad p_n \neq p, \quad \lim_{n \rightarrow \infty} p_n = p.$$

Proof Suppose (4) holds. Choose $\{p_n\}$ in E satisfying (6). Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that $d_Y(f(x), q) < \varepsilon$ if $x \in E$ and $0 < d_X(x, p) < \delta$. Also, there exists N such that $n > N$ implies $0 < d_X(p_n, p) < \delta$. Thus, for $n > N$, we have $d_Y(f(p_n), q) < \varepsilon$, which shows that (5) holds.

Conversely, suppose (4) is false. Then there exists some $\varepsilon > 0$ such that for every $\delta > 0$ there exists a point $x \in E$ (depending on δ), for which $d_Y(f(x), q) \geq \varepsilon$ but $0 < d_X(x, p) < \delta$. Taking $\delta_n = 1/n$ ($n = 1, 2, 3, \dots$), we thus find a sequence in E satisfying (6) for which (5) is false.

Corollary If f has a limit at p , this limit is unique.

This follows from Theorems 3.2(b) and 4.2.

4.3 Definition Suppose we have two complex functions, f and g , both defined on E . By $f + g$ we mean the function which assigns to each point x of E the number $f(x) + g(x)$. Similarly we define the difference $f - g$, the product fg , and the quotient f/g of the two functions, with the understanding that the quotient is defined only at those points x of E at which $g(x) \neq 0$. If f assigns to each point x of E the same number c , then f is said to be a constant function, or simply a constant, and we write $f = c$. If f and g are real functions, and if $f(x) \geq g(x)$ for every $x \in E$, we shall sometimes write $f \geq g$, for brevity.

Similarly, if \mathbf{f} and \mathbf{g} map E into R^k , we define $\mathbf{f} + \mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$ by

$$(\mathbf{f} + \mathbf{g})(x) = \mathbf{f}(x) + \mathbf{g}(x), \quad (\mathbf{f} \cdot \mathbf{g})(x) = \mathbf{f}(x) \cdot \mathbf{g}(x);$$

and if λ is a real number, $(\lambda \mathbf{f})(x) = \lambda \mathbf{f}(x)$.

4.4 Theorem Suppose $E \subset X$, a metric space, p is a limit point of E , f and g are complex functions on E , and

$$\lim_{x \rightarrow p} f(x) = A, \quad \lim_{x \rightarrow p} g(x) = B.$$

Then (a) $\lim_{x \rightarrow p} (f + g)(x) = A + B$;

(b) $\lim_{x \rightarrow p} (fg)(x) = AB$;

(c) $\lim_{x \rightarrow p} \left(\frac{f}{g} \right)(x) = \frac{A}{B}$, if $B \neq 0$.

Proof In view of Theorem 4.2, these assertions follow immediately from the analogous properties of sequences (Theorem 3.3).

Remark If \mathbf{f} and \mathbf{g} map E into R^k , then (a) remains true, and (b) becomes

$$(b') \quad \lim_{x \rightarrow p} (\mathbf{f} \cdot \mathbf{g})(x) = \mathbf{A} \cdot \mathbf{B}.$$

(Compare Theorem 3.4.)

CONTINUOUS FUNCTIONS

4.5 Definition Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y . Then f is said to be *continuous at p* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \varepsilon$$

for all points $x \in E$ for which $d_X(x, p) < \delta$.

If f is continuous at every point of E , then f is said to be *continuous on E* .

It should be noted that f has to be defined at the point p in order to be continuous at p . (Compare this with the remark following Definition 4.1.)

If p is an isolated point of E , then our definition implies that every function f which has E as its domain of definition is continuous at p . For, no matter which $\varepsilon > 0$ we choose, we can pick $\delta > 0$ so that the only point $x \in E$ for which $d_X(x, p) < \delta$ is $x = p$; then

$$d_Y(f(x), f(p)) = 0 < \varepsilon.$$

4.6 Theorem *In the situation given in Definition 4.5, assume also that p is a limit point of E . Then f is continuous at p if and only if $\lim_{x \rightarrow p} f(x) = f(p)$.*

Proof This is clear if we compare Definitions 4.1 and 4.5.

We now turn to compositions of functions. A brief statement of the following theorem is that a continuous function of a continuous function is continuous.

4.7 Theorem *Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y , g maps the range of f , $f(E)$, into Z , and h is the mapping of E into Z defined by*

$$h(x) = g(f(x)) \quad (x \in E).$$

If f is continuous at a point $p \in E$ and if g is continuous at the point $f(p)$, then h is continuous at p .

This function h is called the *composition* or the *composite* of f and g . The notation

$$h = g \circ f$$

is frequently used in this context.

Proof Let $\varepsilon > 0$ be given. Since g is continuous at $f(p)$, there exists $\eta > 0$ such that

$$d_Z(g(y), g(f(p))) < \varepsilon \text{ if } d_Y(y, f(p)) < \eta \text{ and } y \in f(E).$$

Since f is continuous at p , there exists $\delta > 0$ such that

$$d_Y(f(x), f(p)) < \eta \text{ if } d_X(x, p) < \delta \text{ and } x \in E.$$

It follows that

$$d_Z(h(x), h(p)) = d_Z(g(f(x)), g(f(p))) < \varepsilon$$

if $d_X(x, p) < \delta$ and $x \in E$. Thus h is continuous at p .

4.8 Theorem *A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y .*

(Inverse images are defined in Definition 2.2.) This is a very useful characterization of continuity.

Proof Suppose f is continuous on X and V is an open set in Y . We have to show that every point of $f^{-1}(V)$ is an interior point of $f^{-1}(V)$. So, suppose $p \in X$ and $f(p) \in V$. Since V is open, there exists $\varepsilon > 0$ such that $y \in V$ if $d_Y(f(p), y) < \varepsilon$; and since f is continuous at p , there exists $\delta > 0$ such that $d_Y(f(x), f(p)) < \varepsilon$ if $d_X(x, p) < \delta$. Thus $x \in f^{-1}(V)$ as soon as $d_X(x, p) < \delta$.

Conversely, suppose $f^{-1}(V)$ is open in X for every open set V in Y . Fix $p \in X$ and $\varepsilon > 0$, let V be the set of all $y \in Y$ such that $d_Y(y, f(p)) < \varepsilon$. Then V is open; hence $f^{-1}(V)$ is open; hence there exists $\delta > 0$ such that $x \in f^{-1}(V)$ as soon as $d_X(p, x) < \delta$. But if $x \in f^{-1}(V)$, then $f(x) \in V$, so that $d_Y(f(x), f(p)) < \varepsilon$.

This completes the proof.

Corollary *A mapping f of a metric space X into a metric space Y is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y .*

This follows from the theorem, since a set is closed if and only if its complement is open, and since $f^{-1}(E^c) = [f^{-1}(E)]^c$ for every $E \subset Y$.

We now turn to complex-valued and vector-valued functions, and to functions defined on subsets of R^k .

4.9 Theorem *Let f and g be complex continuous functions on a metric space X . Then $f + g$, fg , and f/g are continuous on X .*

In the last case, we must of course assume that $g(x) \neq 0$, for all $x \in X$.

Proof At isolated points of X there is nothing to prove. At limit points, the statement follows from Theorems 4.4 and 4.6.

4.10 Theorem

(a) *Let f_1, \dots, f_k be real functions on a metric space X , and let \mathbf{f} be the mapping of X into R^k defined by*

$$(7) \quad \mathbf{f}(x) = (f_1(x), \dots, f_k(x)) \quad (x \in X);$$

then \mathbf{f} is continuous if and only if each of the functions f_1, \dots, f_k is continuous.

(b) *If \mathbf{f} and \mathbf{g} are continuous mappings of X into R^k , then $\mathbf{f} + \mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$ are continuous on X .*

The functions f_1, \dots, f_k are called the *components* of \mathbf{f} . Note that $\mathbf{f} + \mathbf{g}$ is a mapping into R^k , whereas $\mathbf{f} \cdot \mathbf{g}$ is a *real* function on X .

Proof Part (a) follows from the inequalities

$$|f_j(x) - f_j(y)| \leq |\mathbf{f}(x) - \mathbf{f}(y)| = \left\{ \sum_{i=1}^k |f_i(x) - f_i(y)|^2 \right\}^{\frac{1}{2}},$$

for $j = 1, \dots, k$. Part (b) follows from (a) and Theorem 4.9.

4.11 Examples If x_1, \dots, x_k are the coordinates of the point $\mathbf{x} \in R^k$, the functions ϕ_i defined by

$$(8) \quad \phi_i(\mathbf{x}) = x_i \quad (\mathbf{x} \in R^k)$$

are continuous on R^k , since the inequality

$$|\phi_i(\mathbf{x}) - \phi_i(\mathbf{y})| \leq |\mathbf{x} - \mathbf{y}|$$

shows that we may take $\delta = \varepsilon$ in Definition 4.5. The functions ϕ_i are sometimes called the *coordinate functions*.

Repeated application of Theorem 4.9 then shows that every monomial

$$(9) \quad x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$$

where n_1, \dots, n_k are nonnegative integers, is continuous on R^k . The same is true of constant multiples of (9), since constants are evidently continuous. It follows that every polynomial P , given by

$$(10) \quad P(\mathbf{x}) = \sum c_{n_1 \dots n_k} x_1^{n_1} \dots x_k^{n_k} \quad (\mathbf{x} \in R^k),$$

is continuous on R^k . Here the coefficients $c_{n_1 \dots n_k}$ are complex numbers, n_1, \dots, n_k are nonnegative integers, and the sum in (10) has finitely many terms.

Furthermore, every rational function in x_1, \dots, x_k , that is, every quotient of two polynomials of the form (10), is continuous on R^k wherever the denominator is different from zero.

From the triangle inequality one sees easily that

$$(11) \quad ||\mathbf{x}| - |\mathbf{y}|| \leq |\mathbf{x} - \mathbf{y}| \quad (\mathbf{x}, \mathbf{y} \in R^k).$$

Hence the mapping $\mathbf{x} \rightarrow |\mathbf{x}|$ is a continuous real function on R^k .

If now \mathbf{f} is a continuous mapping from a metric space X into R^k , and if ϕ is defined on X by setting $\phi(p) = |\mathbf{f}(p)|$, it follows, by Theorem 4.7, that ϕ is a continuous real function on X .

4.12 Remark We defined the notion of continuity for functions defined on a subset E of a metric space X . However, the complement of E in X plays no role whatever in this definition (note that the situation was somewhat different for limits of functions). Accordingly, we lose nothing of interest by discarding the complement of the domain of f . This means that we may just as well talk only about continuous mappings of one metric space into another, rather than

of mappings of subsets. This simplifies statements and proofs of some theorems. We have already made use of this principle in Theorems 4.8 to 4.10, and will continue to do so in the following section on compactness.

CONTINUITY AND COMPACTNESS

4.13 Definition A mapping f of a set E into R^k is said to be *bounded* if there is a real number M such that $|f(x)| \leq M$ for all $x \in E$.

4.14 Theorem Suppose f is a continuous mapping of a compact metric space X into a metric space Y . Then $f(X)$ is compact.

Proof Let $\{V_\alpha\}$ be an open cover of $f(X)$. Since f is continuous, Theorem 4.8 shows that each of the sets $f^{-1}(V_\alpha)$ is open. Since X is compact, there are finitely many indices, say $\alpha_1, \dots, \alpha_n$, such that

$$(12) \quad X \subset f^{-1}(V_{\alpha_1}) \cup \dots \cup f^{-1}(V_{\alpha_n}).$$

Since $f(f^{-1}(E)) \subset E$ for every $E \subset Y$, (12) implies that

$$(13) \quad f(X) \subset V_{\alpha_1} \cup \dots \cup V_{\alpha_n}.$$

This completes the proof.

Note: We have used the relation $f(f^{-1}(E)) \subset E$, valid for $E \subset Y$. If $E \subset X$, then $f^{-1}(f(E)) \supset E$; equality need not hold in either case.

We shall now deduce some consequences of Theorem 4.14.

4.15 Theorem If f is a continuous mapping of a compact metric space X into R^k , then $f(X)$ is closed and bounded. Thus, f is bounded.

This follows from Theorem 2.41. The result is particularly important when f is real:

4.16 Theorem Suppose f is a continuous real function on a compact metric space X , and

$$(14) \quad M = \sup_{p \in X} f(p), \quad m = \inf_{p \in X} f(p).$$

Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.

The notation in (14) means that M is the least upper bound of the set of all numbers $f(p)$, where p ranges over X , and that m is the greatest lower bound of this set of numbers.

The conclusion may also be stated as follows: *There exist points p and q in X such that $f(q) \leq f(x) \leq f(p)$ for all $x \in X$; that is, f attains its maximum (at p) and its minimum (at q).*

Proof By Theorem 4.15, $f(X)$ is a closed and bounded set of real numbers; hence $f(X)$ contains

$$M = \sup f(X) \quad \text{and} \quad m = \inf f(X),$$

by Theorem 2.28.

4.17 Theorem *Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y . Then the inverse mapping f^{-1} defined on Y by*

$$f^{-1}(f(x)) = x \quad (x \in X)$$

is a continuous mapping of Y onto X .

Proof Applying Theorem 4.8 to f^{-1} in place of f , we see that it suffices to prove that $f(V)$ is an open set in Y for every open set V in X . Fix such a set V .

The complement V^c of V is closed in X , hence compact (Theorem 2.35); hence $f(V^c)$ is a compact subset of Y (Theorem 4.14) and so is closed in Y (Theorem 2.34). Since f is one-to-one and onto, $f(V)$ is the complement of $f(V^c)$. Hence $f(V)$ is open.

4.18 Definition Let f be a mapping of a metric space X into a metric space Y . We say that f is *uniformly continuous* on X if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$(15) \quad d_Y(f(p), f(q)) < \varepsilon$$

for all p and q in X for which $d_X(p, q) < \delta$.

Let us consider the differences between the concepts of continuity and of uniform continuity. First, uniform continuity is a property of a function on a set, whereas continuity can be defined at a single point. To ask whether a given function is uniformly continuous at a certain point is meaningless. Second, if f is continuous on X , then it is possible to find, for each $\varepsilon > 0$ and for each point p of X , a number $\delta > 0$ having the property specified in Definition 4.5. This δ depends on ε and on p . If f is, however, uniformly continuous on X , then it is possible, for each $\varepsilon > 0$, to find *one* number $\delta > 0$ which will do for *all* points p of X .

Evidently, every uniformly continuous function is continuous. That the two concepts are equivalent on compact sets follows from the next theorem.

4.19 Theorem *Let f be a continuous mapping of a compact metric space X into a metric space Y . Then f is uniformly continuous on X .*

Proof Let $\varepsilon > 0$ be given. Since f is continuous, we can associate to each point $p \in X$ a positive number $\phi(p)$ such that

$$(16) \quad q \in X, d_X(p, q) < \phi(p) \text{ implies } d_Y(f(p), f(q)) < \frac{\varepsilon}{2}.$$

Let $J(p)$ be the set of all $q \in X$ for which

$$(17) \quad d_X(p, q) < \frac{1}{2}\phi(p).$$

Since $p \in J(p)$, the collection of all sets $J(p)$ is an open cover of X ; and since X is compact, there is a finite set of points p_1, \dots, p_n in X , such that

$$(18) \quad X \subset J(p_1) \cup \dots \cup J(p_n).$$

We put

$$(19) \quad \delta = \frac{1}{2} \min [\phi(p_1), \dots, \phi(p_n)].$$

Then $\delta > 0$. (This is one point where the finiteness of the covering, inherent in the definition of compactness, is essential. The minimum of a finite set of positive numbers is positive, whereas the inf of an infinite set of positive numbers may very well be 0.)

Now let q and p be points of X , such that $d_X(p, q) < \delta$. By (18), there is an integer m , $1 \leq m \leq n$, such that $p \in J(p_m)$; hence

$$(20) \quad d_X(p, p_m) < \frac{1}{2}\phi(p_m),$$

and we also have

$$d_X(q, p_m) \leq d_X(p, q) + d_X(p, p_m) < \delta + \frac{1}{2}\phi(p_m) \leq \phi(p_m).$$

Finally, (16) shows that therefore

$$d_Y(f(p), f(q)) \leq d_Y(f(p), f(p_m)) + d_Y(f(q), f(p_m)) < \varepsilon.$$

This completes the proof.

An alternative proof is sketched in Exercise 10.

We now proceed to show that compactness is essential in the hypotheses of Theorems 4.14, 4.15, 4.16, and 4.19.

4.20 Theorem *Let E be a noncompact set in R^1 . Then*

- (a) *there exists a continuous function on E which is not bounded;*
- (b) *there exists a continuous and bounded function on E which has no maximum.*

If, in addition, E is bounded, then

(c) *there exists a continuous function on E which is not uniformly continuous.*

Proof Suppose first that E is bounded, so that there exists a limit point x_0 of E which is not a point of E . Consider

$$(21) \quad f(x) = \frac{1}{x - x_0} \quad (x \in E).$$

This is continuous on E (Theorem 4.9), but evidently unbounded. To see that (21) is not uniformly continuous, let $\varepsilon > 0$ and $\delta > 0$ be arbitrary, and choose a point $x \in E$ such that $|x - x_0| < \delta$. Taking t close enough to x_0 , we can then make the difference $|f(t) - f(x)|$ greater than ε , although $|t - x| < \delta$. Since this is true for every $\delta > 0$, f is not uniformly continuous on E .

The function g given by

$$(22) \quad g(x) = \frac{1}{1 + (x - x_0)^2} \quad (x \in E)$$

is continuous on E , and is bounded, since $0 < g(x) < 1$. It is clear that

$$\sup_{x \in E} g(x) = 1,$$

whereas $g(x) < 1$ for all $x \in E$. Thus g has no maximum on E .

Having proved the theorem for bounded sets E , let us now suppose that E is unbounded. Then $f(x) = x$ establishes (a), whereas

$$(23) \quad h(x) = \frac{x^2}{1 + x^2} \quad (x \in E)$$

establishes (b), since

$$\sup_{x \in E} h(x) = 1$$

and $h(x) < 1$ for all $x \in E$.

Assertion (c) would be false if boundedness were omitted from the hypotheses. For, let E be the set of all integers. Then every function defined on E is uniformly continuous on E . To see this, we need merely take $\delta < 1$ in Definition 4.18.

We conclude this section by showing that compactness is also essential in Theorem 4.17.

4.21 Example Let X be the half-open interval $[0, 2\pi)$ on the real line, and let f be the mapping of X onto the circle Y consisting of all points whose distance from the origin is 1, given by

$$(24) \quad f(t) = (\cos t, \sin t) \quad (0 \leq t < 2\pi).$$

The continuity of the trigonometric functions cosine and sine, as well as their periodicity properties, will be established in Chap. 8. These results show that f is a continuous 1-1 mapping of X onto Y .

However, the inverse mapping (which exists, since f is one-to-one and onto) fails to be continuous at the point $(1, 0) = f(0)$. Of course, X is not compact in this example. (It may be of interest to observe that f^{-1} fails to be continuous in spite of the fact that Y is compact!)

CONTINUITY AND CONNECTEDNESS

4.22 Theorem *If f is a continuous mapping of a metric space X into a metric space Y , and if E is a connected subset of X , then $f(E)$ is connected.*

Proof Assume, on the contrary, that $f(E) = A \cup B$, where A and B are nonempty separated subsets of Y . Put $G = E \cap f^{-1}(A)$, $H = E \cap f^{-1}(B)$.

Then $E = G \cup H$, and neither G nor H is empty.

Since $A \subset \bar{A}$ (the closure of A), we have $G \subset f^{-1}(\bar{A})$; the latter set is closed, since f is continuous; hence $\bar{G} \subset f^{-1}(\bar{A})$. It follows that $f(\bar{G}) \subset \bar{A}$. Since $f(H) = B$ and $\bar{A} \cap B$ is empty, we conclude that $\bar{G} \cap H$ is empty.

The same argument shows that $G \cap \bar{H}$ is empty. Thus G and H are separated. This is impossible if E is connected.

4.23 Theorem *Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$ and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.*

A similar result holds, of course, if $f(a) > f(b)$. Roughly speaking, the theorem says that a continuous real function assumes all intermediate values on an interval.

Proof By Theorem 2.47, $[a, b]$ is connected; hence Theorem 4.22 shows that $f([a, b])$ is a connected subset of R^1 , and the assertion follows if we appeal once more to Theorem 2.47.

4.24 Remark At first glance, it might seem that Theorem 4.23 has a converse. That is, one might think that if for any two points $x_1 < x_2$ and for any number c between $f(x_1)$ and $f(x_2)$ there is a point x in (x_1, x_2) such that $f(x) = c$, then f must be continuous.

That this is not so may be concluded from Example 4.27(d).

DISCONTINUITIES

If x is a point in the domain of definition of the function f at which f is not continuous, we say that f is *discontinuous* at x , or that f has a *discontinuity* at x . If f is defined on an interval or on a segment, it is customary to divide discontinuities into two types. Before giving this classification, we have to define the *right-hand* and the *left-hand limits* of f at x , which we denote by $f(x+)$ and $f(x-)$, respectively.

4.25 Definition Let f be defined on (a, b) . Consider any point x such that $a \leq x < b$. We write

$$f(x+) = q$$

if $f(t_n) \rightarrow q$ as $n \rightarrow \infty$, for all sequences $\{t_n\}$ in (x, b) such that $t_n \rightarrow x$. To obtain the definition of $f(x-)$, for $a < x \leq b$, we restrict ourselves to sequences $\{t_n\}$ in (a, x) .

It is clear that any point x of (a, b) , $\lim_{t \rightarrow x} f(t)$ exists if and only if

$$f(x+) = f(x-) = \lim_{t \rightarrow x} f(t).$$

4.26 Definition Let f be defined on (a, b) . If f is discontinuous at a point x , and if $f(x+)$ and $f(x-)$ exist, then f is said to have a discontinuity of the *first kind*, or a *simple discontinuity*, at x . Otherwise the discontinuity is said to be of the *second kind*.

There are two ways in which a function can have a simple discontinuity: either $f(x+) \neq f(x-)$ [in which case the value $f(x)$ is immaterial], or $f(x+) = f(x-) \neq f(x)$.

4.27 Examples

(a) Define

$$f(x) = \begin{cases} 1 & (x \text{ rational}), \\ 0 & (x \text{ irrational}). \end{cases}$$

Then f has a discontinuity of the second kind at every point x , since neither $f(x+)$ nor $f(x-)$ exists.

(b) Define

$$f(x) = \begin{cases} x & (x \text{ rational}), \\ 0 & (x \text{ irrational}). \end{cases}$$

Then f is continuous at $x = 0$ and has a discontinuity of the second kind at every other point.

(c) Define

$$f(x) = \begin{cases} x + 2 & (-3 < x < -2), \\ -x - 2 & (-2 \leq x < 0), \\ x + 2 & (0 \leq x < 1). \end{cases}$$

Then f has a simple discontinuity at $x = 0$ and is continuous at every other point of $(-3, 1)$.

(d) Define

$$f(x) = \begin{cases} \sin \frac{1}{x} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Since neither $f(0+)$ nor $f(0-)$ exists, f has a discontinuity of the second kind at $x = 0$. We have not yet shown that $\sin x$ is a continuous function. If we assume this result for the moment, Theorem 4.7 implies that f is continuous at every point $x \neq 0$.

MONOTONIC FUNCTIONS

We shall now study those functions which never decrease (or never increase) on a given segment.

4.28 Definition Let f be real on (a, b) . Then f is said to be *monotonically increasing* on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$. If the last inequality is reversed, we obtain the definition of a *monotonically decreasing* function. The class of monotonic functions consists of both the increasing and the decreasing functions.

4.29 Theorem Let f be monotonically increasing on (a, b) . Then $f(x+)$ and $f(x-)$ exist at every point of (a, b) . More precisely,

$$(25) \quad \sup_{a < t < x} f(t) = f(x-) \leq f(x) \leq f(x+) = \inf_{x < t < b} f(t).$$

Furthermore, if $a < x < y < b$, then

$$(26) \quad f(x+) \leq f(y-).$$

Analogous results evidently hold for monotonically decreasing functions.

Proof By hypothesis, the set of numbers $f(t)$, where $a < t < x$, is bounded above by the number $f(x)$, and therefore has a least upper bound which we shall denote by A . Evidently $A \leq f(x)$. We have to show that $A = f(x-)$.

Let $\varepsilon > 0$ be given. It follows from the definition of A as a least upper bound that there exists $\delta > 0$ such that $a < x - \delta < x$ and

$$(27) \quad A - \varepsilon < f(x - \delta) \leq A.$$

Since f is monotonic, we have

$$(28) \quad f(x - \delta) \leq f(t) \leq A \quad (x - \delta < t < x).$$

Combining (27) and (28), we see that

$$|f(t) - A| < \varepsilon \quad (x - \delta < t < x).$$

Hence $f(x-) = A$.

The second half of (25) is proved in precisely the same way.

Next, if $a < x < y < b$, we see from (25) that

$$(29) \quad f(x+) = \inf_{x < t < b} f(t) = \inf_{x < t < y} f(t).$$

The last equality is obtained by applying (25) to (a, y) in place of (a, b) . Similarly,

$$(30) \quad f(y-) = \sup_{a < t < y} f(t) = \sup_{x < t < y} f(t).$$

Comparison of (29) and (30) gives (26).

Corollary *Monotonic functions have no discontinuities of the second kind.*

This corollary implies that every monotonic function is discontinuous at a countable set of points at most. Instead of appealing to the general theorem whose proof is sketched in Exercise 17, we give here a simple proof which is applicable to monotonic functions.

4.30 Theorem *Let f be monotonic on (a, b) . Then the set of points of (a, b) at which f is discontinuous is at most countable.*

Proof Suppose, for the sake of definiteness, that f is increasing, and let E be the set of points at which f is discontinuous.

With every point x of E we associate a rational number $r(x)$ such that

$$f(x-) < r(x) < f(x+).$$

Since $x_1 < x_2$ implies $f(x_1+) \leq f(x_2-)$, we see that $r(x_1) \neq r(x_2)$ if $x_1 \neq x_2$.

We have thus established a 1-1 correspondence between the set E and a subset of the set of rational numbers. The latter, as we know, is countable.

4.31 Remark It should be noted that the discontinuities of a monotonic function need not be isolated. In fact, given any countable subset E of (a, b) , which may even be dense, we can construct a function f , monotonic on (a, b) , discontinuous at every point of E , and at no other point of (a, b) .

To show this, let the points of E be arranged in a sequence $\{x_n\}$, $n = 1, 2, 3, \dots$. Let $\{c_n\}$ be a sequence of positive numbers such that $\sum c_n$ converges. Define

$$(31) \quad f(x) = \sum_{x_n < x} c_n \quad (a < x < b).$$

The summation is to be understood as follows: Sum over those indices n for which $x_n < x$. If there are no points x_n to the left of x , the sum is empty; following the usual convention, we define it to be zero. Since (31) converges absolutely, the order in which the terms are arranged is immaterial.

We leave the verification of the following properties of f to the reader:

- (a) f is monotonically increasing on (a, b) ;
- (b) f is discontinuous at every point of E ; in fact,

$$f(x_n+) - f(x_n-) = c_n.$$

- (c) f is continuous at every other point of (a, b) .

Moreover, it is not hard to see that $f(x-) = f(x)$ at all points of (a, b) . If a function satisfies this condition, we say that f is *continuous from the left*. If the summation in (31) were taken over all indices n for which $x_n \leq x$, we would have $f(x+) = f(x)$ at every point of (a, b) ; that is, f would be *continuous from the right*.

Functions of this sort can also be defined by another method; for an example we refer to Theorem 6.16.

INFINITE LIMITS AND LIMITS AT INFINITY

To enable us to operate in the extended real number system, we shall now enlarge the scope of Definition 4.1, by reformulating it in terms of neighborhoods.

For any real number x , we have already defined a neighborhood of x to be any segment $(x - \delta, x + \delta)$.

4.32 Definition For any real c , the set of real numbers x such that $x > c$ is called a neighborhood of $+\infty$ and is written $(c, +\infty)$. Similarly, the set $(-\infty, c)$ is a neighborhood of $-\infty$.

4.33 Definition Let f be a real function defined on $E \subset \mathbb{R}$. We say that

$$f(t) \rightarrow A \text{ as } t \rightarrow x,$$

where A and x are in the extended real number system, if for every neighborhood U of A there is a neighborhood V of x such that $V \cap E$ is not empty, and such that $f(t) \in U$ for all $t \in V \cap E$, $t \neq x$.

A moment's consideration will show that this coincides with Definition 4.1 when A and x are real.

The analogue of Theorem 4.4 is still true, and the proof offers nothing new. We state it, for the sake of completeness.

4.34 Theorem Let f and g be defined on $E \subset \mathbb{R}$. Suppose

$$f(t) \rightarrow A, \quad g(t) \rightarrow B \quad \text{as } t \rightarrow x.$$

Then

- (a) $f(t) \rightarrow A'$ implies $A' = A$.
- (b) $(f+g)(t) \rightarrow A+B$,
- (c) $(fg)(t) \rightarrow AB$,
- (d) $(f/g)(t) \rightarrow A/B$,

provided the right members of (b), (c), and (d) are defined.

Note that $\infty - \infty$, $0 \cdot \infty$, ∞/∞ , $A/0$ are not defined (see Definition 1.23).

EXERCISES

1. Suppose f is a real function defined on \mathbb{R}^1 which satisfies

$$\lim_{h \rightarrow 0} [f(x+h) - f(x-h)] = 0$$

for every $x \in \mathbb{R}^1$. Does this imply that f is continuous?

2. If f is a continuous mapping of a metric space X into a metric space Y , prove that

$$f(\bar{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. (\bar{E} denotes the closure of E .) Show, by an example, that $f(\bar{E})$ can be a proper subset of $\overline{f(E)}$.

3. Let f be a continuous real function on a metric space X . Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.
4. Let f and g be continuous mappings of a metric space X into a metric space Y ,

and let E be a dense subset of X . Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)

5. If f is a real continuous function defined on a closed set $E \subset R^1$, prove that there exist continuous real functions g on R^1 such that $g(x) = f(x)$ for all $x \in E$. (Such functions g are called *continuous extensions* of f from E to R^1 .) Show that the result becomes false if the word "closed" is omitted. Extend the result to vector-valued functions. *Hint*: Let the graph of g be a straight line on each of the segments which constitute the complement of E (compare Exercise 29, Chap. 2). The result remains true if R^1 is replaced by any metric space, but the proof is not so simple.
6. If f is defined on E , the *graph* of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plane.

Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.

7. If $E \subset X$ and if f is a function defined on X , the *restriction* of f to E is the function g whose domain of definition is E , such that $g(p) = f(p)$ for $p \in E$. Define f and g on R^2 by: $f(0, 0) = g(0, 0) = 0$, $f(x, y) = xy^2/(x^2 + y^4)$, $g(x, y) = xy^2/(x^2 + y^6)$ if $(x, y) \neq (0, 0)$. Prove that f is bounded on R^2 , that g is unbounded in every neighborhood of $(0, 0)$, and that f is not continuous at $(0, 0)$; nevertheless, the restrictions of both f and g to every straight line in R^2 are continuous!
8. Let f be a real uniformly continuous function on the bounded set E in R^1 . Prove that f is bounded on E .

Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

9. Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets: To every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\text{diam } f(E) < \varepsilon$ for all $E \subset X$ with $\text{diam } E < \delta$.
10. Complete the details of the following alternative proof of Theorem 4.19: If f is not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}, \{q_n\}$ in X such that $d_X(p_n, q_n) \rightarrow 0$ but $d_Y(f(p_n), f(q_n)) > \varepsilon$. Use Theorem 2.37 to obtain a contradiction.
11. Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X . Use this result to give an alternative proof of the theorem stated in Exercise 13.
12. A uniformly continuous function of a uniformly continuous function is uniformly continuous.

State this more precisely and prove it.

13. Let E be a dense subset of a metric space X , and let f be a uniformly continuous real function defined on E . Prove that f has a continuous extension from E to X .

(see Exercise 5 for terminology). (Uniqueness follows from Exercise 4.) *Hint:* For each $p \in X$ and each positive integer n , let $V_n(p)$ be the set of all $q \in E$ with $d(p, q) < 1/n$. Use Exercise 9 to show that the intersection of the closures of the sets $f(V_1(p)), f(V_2(p)), \dots$, consists of a single point, say $g(p)$, of R^1 . Prove that the function g so defined on X is the desired extension of f .

Could the range space R^1 be replaced by R^k ? By any compact metric space? By any complete metric space? By any metric space?

14. Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I . Prove that $f(x) = x$ for at least one $x \in I$.
15. Call a mapping of X into Y *open* if $f(V)$ is an open set in Y whenever V is an open set in X .

Prove that every continuous open mapping of R^1 into R^1 is monotonic.

16. Let $[x]$ denote the largest integer contained in x , that is, $[x]$ is the integer such that $x - 1 < [x] \leq x$; and let $(x) = x - [x]$ denote the fractional part of x . What discontinuities do the functions $[x]$ and (x) have?
17. Let f be a real function defined on (a, b) . Prove that the set of points at which f has a simple discontinuity is at most countable. *Hint:* Let E be the set on which $f(x-) < f(x+)$. With each point x of E , associate a triple (p, q, r) of rational numbers such that
- (a) $f(x-) < p < f(x+)$,
 - (b) $a < q < t < x$ implies $f(t) < p$,
 - (c) $x < t < r < b$ implies $f(t) > p$.

The set of all such triples is countable. Show that each triple is associated with at most one point of E . Deal similarly with the other possible types of simple discontinuities.

18. Every rational x can be written in the form $x = m/n$, where $n > 0$, and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on R^1 by

$$f(x) = \begin{cases} 0 & (x \text{ irrational}), \\ \frac{1}{n} & \left(x = \frac{m}{n}\right). \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

19. Suppose f is a real function with domain R^1 which has the intermediate value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b .

Suppose also, for every rational r , that the set of all x with $f(x) = r$ is closed. Prove that f is continuous.

Hint: If $x_n \rightarrow x_0$ but $f(x_n) > r > f(x_0)$ for some r and all n , then $f(t_n) = r$ for some t_n between x_0 and x_n ; thus $t_n \rightarrow x_0$. Find a contradiction. (N. J. Fine, *Amer. Math. Monthly*, vol. 73, 1966, p. 782.)

20. If E is a nonempty subset of a metric space X , define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

- (a) Prove that $\rho_E(x) = 0$ if and only if $x \in \bar{E}$.
 (b) Prove that ρ_E is a uniformly continuous function on X , by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

for all $x \in X, y \in X$.

Hint: $\rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z)$, so that

$$\rho_E(x) \leq d(x, y) + \rho_E(y).$$

21. Suppose K and F are disjoint sets in a metric space X , K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K, q \in F$. *Hint:* ρ_F is a continuous positive function on K .

Show that the conclusion may fail for two disjoint closed sets if neither is compact.

22. Let A and B be disjoint nonempty closed sets in a metric space X , and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)} \quad (p \in X).$$

Show that f is a continuous function on X whose range lies in $[0, 1]$, that $f(p) = 0$ precisely on A and $f(p) = 1$ precisely on B . This establishes a converse of Exercise 3: Every closed set $A \subset X$ is $Z(f)$ for some continuous real f on X . Setting

$$V = f^{-1}([0, \tfrac{1}{2})), \quad W = f^{-1}((\tfrac{1}{2}, 1]),$$

show that V and W are open and disjoint, and that $A \subset V, B \subset W$. (Thus pairs of disjoint closed sets in a metric space can be covered by pairs of disjoint open sets. This property of metric spaces is called *normality*.)

23. A real-valued function f defined in (a, b) is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever $a < x < b, a < y < b, 0 < \lambda < 1$. Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex. (For example, if f is convex, so is e^f .)

If f is convex in (a, b) and if $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

24. Assume that f is a continuous real function defined in (a, b) such that

$$f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2}$$

for all $x, y \in (a, b)$. Prove that f is convex.

25. If $A \subset R^k$ and $B \subset R^k$, define $A + B$ to be the set of all sums $\mathbf{x} + \mathbf{y}$ with $\mathbf{x} \in A$, $\mathbf{y} \in B$.

(a) If K is compact and C is closed in R^k , prove that $K + C$ is closed.

Hint: Take $\mathbf{z} \notin K + C$, put $F = \mathbf{z} - C$, the set of all $\mathbf{z} - \mathbf{y}$ with $\mathbf{y} \in C$. Then K and F are disjoint. Choose δ as in Exercise 21. Show that the open ball with center \mathbf{z} and radius δ does not intersect $K + C$.

(b) Let α be an irrational real number. Let C_1 be the set of all integers, let C_2 be the set of all $n\alpha$ with $n \in C_1$. Show that C_1 and C_2 are closed subsets of R^1 whose sum $C_1 + C_2$ is *not* closed, by showing that $C_1 + C_2$ is a countable dense subset of R^1 .

26. Suppose X, Y, Z are metric spaces, and Y is compact. Let f map X into Y , let g be a continuous one-to-one mapping of Y into Z , and put $h(x) = g(f(x))$ for $x \in X$.

Prove that f is uniformly continuous if h is uniformly continuous.

Hint: g^{-1} has compact domain $g(Y)$, and $f(x) = g^{-1}(h(x))$.

Prove also that f is continuous if h is continuous.

Show (by modifying Example 4.21, or by finding a different example) that the compactness of Y cannot be omitted from the hypotheses, even when X and Z are compact.

DIFFERENTIATION

In this chapter we shall (except in the final section) confine our attention to *real* functions defined on intervals or segments. This is not just a matter of convenience, since genuine differences appear when we pass from real functions to vector-valued ones. Differentiation of functions defined on R^k will be discussed in Chap. 9.

THE DERIVATIVE OF A REAL FUNCTION

5.1 Definition Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$ form the quotient

$$(1) \quad \phi(t) = \frac{f(t) - f(x)}{t - x} \quad (a < t < b, t \neq x),$$

and define

$$(2) \quad f'(x) = \lim_{t \rightarrow x} \phi(t),$$

provided this limit exists in accordance with Definition 4.1.

We thus associate with the function f a function f' whose domain is the set of points x at which the limit (2) exists; f' is called the *derivative* of f .

If f' is defined at a point x , we say that f is *differentiable* at x . If f' is defined at every point of a set $E \subset [a, b]$, we say that f is differentiable on E .

It is possible to consider right-hand and left-hand limits in (2); this leads to the definition of right-hand and left-hand derivatives. In particular, at the endpoints a and b , the derivative, if it exists, is a right-hand or left-hand derivative, respectively. We shall not, however, discuss one-sided derivatives in any detail.

If f is defined on a segment (a, b) and if $a < x < b$, then $f'(x)$ is defined by (1) and (2), as above. But $f'(a)$ and $f'(b)$ are not defined in this case.

5.2 Theorem *Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$, then f is continuous at x .*

Proof As $t \rightarrow x$, we have, by Theorem 4.4,

$$f(t) - f(x) = \frac{f(t) - f(x)}{t - x} \cdot (t - x) \rightarrow f'(x) \cdot 0 = 0.$$

The converse of this theorem is not true. It is easy to construct continuous functions which fail to be differentiable at isolated points. In Chap. 7 we shall even become acquainted with a function which is continuous on the whole line without being differentiable at any point!

5.3 Theorem *Suppose f and g are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$. Then $f + g$, fg , and f/g are differentiable at x , and*

- (a) $(f + g)'(x) = f'(x) + g'(x)$;
- (b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$;
- (c) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}.$

In (c), we assume of course that $g(x) \neq 0$.

Proof (a) is clear, by Theorem 4.4. Let $h = fg$. Then

$$h(t) - h(x) = f(t)[g(t) - g(x)] + g(x)[f(t) - f(x)].$$

If we divide this by $t - x$ and note that $f(t) \rightarrow f(x)$ as $t \rightarrow x$ (Theorem 5.2), (b) follows. Next, let $h = f/g$. Then

$$\frac{h(t) - h(x)}{t - x} = \frac{1}{g(t)g(x)} \left[g(x) \frac{f(t) - f(x)}{t - x} - f(x) \frac{g(t) - g(x)}{t - x} \right].$$

Letting $t \rightarrow x$, and applying Theorems 4.4 and 5.2, we obtain (c).

5.4 Examples The derivative of any constant is clearly zero. If f is defined by $f(x) = x$, then $f'(x) = 1$. Repeated application of (b) and (c) then shows that x^n is differentiable, and that its derivative is nx^{n-1} , for any integer n (if $n < 0$, we have to restrict ourselves to $x \neq 0$). Thus every polynomial is differentiable, and so is every rational function, except at the points where the denominator is zero.

The following theorem is known as the “chain rule” for differentiation. It deals with differentiation of composite functions and is probably the most important theorem about derivatives. We shall meet more general versions of it in Chap. 9.

5.5 Theorem Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f , and g is differentiable at the point $f(x)$. If

$$h(t) = g(f(t)) \quad (a \leq t \leq b),$$

then h is differentiable at x , and

$$(3) \quad h'(x) = g'(f(x))f'(x).$$

Proof Let $y = f(x)$. By the definition of the derivative, we have

$$(4) \quad f(t) - f(x) = (t - x)[f'(x) + u(t)],$$

$$(5) \quad g(s) - g(y) = (s - y)[g'(y) + v(s)],$$

where $t \in [a, b]$, $s \in I$, and $u(t) \rightarrow 0$ as $t \rightarrow x$, $v(s) \rightarrow 0$ as $s \rightarrow y$. Let $s = f(t)$.

Using first (5) and then (4), we obtain

$$\begin{aligned} h(t) - h(x) &= g(f(t)) - g(f(x)) \\ &= [f(t) - f(x)] \cdot [g'(y) + v(s)] \\ &= (t - x) \cdot [f'(x) + u(t)] \cdot [g'(y) + v(s)], \end{aligned}$$

or, if $t \neq x$,

$$(6) \quad \frac{h(t) - h(x)}{t - x} = [g'(y) + v(s)] \cdot [f'(x) + u(t)].$$

Letting $t \rightarrow x$, we see that $s \rightarrow y$, by the continuity of f , so that the right side of (6) tends to $g'(y)f'(x)$, which gives (3).

5.6 Examples

(a) Let f be defined by

$$(7) \quad f(x) = \begin{cases} x \sin \frac{1}{x} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Taking for granted that the derivative of $\sin x$ is $\cos x$ (we shall discuss the trigonometric functions in Chap. 8), we can apply Theorems 5.3 and 5.5 whenever $x \neq 0$, and obtain

$$(8) \quad f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x} \quad (x \neq 0).$$

At $x = 0$, these theorems do not apply any longer, since $1/x$ is not defined there, and we appeal directly to the definition: for $t \neq 0$,

$$\frac{f(t) - f(0)}{t - 0} = \sin \frac{1}{t}.$$

As $t \rightarrow 0$, this does not tend to any limit, so that $f'(0)$ does not exist.

(b) Let f be defined by

$$(9) \quad f(x) = \begin{cases} x^2 \sin \frac{1}{x} & (x \neq 0), \\ 0 & (x = 0), \end{cases}$$

As above, we obtain

$$(10) \quad f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} \quad (x \neq 0).$$

At $x = 0$, we appeal to the definition, and obtain

$$\left| \frac{f(t) - f(0)}{t - 0} \right| = \left| t \sin \frac{1}{t} \right| \leq |t| \quad (t \neq 0);$$

letting $t \rightarrow 0$, we see that

$$(11) \quad f'(0) = 0.$$

Thus f is differentiable at all points x , but f' is not a continuous function, since $\cos(1/x)$ in (10) does not tend to a limit as $x \rightarrow 0$.

MEAN VALUE THEOREMS

5.7 Definition Let f be a real function defined on a metric space X . We say that f has a *local maximum* at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$.

Local minima are defined likewise.

Our next theorem is the basis of many applications of differentiation.

5.8 Theorem Let f be defined on $[a, b]$; if f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then $f'(x) = 0$.

The analogous statement for local minima is of course also true.

Proof Choose δ in accordance with Definition 5.7, so that

$$a < x - \delta < x < x + \delta < b.$$

If $x - \delta < t < x$, then

$$\frac{f(t) - f(x)}{t - x} \geq 0.$$

Letting $t \rightarrow x$, we see that $f'(x) \geq 0$.

If $x < t < x + \delta$, then

$$\frac{f(t) - f(x)}{t - x} \leq 0,$$

which shows that $f'(x) \leq 0$. Hence $f'(x) = 0$.

5.9 Theorem If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b) , then there is a point $x \in (a, b)$ at which

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x).$$

Note that differentiability is not required at the endpoints.

Proof Put

$$h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t) \quad (a \leq t \leq b).$$

Then h is continuous on $[a, b]$, h is differentiable in (a, b) , and

$$(12) \quad h(a) = f(b)g(a) - f(a)g(b) = h(b).$$

To prove the theorem, we have to show that $h'(x) = 0$ for some $x \in (a, b)$.

If h is constant, this holds for every $x \in (a, b)$. If $h(t) > h(a)$ for some $t \in (a, b)$, let x be a point on $[a, b]$ at which h attains its maximum

(Theorem 4.16). By (12), $x \in (a, b)$, and Theorem 5.8 shows that $h'(x) = 0$. If $h(t) < h(a)$ for some $t \in (a, b)$, the same argument applies if we choose for x a point on $[a, b]$ where h attains its minimum.

This theorem is often called a *generalized mean value theorem*; the following special case is usually referred to as “the” mean value theorem:

5.10 Theorem *If f is a real continuous function on $[a, b]$ which is differentiable in (a, b) , then there is a point $x \in (a, b)$ at which*

$$f(b) - f(a) = (b - a)f'(x).$$

Proof Take $g(x) = x$ in Theorem 5.9.

5.11 Theorem *Suppose f is differentiable in (a, b) .*

(a) *If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.*

(b) *If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.*

(c) *If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.*

Proof All conclusions can be read off from the equation

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(x),$$

which is valid, for each pair of numbers x_1, x_2 in (a, b) , for *some* x between x_1 and x_2 .

THE CONTINUITY OF DERIVATIVES

We have already seen [Example 5.6(b)] that a function f may have a derivative f' which exists at every point, but is discontinuous at some point. However, not every function is a derivative. In particular, derivatives which exist at every point of an interval have one important property in common with functions which are continuous on an interval: Intermediate values are assumed (compare Theorem 4.23). The precise statement follows.

5.12 Theorem *Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.*

A similar result holds of course if $f'(a) > f'(b)$.

Proof Put $g(t) = f(t) - \lambda t$. Then $g'(a) < 0$, so that $g(t_1) < g(a)$ for some $t_1 \in (a, b)$, and $g'(b) > 0$, so that $g(t_2) < g(b)$ for some $t_2 \in (a, b)$. Hence g attains its minimum on $[a, b]$ (Theorem 4.16) at some point x such that $a < x < b$. By Theorem 5.8, $g'(x) = 0$. Hence $f'(x) = \lambda$.

Corollary *If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuities on $[a, b]$.*

But f' may very well have discontinuities of the second kind.

L'HOSPITAL'S RULE

The following theorem is frequently useful in the evaluation of limits.

5.13 Theorem *Suppose f and g are real and differentiable in (a, b) , and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty \leq a < b \leq +\infty$. Suppose*

$$(13) \quad \frac{f'(x)}{g'(x)} \rightarrow A \text{ as } x \rightarrow a.$$

If

$$(14) \quad f(x) \rightarrow 0 \text{ and } g(x) \rightarrow 0 \text{ as } x \rightarrow a,$$

or if

$$(15) \quad g(x) \rightarrow +\infty \text{ as } x \rightarrow a,$$

then

$$(16) \quad \frac{f(x)}{g(x)} \rightarrow A \text{ as } x \rightarrow a.$$

The analogous statement is of course also true if $x \rightarrow b$, or if $g(x) \rightarrow -\infty$ in (15). Let us note that we now use the limit concept in the extended sense of Definition 4.33.

Proof We first consider the case in which $-\infty \leq A < +\infty$. Choose a real number q such that $A < q$, and then choose r such that $A < r < q$. By (13) there is a point $c \in (a, b)$ such that $a < x < c$ implies

$$(17) \quad \frac{f'(x)}{g'(x)} < r.$$

If $a < x < y < c$, then Theorem 5.9 shows that there is a point $t \in (x, y)$ such that

$$(18) \quad \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(t)}{g'(t)} < r.$$

Suppose (14) holds. Letting $x \rightarrow a$ in (18), we see that

$$(19) \quad \frac{f(y)}{g(y)} \leq r < q \quad (a < y < c).$$

Next, suppose (15) holds. Keeping y fixed in (18), we can choose a point $c_1 \in (a, y)$ such that $g(x) > g(y)$ and $g(x) > 0$ if $a < x < c_1$. Multiplying (18) by $[g(x) - g(y)]/g(x)$, we obtain

$$(20) \quad \frac{f(x)}{g(x)} < r - r \frac{g(y)}{g(x)} + \frac{f(y)}{g(x)} \quad (a < x < c_1).$$

If we let $x \rightarrow a$ in (20), (15) shows that there is a point $c_2 \in (a, c_1)$ such that

$$(21) \quad \frac{f(x)}{g(x)} < q \quad (a < x < c_2).$$

Summing up, (19) and (21) show that for any q , subject only to the condition $A < q$, there is a point c_2 such that $f(x)/g(x) < q$ if $a < x < c_2$.

In the same manner, if $-\infty < A \leq +\infty$, and p is chosen so that $p < A$, we can find a point c_3 such that

$$(22) \quad p < \frac{f(x)}{g(x)} \quad (a < x < c_3),$$

and (16) follows from these two statements.

DERIVATIVES OF HIGHER ORDER

5.14 Definition If f has a derivative f' on an interval, and if f' is itself differentiable, we denote the derivative of f' by f'' and call f'' the second derivative of f . Continuing in this manner, we obtain functions

$$f, f', f'', f^{(3)}, \dots, f^{(n)},$$

each of which is the derivative of the preceding one. $f^{(n)}$ is called the n th derivative, or the derivative of order n , of f .

In order for $f^{(n)}(x)$ to exist at a point x , $f^{(n-1)}(t)$ must exist in a neighborhood of x (or in a one-sided neighborhood, if x is an endpoint of the interval on which f is defined), and $f^{(n-1)}$ must be differentiable at x . Since $f^{(n-1)}$ must exist in a neighborhood of x , $f^{(n-2)}$ must be differentiable in that neighborhood.

TAYLOR'S THEOREM

5.15 Theorem Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define

$$(23) \quad P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k.$$

Then there exists a point x between α and β such that

$$(24) \quad f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!} (\beta - \alpha)^n.$$

For $n = 1$, this is just the mean value theorem. In general, the theorem shows that f can be approximated by a polynomial of degree $n - 1$, and that (24) allows us to estimate the error, if we know bounds on $|f^{(n)}(x)|$.

Proof Let M be the number defined by

$$(25) \quad f(\beta) = P(\beta) + M(\beta - \alpha)^n$$

and put

$$(26) \quad g(t) = f(t) - P(t) - M(t - \alpha)^n \quad (a \leq t \leq b).$$

We have to show that $n!M = f^{(n)}(x)$ for some x between α and β . By (23) and (26),

$$(27) \quad g^{(n)}(t) = f^{(n)}(t) - n!M \quad (a < t < b).$$

Hence the proof will be complete if we can show that $g^{(n)}(x) = 0$ for some x between α and β .

Since $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, \dots, n - 1$, we have

$$(28) \quad g(\alpha) = g'(\alpha) = \dots = g^{(n-1)}(\alpha) = 0.$$

Our choice of M shows that $g(\beta) = 0$, so that $g'(x_1) = 0$ for some x_1 between α and β , by the mean value theorem. Since $g'(\alpha) = 0$, we conclude similarly that $g''(x_2) = 0$ for some x_2 between α and x_1 . After n steps we arrive at the conclusion that $g^{(n)}(x_n) = 0$ for some x_n between α and x_{n-1} , that is, between α and β .

DIFFERENTIATION OF VECTOR-VALUED FUNCTIONS

5.16 Remarks Definition 5.1 applies without any change to complex functions f defined on $[a, b]$, and Theorems 5.2 and 5.3, as well as their proofs, remain valid. If f_1 and f_2 are the real and imaginary parts of f , that is, if

$$f(t) = f_1(t) + if_2(t)$$

for $a \leq t \leq b$, where $f_1(t)$ and $f_2(t)$ are real, then we clearly have

$$(29) \quad f'(x) = f_1'(x) + if_2'(x);$$

also, f is differentiable at x if and only if both f_1 and f_2 are differentiable at x .

Passing to vector-valued functions in general, i.e., to functions \mathbf{f} which map $[a, b]$ into some R^k , we may still apply Definition 5.1 to define $\mathbf{f}'(x)$. The term $\phi(t)$ in (1) is now, for each t , a point in R^k , and the limit in (2) is taken with respect to the norm of R^k . In other words, $\mathbf{f}'(x)$ is that point of R^k (if there is one) for which

$$(30) \quad \lim_{t \rightarrow x} \left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right| = 0,$$

and \mathbf{f}' is again a function with values in R^k .

If f_1, \dots, f_k are the components of \mathbf{f} , as defined in Theorem 4.10, then

$$(31) \quad \mathbf{f}' = (f'_1, \dots, f'_k),$$

and \mathbf{f} is differentiable at a point x if and only if each of the functions f_1, \dots, f_k is differentiable at x .

Theorem 5.2 is true in this context as well, and so is Theorem 5.3(a) and (b), if fg is replaced by the inner product $\mathbf{f} \cdot \mathbf{g}$ (see Definition 4.3).

When we turn to the mean value theorem, however, and to one of its consequences, namely, L'Hospital's rule, the situation changes. The next two examples will show that each of these results fails to be true for complex-valued functions.

5.17 Example Define, for real x ,

$$(32) \quad f(x) = e^{ix} = \cos x + i \sin x.$$

(The last expression may be taken as the definition of the complex exponential e^{ix} ; see Chap. 8 for a full discussion of these functions.) Then

$$(33) \quad f(2\pi) - f(0) = 1 - 1 = 0,$$

but

$$(34) \quad f'(x) = ie^{ix},$$

so that $|f'(x)| = 1$ for all real x .

Thus Theorem 5.10 fails to hold in this case.

5.18 Example On the segment $(0, 1)$, define $f(x) = x$ and

$$(35) \quad g(x) = x + x^2 e^{i/x^2}.$$

Since $|e^{it}| = 1$ for all real t , we see that

$$(36) \quad \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = 1.$$

Next,

$$(37) \quad g'(x) = 1 + \left\{ 2x - \frac{2i}{x} \right\} e^{i/x^2} \quad (0 < x < 1),$$

so that

$$(38) \quad |g'(x)| \geq \left| 2x - \frac{2i}{x} \right| - 1 \geq \frac{2}{x} - 1.$$

Hence

$$(39) \quad \left| \frac{f'(x)}{g'(x)} \right| = \frac{1}{|g'(x)|} \leq \frac{x}{2-x}$$

and so

$$(40) \quad \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} = 0.$$

By (36) and (40), L'Hospital's rule fails in this case. Note also that $g'(x) \neq 0$ on $(0, 1)$, by (38).

However, there is a consequence of the mean value theorem which, for purposes of applications, is almost as useful as Theorem 5.10, and which remains true for vector-valued functions: From Theorem 5.10 it follows that

$$(41) \quad |f(b) - f(a)| \leq (b - a) \sup_{a < x < b} |f'(x)|.$$

5.19 Theorem Suppose \mathbf{f} is a continuous mapping of $[a, b]$ into R^k and \mathbf{f} is differentiable in (a, b) . Then there exists $x \in (a, b)$ such that

$$|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b - a) |\mathbf{f}'(x)|.$$

Proof¹ Put $\mathbf{z} = \mathbf{f}(b) - \mathbf{f}(a)$, and define

$$\varphi(t) = \mathbf{z} \cdot \mathbf{f}(t) \quad (a \leq t \leq b).$$

Then φ is a real-valued continuous function on $[a, b]$ which is differentiable in (a, b) . The mean value theorem shows therefore that

$$\varphi(b) - \varphi(a) = (b - a)\varphi'(x) = (b - a)\mathbf{z} \cdot \mathbf{f}'(x)$$

for some $x \in (a, b)$. On the other hand,

$$\varphi(b) - \varphi(a) = \mathbf{z} \cdot \mathbf{f}(b) - \mathbf{z} \cdot \mathbf{f}(a) = \mathbf{z} \cdot \mathbf{z} = |\mathbf{z}|^2.$$

The Schwarz inequality now gives

$$|\mathbf{z}|^2 = (b - a) |\mathbf{z} \cdot \mathbf{f}'(x)| \leq (b - a) |\mathbf{z}| |\mathbf{f}'(x)|.$$

Hence $|\mathbf{z}| \leq (b - a) |\mathbf{f}'(x)|$, which is the desired conclusion.

¹ V. P. Havin translated the second edition of this book into Russian and added this proof to the original one.

EXERCISES

1. Let f be defined for all real x , and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y . Prove that f is constant.

2. Suppose $f'(x) > 0$ in (a, b) . Prove that f is strictly increasing in (a, b) , and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

3. Suppose g is a real function on R^1 , with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough. (A set of admissible values of ε can be determined which depends only on M .)
4. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where C_0, \dots, C_n are real constants, prove that the equation

$$C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$$

has at least one real root between 0 and 1.

5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \rightarrow 0$ as $x \rightarrow +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \rightarrow 0$ as $x \rightarrow +\infty$.

6. Suppose

- (a) f is continuous for $x \geq 0$,
- (b) $f'(x)$ exists for $x > 0$,
- (c) $f(0) = 0$,
- (d) f' is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that g is monotonically increasing.

7. Suppose $f'(x), g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \rightarrow x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

(This holds also for complex functions.)

8. Suppose f' is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying that f is *uniformly differentiable* on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?

9. Let f be a continuous real function on \mathbb{R}^1 , of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \rightarrow 3$ as $x \rightarrow 0$. Does it follow that $f'(0)$ exists?
10. Suppose f and g are complex differentiable functions on $(0, 1)$, $f(x) \rightarrow 0$, $g(x) \rightarrow 0$, $f'(x) \rightarrow A$, $g'(x) \rightarrow B$ as $x \rightarrow 0$, where A and B are complex numbers, $B \neq 0$. Prove that

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

Compare with Example 5.18. *Hint:*

$$\frac{f(x)}{g(x)} = \left\{ \frac{f(x)}{x} - A \right\} \cdot \frac{x}{g(x)} + A \cdot \frac{x}{g(x)}.$$

Apply Theorem 5.13 to the real and imaginary parts of $f(x)/x$ and $g(x)/x$.

11. Suppose f is defined in a neighborhood of x , and suppose $f''(x)$ exists. Show that

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Show by an example that the limit may exist even if $f''(x)$ does not.

Hint: Use Theorem 5.13.

12. If $f(x) = |x|^3$, compute $f'(x)$, $f''(x)$ for all real x , and show that $f^{(3)}(0)$ does not exist.
13. Suppose a and c are real numbers, $c > 0$, and f is defined on $[-1, 1]$ by

$$f(x) = \begin{cases} x^a \sin(|x|^{-c}) & (\text{if } x \neq 0), \\ 0 & (\text{if } x = 0). \end{cases}$$

Prove the following statements:

- (a) f is continuous if and only if $a > 0$.
- (b) $f'(0)$ exists if and only if $a > 1$.
- (c) f' is bounded if and only if $a \geq 1 + c$.
- (d) f' is continuous if and only if $a > 1 + c$.
- (e) $f''(0)$ exists if and only if $a > 2 + c$.
- (f) f'' is bounded if and only if $a \geq 2 + 2c$.
- (g) f'' is continuous if and only if $a > 2 + 2c$.
14. Let f be a differentiable real function defined in (a, b) . Prove that f is convex if and only if f' is monotonically increasing. Assume next that $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.
15. Suppose $a \in \mathbb{R}^1$, f is a twice-differentiable real function on (a, ∞) , and M_0 , M_1 , M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, on (a, ∞) . Prove that

$$M_1^2 \leq 4M_0M_2.$$

Hint: If $h > 0$, Taylor's theorem shows that

$$f'(x) = \frac{1}{2h} [f(x+2h) - f(x)] - hf''(\xi)$$

for some $\xi \in (x, x+2h)$. Hence

$$|f'(x)| \leq hM_2 + \frac{M_0}{h}.$$

To show that $M_1^2 = 4M_0M_2$ can actually happen, take $a = -1$, define

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \frac{x^2 - 1}{x^2 + 1} & (0 \leq x < \infty), \end{cases}$$

and show that $M_0 = 1$, $M_1 = 4$, $M_2 = 4$.

Does $M_1^2 \leq 4M_0M_2$ hold for vector-valued functions too?

16. Suppose f is twice-differentiable on $(0, \infty)$, f'' is bounded on $(0, \infty)$, and $f(x) \rightarrow 0$ as $x \rightarrow \infty$. Prove that $f'(x) \rightarrow 0$ as $x \rightarrow \infty$.

Hint: Let $a \rightarrow \infty$ in Exercise 15.

17. Suppose f is a real, three times differentiable function on $[-1, 1]$, such that

$$f(-1) = 0, \quad f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0.$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$.

Note that equality holds for $\frac{1}{2}(x^3 + x^2)$.

Hint: Use Theorem 5.15, with $\alpha = 0$ and $\beta = \pm 1$, to show that there exist $s \in (0, 1)$ and $t \in (-1, 0)$ such that

$$f^{(3)}(s) + f^{(3)}(t) = 6.$$

18. Suppose f is a real function on $[a, b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let α , β , and P be as in Taylor's theorem (5.15). Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a, b]$, $t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

$n - 1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!} (\beta - \alpha)^n.$$

19. Suppose f is defined in $(-1, 1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \rightarrow 0$, and $\beta_n \rightarrow 0$ as $n \rightarrow \infty$. Define the difference quotients

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}.$$

Prove the following statements:

- (a) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$.
- (b) If $0 < \alpha_n < \beta_n$ and $\{\beta_n/(\beta_n - \alpha_n)\}$ is bounded, then $\lim D_n = f'(0)$.
- (c) If f' is continuous in $(-1, 1)$, then $\lim D_n = f'(0)$.

Give an example in which f is differentiable in $(-1, 1)$ (but f' is not continuous at 0) and in which α_n, β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.

- 20. Formulate and prove an inequality which follows from Taylor's theorem and which remains valid for vector-valued functions.
- 21. Let E be a closed subset of R^1 . We saw in Exercise 22, Chap. 4, that there is a real continuous function f on R^1 whose zero set is E . Is it possible, for each closed set E , to find such an f which is differentiable on R^1 , or one which is n times differentiable, or even one which has derivatives of all orders on R^1 ?
- 22. Suppose f is a real function on $(-\infty, \infty)$. Call x a *fixed point* of f if $f(x) = x$.
 - (a) If f is differentiable and $f'(t) \neq 1$ for every real t , prove that f has at most one fixed point.
 - (b) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t .

- (c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t , prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \dots$

- (d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \rightarrow (x_2, x_2) \rightarrow (x_2, x_3) \rightarrow (x_3, x_3) \rightarrow (x_3, x_4) \rightarrow \dots$$

- 23. The function f defined by

$$f(x) = \frac{x^3 + 1}{3}$$

has three fixed points, say α, β, γ , where

$$-2 < \alpha < -1, \quad 0 < \beta < 1, \quad 1 < \gamma < 2.$$

For arbitrarily chosen x_1 , define $\{x_n\}$ by setting $x_{n+1} = f(x_n)$.

- (a) If $x_1 < \alpha$, prove that $x_n \rightarrow -\infty$ as $n \rightarrow \infty$.
- (b) If $\alpha < x_1 < \gamma$, prove that $x_n \rightarrow \beta$ as $n \rightarrow \infty$.
- (c) If $\gamma < x_1$, prove that $x_n \rightarrow +\infty$ as $n \rightarrow \infty$.

Thus β can be located by this method, but α and γ cannot.

24. The process described in part (c) of Exercise 22 can of course also be applied to functions that map $(0, \infty)$ to $(0, \infty)$.

Fix some $\alpha > 1$, and put

$$f(x) = \frac{1}{2} \left(x + \frac{\alpha}{x} \right), \quad g(x) = \frac{\alpha + x}{1 + x}.$$

Both f and g have $\sqrt{\alpha}$ as their only fixed point in $(0, \infty)$. Try to explain, on the basis of properties of f and g , why the convergence in Exercise 16, Chap. 3, is so much more rapid than it is in Exercise 17. (Compare f' and g' , draw the zig-zags suggested in Exercise 22.)

Do the same when $0 < \alpha < 1$.

25. Suppose f is twice differentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$.

Complete the details in the following outline of *Newton's method* for computing ξ .

- (a) Choose $x_1 \in (\xi, b)$, and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Interpret this geometrically, in terms of a tangent to the graph of f .

- (b) Prove that $x_{n+1} < x_n$ and that

$$\lim_{n \rightarrow \infty} x_n = \xi.$$

- (c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)} (x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

- (d) If $A = M/2\delta$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A} [A(x_1 - \xi)]^{2^n}.$$

(Compare with Exercises 16 and 18, Chap. 3.)

- (e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

How does $g'(x)$ behave for x near ξ ?

- (f) Put $f(x) = x^{1/3}$ on $(-\infty, \infty)$ and try Newton's method. What happens?

26. Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$. *Hint:* Fix $x_0 \in [a, b]$, let

$$M_0 = \sup |f(x)|, \quad M_1 = \sup |f'(x)|$$

for $a \leq x \leq x_0$. For any such x ,

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_0.$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$. Proceed.

27. Let ϕ be a real function defined on a rectangle R in the plane, given by $a \leq x \leq b$, $\alpha \leq y \leq \beta$. A *solution* of the initial-value problem

$$y' = \phi(x, y), \quad y(a) = c \quad (\alpha \leq c \leq \beta)$$

is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c$, $\alpha \leq f(x) \leq \beta$, and

$$f'(x) = \phi(x, f(x)) \quad (a \leq x \leq b).$$

Prove that such a problem has at most one solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

whenever $(x, y_1) \in R$ and $(x, y_2) \in R$.

Hint: Apply Exercise 26 to the difference of two solutions. Note that this uniqueness theorem does not hold for the initial-value problem

$$y' = y^{1/2}, \quad y(0) = 0,$$

which has two solutions: $f(x) = 0$ and $f(x) = x^2/4$. Find all other solutions.

28. Formulate and prove an analogous uniqueness theorem for systems of differential equations of the form

$$y'_j = \phi_j(x, y_1, \dots, y_k), \quad y_j(a) = c_j \quad (j = 1, \dots, k).$$

Note that this can be rewritten in the form

$$\mathbf{y}' = \Phi(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c}$$

where $\mathbf{y} = (y_1, \dots, y_k)$ ranges over a k -cell, Φ is the mapping of a $(k+1)$ -cell into the Euclidean k -space whose components are the functions ϕ_1, \dots, ϕ_k , and \mathbf{c} is the vector (c_1, \dots, c_k) . Use Exercise 26, for vector-valued functions.

29. Specialize Exercise 28 by considering the system

$$y'_j = y_{j+1} \quad (j = 1, \dots, k-1),$$

$$y'_k = f(x) - \sum_{j=1}^k g_j(x)y_j,$$

where f, g_1, \dots, g_k are continuous real functions on $[a, b]$, and derive a uniqueness theorem for solutions of the equation

$$y^{(k)} + g_k(x)y^{(k-1)} + \dots + g_2(x)y' + g_1(x)y = f(x),$$

subject to initial conditions

$$y(a) = c_1, \quad y'(a) = c_2, \quad \dots, \quad y^{(k-1)}(a) = c_k.$$

6

THE RIEMANN-STIELTJES INTEGRAL

The present chapter is based on a definition of the Riemann integral which depends very explicitly on the order structure of the real line. Accordingly, we begin by discussing integration of real-valued functions on intervals. Extensions to complex- and vector-valued functions on intervals follow in later sections. Integration over sets other than intervals is discussed in Chaps. 10 and 11.

DEFINITION AND EXISTENCE OF THE INTEGRAL

6.1 Definition Let $[a, b]$ be a given interval. By a *partition* P of $[a, b]$ we mean a finite set of points x_0, x_1, \dots, x_n , where

$$a = x_0 \leq x_1 \leq \dots \leq x_{n-1} \leq x_n = b.$$

We write

$$\Delta x_i = x_i - x_{i-1} \quad (i = 1, \dots, n).$$

Now suppose f is a bounded real function defined on $[a, b]$. Corresponding to each partition P of $[a, b]$ we put

$$\begin{aligned} M_i &= \sup f(x) & (x_{i-1} \leq x \leq x_i), \\ m_i &= \inf f(x) & (x_{i-1} \leq x \leq x_i), \\ U(P, f) &= \sum_{i=1}^n M_i \Delta x_i, \\ L(P, f) &= \sum_{i=1}^n m_i \Delta x_i, \end{aligned}$$

and finally

$$(1) \quad \int_a^b f \, dx = \inf U(P, f),$$

$$(2) \quad \int_a^b f \, dx = \sup L(P, f),$$

where the inf and the sup are taken over all partitions P of $[a, b]$. The left members of (1) and (2) are called the *upper* and *lower Riemann integrals* of f over $[a, b]$, respectively.

If the upper and lower integrals are equal, we say that f is *Riemann-integrable* on $[a, b]$, we write $f \in \mathcal{R}$ (that is, \mathcal{R} denotes the set of Riemann-integrable functions), and we denote the common value of (1) and (2) by

$$(3) \quad \int_a^b f \, dx,$$

or by

$$(4) \quad \int_a^b f(x) \, dx.$$

This is the *Riemann integral* of f over $[a, b]$. Since f is bounded, there exist two numbers, m and M , such that

$$m \leq f(x) \leq M \quad (a \leq x \leq b).$$

Hence, for every P ,

$$m(b-a) \leq L(P, f) \leq U(P, f) \leq M(b-a),$$

so that the numbers $L(P, f)$ and $U(P, f)$ form a bounded set. This shows that *the upper and lower integrals are defined for every bounded function f* . The question of their equality, and hence the question of the integrability of f , is a more delicate one. Instead of investigating it separately for the Riemann integral, we shall immediately consider a more general situation.

6.2 Definition Let α be a monotonically increasing function on $[a, b]$ (since $\alpha(a)$ and $\alpha(b)$ are finite, it follows that α is bounded on $[a, b]$). Corresponding to each partition P of $[a, b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}).$$

It is clear that $\Delta\alpha_i \geq 0$. For any real function f which is bounded on $[a, b]$ we put

$$U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i,$$

$$L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta\alpha_i,$$

where M_i, m_i have the same meaning as in Definition 6.1, and we define

$$(5) \quad \int_a^b f d\alpha = \inf U(P, f, \alpha),$$

$$(6) \quad \int_a^b f d\alpha = \sup L(P, f, \alpha),$$

the inf and sup again being taken over all partitions.

If the left members of (5) and (6) are equal, we denote their common value by

$$(7) \quad \int_a^b f d\alpha$$

or sometimes by

$$(8) \quad \int_a^b f(x) d\alpha(x).$$

This is the *Riemann-Stieltjes integral* (or simply the *Stieltjes integral*) of f with respect to α , over $[a, b]$.

If (7) exists, i.e., if (5) and (6) are equal, we say that f is integrable with respect to α , in the Riemann sense, and write $f \in \mathcal{R}(\alpha)$.

By taking $\alpha(x) = x$, the Riemann integral is seen to be a special case of the Riemann-Stieltjes integral. Let us mention explicitly, however, that in the general case α need not even be continuous.

A few words should be said about the notation. We prefer (7) to (8), since the letter x which appears in (8) adds nothing to the content of (7). It is immaterial which letter we use to represent the so-called “variable of integration.” For instance, (8) is the same as

$$\int_a^b f(y) d\alpha(y).$$

The integral depends on f , α , a and b , but not on the variable of integration, which may as well be omitted.

The role played by the variable of integration is quite analogous to that of the index of summation: The two symbols

$$\sum_{i=1}^n c_i, \quad \sum_{k=1}^n c_k$$

mean the same thing, since each means $c_1 + c_2 + \cdots + c_n$.

Of course, no harm is done by inserting the variable of integration, and in many cases it is actually convenient to do so.

We shall now investigate the existence of the integral (7). Without saying so every time, f will be assumed real and bounded, and α monotonically increasing on $[a, b]$; and, when there can be no misunderstanding, we shall write \int in place of \int_a^b .

6.3 Definition We say that the partition P^* is a *refinement* of P if $P^* \supset P$ (that is, if every point of P is a point of P^*). Given two partitions, P_1 and P_2 , we say that P^* is their *common refinement* if $P^* = P_1 \cup P_2$.

6.4 Theorem If P^* is a refinement of P , then

$$(9) \quad L(P, f, \alpha) \leq L(P^*, f, \alpha)$$

and

$$(10) \quad U(P^*, f, \alpha) \leq U(P, f, \alpha).$$

Proof To prove (9), suppose first that P^* contains just one point more than P . Let this extra point be x^* , and suppose $x_{i-1} < x^* < x_i$, where x_{i-1} and x_i are two consecutive points of P . Put

$$\begin{aligned} w_1 &= \inf f(x) & (x_{i-1} \leq x \leq x^*), \\ w_2 &= \inf f(x) & (x^* \leq x \leq x_i). \end{aligned}$$

Clearly $w_1 \geq m_i$ and $w_2 \geq m_i$, where, as before,

$$m_i = \inf f(x) \quad (x_{i-1} \leq x \leq x_i).$$

Hence

$$\begin{aligned} L(P^*, f, \alpha) - L(P, f, \alpha) &= w_1[\alpha(x^*) - \alpha(x_{i-1})] + w_2[\alpha(x_i) - \alpha(x^*)] - m_i[\alpha(x_i) - \alpha(x_{i-1})] \\ &= (w_1 - m_i)[\alpha(x^*) - \alpha(x_{i-1})] + (w_2 - m_i)[\alpha(x_i) - \alpha(x^*)] \geq 0. \end{aligned}$$

If P^* contains k points more than P , we repeat this reasoning k times, and arrive at (9). The proof of (10) is analogous.

6.5 Theorem $\int_a^b f d\alpha \leq \bar{\int}_a^b f d\alpha.$

Proof Let P^* be the common refinement of two partitions P_1 and P_2 . By Theorem 6.4,

$$L(P_1, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P_2, f, \alpha).$$

Hence

$$(11) \quad L(P_1, f, \alpha) \leq U(P_2, f, \alpha).$$

If P_2 is fixed and the sup is taken over all P_1 , (11) gives

$$(12) \quad \int f d\alpha \leq U(P_2, f, \alpha).$$

The theorem follows by taking the inf over all P_2 in (12).

6.6 Theorem $f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that

$$(13) \quad U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon.$$

Proof For every P we have

$$L(P, f, \alpha) \leq \int f d\alpha \leq \bar{\int} f d\alpha \leq U(P, f, \alpha).$$

Thus (13) implies

$$0 \leq \bar{\int} f d\alpha - \int f d\alpha < \varepsilon.$$

Hence, if (13) can be satisfied for every $\varepsilon > 0$, we have

$$\bar{\int} f d\alpha = \int f d\alpha,$$

that is, $f \in \mathcal{R}(\alpha)$.

Conversely, suppose $f \in \mathcal{R}(\alpha)$, and let $\varepsilon > 0$ be given. Then there exist partitions P_1 and P_2 such that

$$(14) \quad U(P_2, f, \alpha) - \int f d\alpha < \frac{\varepsilon}{2},$$

$$(15) \quad \int f d\alpha - L(P_1, f, \alpha) < \frac{\varepsilon}{2}.$$

We choose P to be the common refinement of P_1 and P_2 . Then Theorem 6.4, together with (14) and (15), shows that

$$U(P, f, \alpha) \leq U(P_2, f, \alpha) < \int f d\alpha + \frac{\varepsilon}{2} < L(P_1, f, \alpha) + \varepsilon \leq L(P, f, \alpha) + \varepsilon,$$

so that (13) holds for this partition P .

Theorem 6.6 furnishes a convenient criterion for integrability. Before we apply it, we state some closely related facts.

6.7 Theorem

- (a) If (13) holds for some P and some ε , then (13) holds (with the same ε) for every refinement of P .
 (b) If (13) holds for $P = \{x_0, \dots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then

$$\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta\alpha_i < \varepsilon.$$

- (c) If $f \in \mathcal{R}(\alpha)$ and the hypotheses of (b) hold, then

$$\left| \sum_{i=1}^n f(t_i) \Delta\alpha_i - \int_a^b f d\alpha \right| < \varepsilon.$$

Proof Theorem 6.4 implies (a). Under the assumptions made in (b), both $f(s_i)$ and $f(t_i)$ lie in $[m_i, M_i]$, so that $|f(s_i) - f(t_i)| \leq M_i - m_i$. Thus

$$\sum_{i=1}^n |f(s_i) - f(t_i)| \Delta\alpha_i \leq U(P, f, \alpha) - L(P, f, \alpha),$$

which proves (b). The obvious inequalities

$$L(P, f, \alpha) \leq \sum f(t_i) \Delta\alpha_i \leq U(P, f, \alpha)$$

and

$$L(P, f, \alpha) \leq \int f d\alpha \leq U(P, f, \alpha)$$

prove (c).

6.8 Theorem If f is continuous on $[a, b]$ then $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

Proof Let $\varepsilon > 0$ be given. Choose $\eta > 0$ so that

$$[\alpha(b) - \alpha(a)]\eta < \varepsilon.$$

Since f is uniformly continuous on $[a, b]$ (Theorem 4.19), there exists a $\delta > 0$ such that

$$(16) \quad |f(x) - f(t)| < \eta$$

if $x \in [a, b]$, $t \in [a, b]$, and $|x - t| < \delta$.

If P is any partition of $[a, b]$ such that $\Delta x_i < \delta$ for all i , then (16) implies that

$$(17) \quad M_i - m_i \leq \eta \quad (i = 1, \dots, n)$$

and therefore

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i \\ &\leq \eta \sum_{i=1}^n \Delta \alpha_i = \eta [\alpha(b) - \alpha(a)] < \varepsilon. \end{aligned}$$

By Theorem 6.6, $f \in \mathcal{R}(\alpha)$.

6.9 Theorem *If f is monotonic on $[a, b]$, and if α is continuous on $[a, b]$, then $f \in \mathcal{R}(\alpha)$. (We still assume, of course, that α is monotonic.)*

Proof Let $\varepsilon > 0$ be given. For any positive integer n , choose a partition such that

$$\Delta \alpha_i = \frac{\alpha(b) - \alpha(a)}{n} \quad (i = 1, \dots, n).$$

This is possible since α is continuous (Theorem 4.23).

We suppose that f is monotonically increasing (the proof is analogous in the other case). Then

$$M_i = f(x_i), \quad m_i = f(x_{i-1}) \quad (i = 1, \dots, n),$$

so that

$$\begin{aligned} U(P, f, \alpha) - L(P, f, \alpha) &= \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^n [f(x_i) - f(x_{i-1})] \\ &= \frac{\alpha(b) - \alpha(a)}{n} \cdot [f(b) - f(a)] < \varepsilon \end{aligned}$$

if n is taken large enough. By Theorem 6.6, $f \in \mathcal{R}(\alpha)$.

6.10 Theorem *Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathcal{R}(\alpha)$.*

Proof Let $\varepsilon > 0$ be given. Put $M = \sup |f(x)|$, let E be the set of points at which f is discontinuous. Since E is finite and α is continuous at every point of E , we can cover E by finitely many disjoint intervals $[u_j, v_j] \subset [a, b]$ such that the sum of the corresponding differences $\alpha(v_j) - \alpha(u_j)$ is less than ε . Furthermore, we can place these intervals in such a way that every point of $E \cap (a, b)$ lies in the interior of some $[u_j, v_j]$.

Remove the segments (u_j, v_j) from $[a, b]$. The remaining set K is compact. Hence f is uniformly continuous on K , and there exists $\delta > 0$ such that $|f(s) - f(t)| < \varepsilon$ if $s \in K, t \in K, |s - t| < \delta$.

Now form a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$, as follows: Each u_j occurs in P . Each v_j occurs in P . No point of any segment (u_j, v_j) occurs in P . If x_{i-1} is not one of the u_j , then $\Delta x_i < \delta$.

Note that $M_i - m_i \leq 2M$ for every i , and that $M_i - m_i \leq \varepsilon$ unless x_{i-1} is one of the u_j . Hence, as in the proof of Theorem 6.8,

$$U(P, f, \alpha) - L(P, f, \alpha) \leq [\alpha(b) - \alpha(a)]\varepsilon + 2M\varepsilon.$$

Since ε is arbitrary, Theorem 6.6 shows that $f \in \mathcal{R}(\alpha)$.

Note: If f and α have a common point of discontinuity, then f need not be in $\mathcal{R}(\alpha)$. Exercise 3 shows this.

6.11 Theorem Suppose $f \in \mathcal{R}(\alpha)$ on $[a, b]$, $m \leq f \leq M$, ϕ is continuous on $[m, M]$, and $h(x) = \phi(f(x))$ on $[a, b]$. Then $h \in \mathcal{R}(\alpha)$ on $[a, b]$.

Proof Choose $\varepsilon > 0$. Since ϕ is uniformly continuous on $[m, M]$, there exists $\delta > 0$ such that $\delta < \varepsilon$ and $|\phi(s) - \phi(t)| < \varepsilon$ if $|s - t| \leq \delta$ and $s, t \in [m, M]$.

Since $f \in \mathcal{R}(\alpha)$, there is a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ such that

$$(18) \quad U(P, f, \alpha) - L(P, f, \alpha) < \delta^2.$$

Let M_i, m_i have the same meaning as in Definition 6.1, and let M_i^*, m_i^* be the analogous numbers for h . Divide the numbers $1, \dots, n$ into two classes: $i \in A$ if $M_i - m_i < \delta$, $i \in B$ if $M_i - m_i \geq \delta$.

For $i \in A$, our choice of δ shows that $M_i^* - m_i^* \leq \varepsilon$.

For $i \in B$, $M_i^* - m_i^* \leq 2K$, where $K = \sup|\phi(t)|$, $m \leq t \leq M$. By (18), we have

$$(19) \quad \delta \sum_{i \in B} \Delta \alpha_i \leq \sum_{i \in B} (M_i - m_i) \Delta \alpha_i < \delta^2$$

so that $\sum_{i \in B} \Delta \alpha_i < \delta$. It follows that

$$\begin{aligned} U(P, h, \alpha) - L(P, h, \alpha) &= \sum_{i \in A} (M_i^* - m_i^*) \Delta \alpha_i + \sum_{i \in B} (M_i^* - m_i^*) \Delta \alpha_i \\ &\leq \varepsilon[\alpha(b) - \alpha(a)] + 2K\delta < \varepsilon[\alpha(b) - \alpha(a) + 2K]. \end{aligned}$$

Since ε was arbitrary, Theorem 6.6 implies that $h \in \mathcal{R}(\alpha)$.

Remark: This theorem suggests the question: Just what functions are Riemann-integrable? The answer is given by Theorem 11.33(b).

PROPERTIES OF THE INTEGRAL

6.12 Theorem

(a) If $f_1 \in \mathcal{R}(\alpha)$ and $f_2 \in \mathcal{R}(\alpha)$ on $[a, b]$, then

$$f_1 + f_2 \in \mathcal{R}(\alpha),$$

$cf \in \mathcal{R}(\alpha)$ for every constant c , and

$$\int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha,$$

$$\int_a^b cf d\alpha = c \int_a^b f d\alpha.$$

(b) If $f_1(x) \leq f_2(x)$ on $[a, b]$, then

$$\int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha.$$

(c) If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and if $a < c < b$, then $f \in \mathcal{R}(\alpha)$ on $[a, c]$ and on $[c, b]$, and

$$\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha.$$

(d) If $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and if $|f(x)| \leq M$ on $[a, b]$, then

$$\left| \int_a^b f d\alpha \right| \leq M[\alpha(b) - \alpha(a)].$$

(e) If $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$, then $f \in \mathcal{R}(\alpha_1 + \alpha_2)$ and

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2;$$

if $f \in \mathcal{R}(\alpha)$ and c is a positive constant, then $f \in \mathcal{R}(c\alpha)$ and

$$\int_a^b f d(c\alpha) = c \int_a^b f d\alpha.$$

Proof If $f = f_1 + f_2$ and P is any partition of $[a, b]$, we have

$$(20) \quad \begin{aligned} L(P, f_1, \alpha) + L(P, f_2, \alpha) &\leq L(P, f, \alpha) \\ &\leq U(P, f, \alpha) \leq U(P, f_1, \alpha) + U(P, f_2, \alpha). \end{aligned}$$

If $f_1 \in \mathcal{R}(\alpha)$ and $f_2 \in \mathcal{R}(\alpha)$, let $\varepsilon > 0$ be given. There are partitions P_j ($j = 1, 2$) such that

$$U(P_j, f_j, \alpha) - L(P_j, f_j, \alpha) < \varepsilon.$$

These inequalities persist if P_1 and P_2 are replaced by their common refinement P . Then (20) implies

$$U(P, f, \alpha) - L(P, f, \alpha) < 2\varepsilon,$$

which proves that $f \in \mathcal{R}(\alpha)$.

With this same P we have

$$U(P, f_j, \alpha) < \int f_j d\alpha + \varepsilon \quad (j = 1, 2);$$

hence (20) implies

$$\int f d\alpha \leq U(P, f, \alpha) < \int f_1 d\alpha + \int f_2 d\alpha + 2\varepsilon.$$

Since ε was arbitrary, we conclude that

$$(21) \quad \int f d\alpha \leq \int f_1 d\alpha + \int f_2 d\alpha.$$

If we replace f_1 and f_2 in (21) by $-f_1$ and $-f_2$, the inequality is reversed, and the equality is proved.

The proofs of the other assertions of Theorem 6.12 are so similar that we omit the details. In part (c) the point is that (by passing to refinements) we may restrict ourselves to partitions which contain the point c , in approximating $\int f d\alpha$.

6.13 Theorem *If $f \in \mathcal{R}(\alpha)$ and $g \in \mathcal{R}(\alpha)$ on $[a, b]$, then*

(a) $fg \in \mathcal{R}(\alpha)$;

(b) $|f| \in \mathcal{R}(\alpha)$ and $\left| \int_a^b f d\alpha \right| \leq \int_a^b |f| d\alpha$.

Proof If we take $\phi(t) = t^2$, Theorem 6.11 shows that $f^2 \in \mathcal{R}(\alpha)$ if $f \in \mathcal{R}(\alpha)$. The identity

$$4fg = (f + g)^2 - (f - g)^2$$

completes the proof of (a).

If we take $\phi(t) = |t|$, Theorem 6.11 shows similarly that $|f| \in \mathcal{R}(\alpha)$. Choose $c = \pm 1$, so that

$$c \int f d\alpha \geq 0.$$

Then

$$| \int f d\alpha | = c \int f d\alpha = \int cf d\alpha \leq \int |f| d\alpha,$$

since $cf \leq |f|$.

6.14 Definition The unit step function I is defined by

$$I(x) = \begin{cases} 0 & (x \leq 0), \\ 1 & (x > 0). \end{cases}$$

6.15 Theorem *If $a < s < b$, f is bounded on $[a, b]$, f is continuous at s , and $\alpha(x) = I(x - s)$, then*

$$\int_a^b f d\alpha = f(s).$$

Proof Consider partitions $P = \{x_0, x_1, x_2, x_3\}$, where $x_0 = a$, and $x_1 = s < x_2 < x_3 = b$. Then

$$U(P, f, \alpha) = M_2, \quad L(P, f, \alpha) = m_2.$$

Since f is continuous at s , we see that M_2 and m_2 converge to $f(s)$ as $x_2 \rightarrow s$.

6.16 Theorem *Suppose $c_n \geq 0$ for $1, 2, 3, \dots$, $\sum c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a, b) , and*

$$(22) \quad \alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n).$$

Let f be continuous on $[a, b]$. Then

$$(23) \quad \int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n).$$

Proof The comparison test shows that the series (22) converges for every x . Its sum $\alpha(x)$ is evidently monotonic, and $\alpha(a) = 0$, $\alpha(b) = \sum c_n$. (This is the type of function that occurred in Remark 4.31.)

Let $\varepsilon > 0$ be given, and choose N so that

$$\sum_{n=N+1}^{\infty} c_n < \varepsilon.$$

Put

$$\alpha_1(x) = \sum_{n=1}^N c_n I(x - s_n), \quad \alpha_2(x) = \sum_{n=N+1}^{\infty} c_n I(x - s_n).$$

By Theorems 6.12 and 6.15,

$$(24) \quad \int_a^b f d\alpha_1 = \sum_{i=1}^N c_i f(s_i).$$

Since $\alpha_2(b) - \alpha_2(a) < \varepsilon$,

$$(25) \quad \left| \int_a^b f d\alpha_2 \right| \leq M\varepsilon,$$

where $M = \sup|f(x)|$. Since $\alpha = \alpha_1 + \alpha_2$, it follows from (24) and (25) that

$$(26) \quad \left| \int_a^b f d\alpha - \sum_{i=1}^N c_n f(s_n) \right| \leq M\varepsilon.$$

If we let $N \rightarrow \infty$, we obtain (23).

6.17 Theorem Assume α increases monotonically and $\alpha' \in \mathcal{R}$ on $[a, b]$. Let f be a bounded real function on $[a, b]$.

Then $f \in \mathcal{R}(\alpha)$ if and only if $f\alpha' \in \mathcal{R}$. In that case

$$(27) \quad \int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) dx.$$

Proof Let $\varepsilon > 0$ be given and apply Theorem 6.6 to α' : There is a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ such that

$$(28) \quad U(P, \alpha') - L(P, \alpha') < \varepsilon.$$

The mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$ such that

$$\Delta\alpha_i = \alpha'(t_i) \Delta x_i$$

for $i = 1, \dots, n$. If $s_i \in [x_{i-1}, x_i]$, then

$$(29) \quad \sum_{i=1}^n |\alpha'(s_i) - \alpha'(t_i)| \Delta x_i < \varepsilon,$$

by (28) and Theorem 6.7(b). Put $M = \sup|f(x)|$. Since

$$\sum_{i=1}^n f(s_i) \Delta\alpha_i = \sum_{i=1}^n f(s_i)\alpha'(t_i) \Delta x_i$$

it follows from (29) that

$$(30) \quad \left| \sum_{i=1}^n f(s_i) \Delta\alpha_i - \sum_{i=1}^n f(s_i)\alpha'(s_i) \Delta x_i \right| \leq M\varepsilon.$$

In particular,

$$\sum_{i=1}^n f(s_i) \Delta\alpha_i \leq U(P, f\alpha') + M\varepsilon,$$

for all choices of $s_i \in [x_{i-1}, x_i]$, so that

$$U(P, f, \alpha) \leq U(P, f\alpha') + M\varepsilon.$$

The same argument leads from (30) to

$$U(P, f\alpha') \leq U(P, f, \alpha) + M\varepsilon.$$

Thus

$$(31) \quad |U(P, f, \alpha) - U(P, f\alpha')| \leq M\varepsilon.$$

Now note that (28) remains true if P is replaced by any refinement. Hence (31) also remains true. We conclude that

$$\left| \int_a^b f d\alpha - \int_a^b f(x)\alpha'(x) dx \right| \leq M\varepsilon.$$

But ε is arbitrary. Hence

$$(32) \quad \int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) dx,$$

for any bounded f . The equality of the lower integrals follows from (30) in exactly the same way. The theorem follows.

6.18 Remark The two preceding theorems illustrate the generality and flexibility which are inherent in the Stieltjes process of integration. If α is a pure step function [this is the name often given to functions of the form (22)], the integral reduces to a finite or infinite series. If α has an integrable derivative, the integral reduces to an ordinary Riemann integral. This makes it possible in many cases to study series and integrals simultaneously, rather than separately.

To illustrate this point, consider a physical example. The moment of inertia of a straight wire of unit length, about an axis through an endpoint, at right angles to the wire, is

$$(33) \quad \int_0^1 x^2 dm$$

where $m(x)$ is the mass contained in the interval $[0, x]$. If the wire is regarded as having a continuous density ρ , that is, if $m'(x) = \rho(x)$, then (33) turns into

$$(34) \quad \int_0^1 x^2 \rho(x) dx.$$

On the other hand, if the wire is composed of masses m_i concentrated at points x_i , (33) becomes

$$(35) \quad \sum_i x_i^2 m_i.$$

Thus (33) contains (34) and (35) as special cases, but it contains much more; for instance, the case in which m is continuous but not everywhere differentiable.

6.19 Theorem (change of variable) Suppose φ is a strictly increasing continuous function that maps an interval $[A, B]$ onto $[a, b]$. Suppose α is monotonically increasing on $[a, b]$ and $f \in \mathcal{R}(\alpha)$ on $[a, b]$. Define β and g on $[A, B]$ by

$$(36) \quad \beta(y) = \alpha(\varphi(y)), \quad g(y) = f(\varphi(y)).$$

Then $g \in \mathcal{R}(\beta)$ and

$$(37) \quad \int_A^B g \, d\beta = \int_a^b f \, d\alpha.$$

Proof To each partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ corresponds a partition $Q = \{y_0, \dots, y_n\}$ of $[A, B]$, so that $x_i = \varphi(y_i)$. All partitions of $[A, B]$ are obtained in this way. Since the values taken by f on $[x_{i-1}, x_i]$ are exactly the same as those taken by g on $[y_{i-1}, y_i]$, we see that

$$(38) \quad U(Q, g, \beta) = U(P, f, \alpha), \quad L(Q, g, \beta) = L(P, f, \alpha).$$

Since $f \in \mathcal{R}(\alpha)$, P can be chosen so that both $U(P, f, \alpha)$ and $L(P, f, \alpha)$ are close to $\int f \, d\alpha$. Hence (38), combined with Theorem 6.6, shows that $g \in \mathcal{R}(\beta)$ and that (37) holds. This completes the proof.

Let us note the following special case:

Take $\alpha(x) = x$. Then $\beta = \varphi$. Assume $\varphi' \in \mathcal{R}$ on $[A, B]$. If Theorem 6.17 is applied to the left side of (37), we obtain

$$(39) \quad \int_a^b f(x) \, dx = \int_A^B f(\varphi(y))\varphi'(y) \, dy.$$

INTEGRATION AND DIFFERENTIATION

We still confine ourselves to real functions in this section. We shall show that integration and differentiation are, in a certain sense, inverse operations.

6.20 Theorem Let $f \in \mathcal{R}$ on $[a, b]$. For $a \leq x \leq b$, put

$$F(x) = \int_a^x f(t) \, dt.$$

Then F is continuous on $[a, b]$; furthermore, if f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0 , and

$$F'(x_0) = f(x_0).$$

Proof Since $f \in \mathcal{R}$, f is bounded. Suppose $|f(t)| \leq M$ for $a \leq t \leq b$. If $a \leq x < y \leq b$, then

$$|F(y) - F(x)| = \left| \int_x^y f(t) \, dt \right| \leq M(y - x),$$

by Theorem 6.12(c) and (d). Given $\varepsilon > 0$, we see that

$$|F(y) - F(x)| < \varepsilon,$$

provided that $|y - x| < \varepsilon/M$. This proves continuity (and, in fact, uniform continuity) of F .

Now suppose f is continuous at x_0 . Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$|f(t) - f(x_0)| < \varepsilon$$

if $|t - x_0| < \delta$, and $a \leq t \leq b$. Hence, if

$$x_0 - \delta < s \leq x_0 \leq t < x_0 + \delta \quad \text{and} \quad a \leq s < t \leq b,$$

we have, by Theorem 6.12(d),

$$\left| \frac{F(t) - F(s)}{t - s} - f(x_0) \right| = \left| \frac{1}{t - s} \int_s^t [f(u) - f(x_0)] du \right| < \varepsilon.$$

It follows that $F'(x_0) = f(x_0)$.

6.21 The fundamental theorem of calculus *If $f \in \mathcal{R}$ on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then*

$$\int_a^b f(x) dx = F(b) - F(a).$$

Proof Let $\varepsilon > 0$ be given. Choose a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ so that $U(P, f) - L(P, f) < \varepsilon$. The mean value theorem furnishes points $t_i \in [x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = f(t_i) \Delta x_i$$

for $i = 1, \dots, n$. Thus

$$\sum_{i=1}^n f(t_i) \Delta x_i = F(b) - F(a).$$

It now follows from Theorem 6.7(c) that

$$\left| F(b) - F(a) - \int_a^b f(x) dx \right| < \varepsilon.$$

Since this holds for every $\varepsilon > 0$, the proof is complete.

6.22 Theorem (integration by parts) *Suppose F and G are differentiable functions on $[a, b]$, $F' = f \in \mathcal{R}$, and $G' = g \in \mathcal{R}$. Then*

$$\int_a^b F(x)g(x) dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x) dx.$$

Proof Put $H(x) = F(x)G(x)$ and apply Theorem 6.21 to H and its derivative. Note that $H' \in \mathcal{R}$, by Theorem 6.13.

INTEGRATION OF VECTOR-VALUED FUNCTIONS

6.23 Definition Let f_1, \dots, f_k be real functions on $[a, b]$, and let $\mathbf{f} = (f_1, \dots, f_k)$ be the corresponding mapping of $[a, b]$ into R^k . If α increases monotonically on $[a, b]$, to say that $\mathbf{f} \in \mathcal{R}(\alpha)$ means that $f_j \in \mathcal{R}(\alpha)$ for $j = 1, \dots, k$. If this is the case, we define

$$\int_a^b \mathbf{f} \, d\alpha = \left(\int_a^b f_1 \, d\alpha, \dots, \int_a^b f_k \, d\alpha \right).$$

In other words, $\int \mathbf{f} \, d\alpha$ is the point in R^k whose j th coordinate is $\int f_j \, d\alpha$.

It is clear that parts (a), (c), and (e) of Theorem 6.12 are valid for these vector-valued integrals; we simply apply the earlier results to each coordinate. The same is true of Theorems 6.17, 6.20, and 6.21. To illustrate, we state the analogue of Theorem 6.21.

6.24 Theorem If \mathbf{f} and \mathbf{F} map $[a, b]$ into R^k , if $\mathbf{f} \in \mathcal{R}$ on $[a, b]$, and if $\mathbf{F}' = \mathbf{f}$, then

$$\int_a^b \mathbf{f}(t) \, dt = \mathbf{F}(b) - \mathbf{F}(a).$$

The analogue of Theorem 6.13(b) offers some new features, however, at least in its proof.

6.25 Theorem If \mathbf{f} maps $[a, b]$ into R^k and if $\mathbf{f} \in \mathcal{R}(\alpha)$ for some monotonically increasing function α on $[a, b]$, then $|\mathbf{f}| \in \mathcal{R}(\alpha)$, and

$$(40) \quad \left| \int_a^b \mathbf{f} \, d\alpha \right| \leq \int_a^b |\mathbf{f}| \, d\alpha.$$

Proof If f_1, \dots, f_k are the components of \mathbf{f} , then

$$(41) \quad |\mathbf{f}| = (f_1^2 + \dots + f_k^2)^{1/2}.$$

By Theorem 6.11, each of the functions f_i^2 belongs to $\mathcal{R}(\alpha)$; hence so does their sum. Since x^2 is a continuous function of x , Theorem 4.17 shows that the square-root function is continuous on $[0, M]$, for every real M . If we apply Theorem 6.11 once more, (41) shows that $|\mathbf{f}| \in \mathcal{R}(\alpha)$.

To prove (40), put $\mathbf{y} = (y_1, \dots, y_k)$, where $y_j = \int f_j \, d\alpha$. Then we have $\mathbf{y} = \int \mathbf{f} \, d\alpha$, and

$$|\mathbf{y}|^2 = \sum y_i^2 = \sum y_j \int f_j \, d\alpha = \int \left(\sum y_j f_j \right) d\alpha.$$

By the Schwarz inequality,

$$(42) \quad \sum y_j f_j(t) \leq |\mathbf{y}| |\mathbf{f}(t)| \quad (a \leq t \leq b);$$

hence Theorem 6.12(b) implies

$$(43) \quad |y|^2 \leq |y| \int |f| d\alpha.$$

If $y = 0$, (40) is trivial. If $y \neq 0$, division of (43) by $|y|$ gives (40).

RECTIFIABLE CURVES

We conclude this chapter with a topic of geometric interest which provides an application of some of the preceding theory. The case $k = 2$ (i.e., the case of plane curves) is of considerable importance in the study of analytic functions of a complex variable.

6.26 Definition A continuous mapping γ of an interval $[a, b]$ into R^k is called a *curve* in R^k . To emphasize the parameter interval $[a, b]$, we may also say that γ is a curve on $[a, b]$.

If γ is one-to-one, γ is called an *arc*.

If $\gamma(a) = \gamma(b)$, γ is said to be a *closed curve*.

It should be noted that we define a curve to be a *mapping*, not a point set. Of course, with each curve γ in R^k there is associated a subset of R^k , namely the range of γ , but different curves may have the same range.

We associate to each partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ and to each curve γ on $[a, b]$ the number

$$\Lambda(P, \gamma) = \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})|.$$

The i th term in this sum is the distance (in R^k) between the points $\gamma(x_{i-1})$ and $\gamma(x_i)$. Hence $\Lambda(P, \gamma)$ is the length of a polygonal path with vertices at $\gamma(x_0), \gamma(x_1), \dots, \gamma(x_n)$, in this order. As our partition becomes finer and finer, this polygon approaches the range of γ more and more closely. This makes it seem reasonable to define the *length* of γ as

$$\Lambda(\gamma) = \sup \Lambda(P, \gamma),$$

where the supremum is taken over all partitions of $[a, b]$.

If $\Lambda(\gamma) < \infty$, we say that γ is *rectifiable*.

In certain cases, $\Lambda(\gamma)$ is given by a Riemann integral. We shall prove this for *continuously differentiable* curves, i.e., for curves γ whose derivative γ' is continuous.

6.27 Theorem *If γ' is continuous on $[a, b]$, then γ is rectifiable, and*

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)| dt.$$

Proof If $a \leq x_{i-1} < x_i \leq b$, then

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma'(t) dt \right| \leq \int_{x_{i-1}}^{x_i} |\gamma'(t)| dt.$$

Hence

$$\Lambda(P, \gamma) \leq \int_a^b |\gamma'(t)| dt$$

for every partition P of $[a, b]$. Consequently,

$$\Lambda(\gamma) \leq \int_a^b |\gamma'(t)| dt.$$

To prove the opposite inequality, let $\varepsilon > 0$ be given. Since γ' is uniformly continuous on $[a, b]$, there exists $\delta > 0$ such that

$$|\gamma'(s) - \gamma'(t)| < \varepsilon \quad \text{if } |s - t| < \delta.$$

Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$, with $\Delta x_i < \delta$ for all i . If $x_{i-1} \leq t \leq x_i$, it follows that

$$|\gamma'(t)| \leq |\gamma'(x_i)| + \varepsilon.$$

Hence

$$\begin{aligned} \int_{x_{i-1}}^{x_i} |\gamma'(t)| dt &\leq |\gamma'(x_i)| \Delta x_i + \varepsilon \Delta x_i \\ &= \left| \int_{x_{i-1}}^{x_i} [\gamma'(t) + \gamma'(x_i) - \gamma'(t)] dt \right| + \varepsilon \Delta x_i \\ &\leq \left| \int_{x_{i-1}}^{x_i} \gamma'(t) dt \right| + \left| \int_{x_{i-1}}^{x_i} [\gamma'(x_i) - \gamma'(t)] dt \right| + \varepsilon \Delta x_i \\ &\leq |\gamma(x_i) - \gamma(x_{i-1})| + 2\varepsilon \Delta x_i. \end{aligned}$$

If we add these inequalities, we obtain

$$\begin{aligned} \int_a^b |\gamma'(t)| dt &\leq \Lambda(P, \gamma) + 2\varepsilon(b - a) \\ &\leq \Lambda(\gamma) + 2\varepsilon(b - a). \end{aligned}$$

Since ε was arbitrary,

$$\int_a^b |\gamma'(t)| dt \leq \Lambda(\gamma).$$

This completes the proof.

EXERCISES

1. Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0 , $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and that $\int f d\alpha = 0$.
2. Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x) dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare this with Exercise 1.)
3. Define three functions $\beta_1, \beta_2, \beta_3$ as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.
 - (a) Prove that $f \in \mathcal{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then

$$\int f d\beta_1 = f(0).$$

- (b) State and prove a similar result for β_2 .
- (c) Prove that $f \in \mathcal{R}(\beta_3)$ if and only if f is continuous at 0.
- (d) If f is continuous at 0 prove that

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0).$$

4. If $f(x) = 0$ for all irrational x , $f(x) = 1$ for all rational x , prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.
5. Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume that $f^3 \in \mathcal{R}$?
6. Let P be the Cantor set constructed in Sec. 2.44. Let f be a bounded real function on $[0, 1]$ which is continuous at every point outside P . Prove that $f \in \mathcal{R}$ on $[0, 1]$.
Hint: P can be covered by finitely many segments whose total length can be made as small as desired. Proceed as in Theorem 6.10.
7. Suppose f is a real function on $(0, 1]$ and $f \in \mathcal{R}$ on $[c, 1]$ for every $c > 0$. Define

$$\int_0^1 f(x) dx = \lim_{c \rightarrow 0} \int_c^1 f(x) dx$$

if this limit exists (and is finite).

- (a) If $f \in \mathcal{R}$ on $[0, 1]$, show that this definition of the integral agrees with the old one.
 - (b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f .
8. Suppose $f \in \mathcal{R}$ on $[a, b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after f has been replaced by $|f|$, it is said to *converge absolutely*.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^{\infty} f(x) dx$$

converges if and only if

$$\sum_{n=1}^{\infty} f(n)$$

converges. (This is the so-called “integral test” for convergence of series.)

9. Show that integration by parts can sometimes be applied to the “improper” integrals defined in Exercises 7 and 8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance show that

$$\int_0^{\infty} \frac{\cos x}{1+x} dx = \int_0^{\infty} \frac{\sin x}{(1+x)^2} dx.$$

Show that one of these integrals converges *absolutely*, but that the other does not.

10. Let p and q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

- (a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

- (b) If $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha,$$

then

$$\int_a^b fg d\alpha \leq 1.$$

- (c) If f and g are complex functions in $\mathcal{R}(\alpha)$, then

$$\left| \int_a^b fg d\alpha \right| \leq \left\{ \int_a^b |f|^p d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q d\alpha \right\}^{1/q}.$$

This is *Hölder's inequality*. When $p = q = 2$ it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

- (d) Show that Hölder's inequality is also true for the “improper” integrals described in Exercises 7 and 8.

11. Let α be a fixed increasing function on $[a, b]$. For $u \in \mathcal{R}(\alpha)$, define

$$\|u\|_2 = \left\{ \int_a^b |u|^2 d\alpha \right\}^{1/2}.$$

Suppose $f, g, h \in \mathcal{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

12. With the notations of Exercise 11, suppose $f \in \mathcal{R}(\alpha)$ and $\varepsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $\|f - g\|_2 < \varepsilon$.

Hint: Let $P = \{x_0, \dots, x_n\}$ be a suitable partition of $[a, b]$, define

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if $x_{i-1} \leq t \leq x_i$.

13. Define

$$f(x) = \int_x^{x+1} \sin(t^2) dt.$$

- (a) Prove that $|f(x)| < 1/x$ if $x > 0$.

Hint: Put $t^2 = u$ and integrate by parts, to show that $f(x)$ is equal to

$$\frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du.$$

Replace $\cos u$ by -1 .

- (b) Prove that

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$$

where $|r(x)| < c/x$ and c is a constant.

- (c) Find the upper and lower limits of $xf(x)$, as $x \rightarrow \infty$.

- (d) Does $\int_0^\infty \sin(t^2) dt$ converge?

14. Deal similarly with

$$f(x) = \int_x^{x+1} \sin(e^t) dt.$$

Show that

$$e^x |f(x)| < 2$$

and that

$$e^x f(x) = \cos(e^x) - e^{-1} \cos(e^{x+1}) + r(x),$$

where $|r(x)| < Ce^{-x}$, for some constant C .

15. Suppose f is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f^2(x) dx = 1.$$

Prove that

$$\int_a^b xf'(x)f'(x) dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x) dx > \frac{1}{4}.$$

16. For $1 < s < \infty$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(This is Riemann's zeta function, of great importance in the study of the distribution of prime numbers.) Prove that

$$(a) \quad \zeta(s) = s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx$$

and that

$$(b) \quad \zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{x - [x]}{x^{s+1}} dx,$$

where $[x]$ denotes the greatest integer $\leq x$.

Prove that the integral in (b) converges for all $s > 0$.

Hint: To prove (a), compute the difference between the integral over $[1, N]$ and the N th partial sum of the series that defines $\zeta(s)$.

17. Suppose α increases monotonically on $[a, b]$, g is continuous, and $g(x) = G'(x)$ for $a \leq x \leq b$. Prove that

$$\int_a^b \alpha(x)g(x) dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G d\alpha.$$

Hint: Take g real, without loss of generality. Given $P = \{x_0, x_1, \dots, x_n\}$, choose $t_i \in (x_{i-1}, x_i)$ so that $g(t_i) \Delta x_i = G(x_i) - G(x_{i-1})$. Show that

$$\sum_{i=1}^n \alpha(x_i)g(t_i) \Delta x_i = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G(x_{i-1}) \Delta \alpha_i.$$

18. Let $\gamma_1, \gamma_2, \gamma_3$ be curves in the complex plane, defined on $[0, 2\pi]$ by

$$\gamma_1(t) = e^{it}, \quad \gamma_2(t) = e^{2it}, \quad \gamma_3(t) = e^{2\pi i t \sin(1/t)}.$$

Show that these three curves have the same range, that γ_1 and γ_2 are rectifiable, that the length of γ_1 is 2π , that the length of γ_2 is 4π , and that γ_3 is not rectifiable.

19. Let γ_1 be a curve in R^k , defined on $[a, b]$; let ϕ be a continuous 1-1 mapping of $[c, d]$ onto $[a, b]$, such that $\phi(c) = a$; and define $\gamma_2(s) = \gamma_1(\phi(s))$. Prove that γ_2 is an arc, a closed curve, or a rectifiable curve if and only if the same is true of γ_1 . Prove that γ_2 and γ_1 have the same length.

SEQUENCES AND SERIES OF FUNCTIONS

In the present chapter we confine our attention to complex-valued functions (including the real-valued ones, of course), although many of the theorems and proofs which follow extend without difficulty to vector-valued functions, and even to mappings into general metric spaces. We choose to stay within this simple framework in order to focus attention on the most important aspects of the problems that arise when limit processes are interchanged.

DISCUSSION OF MAIN PROBLEM

7.1 Definition Suppose $\{f_n\}$, $n = 1, 2, 3, \dots$, is a sequence of functions defined on a set E , and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We can then define a function f by

$$(1) \qquad f(x) = \lim_{n \rightarrow \infty} f_n(x) \qquad (x \in E).$$

Under these circumstances we say that $\{f_n\}$ converges on E and that f is the *limit*, or the *limit function*, of $\{f_n\}$. Sometimes we shall use a more descriptive terminology and shall say that “ $\{f_n\}$ converges to f *pointwise* on E ” if (1) holds. Similarly, if $\Sigma f_n(x)$ converges for every $x \in E$, and if we define

$$(2) \quad f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in E),$$

the function f is called the *sum* of the series Σf_n .

The main problem which arises is to determine whether important properties of functions are preserved under the limit operations (1) and (2). For instance, if the functions f_n are continuous, or differentiable, or integrable, is the same true of the limit function? What are the relations between f'_n and f' , say, or between the integrals of f_n and that of f ?

To say that f is continuous at a limit point x means

$$\lim_{t \rightarrow x} f(t) = f(x).$$

Hence, to ask whether the limit of a sequence of continuous functions is continuous is the same as to ask whether

$$(3) \quad \lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t),$$

i.e., whether the order in which limit processes are carried out is immaterial. On the left side of (3), we first let $n \rightarrow \infty$, then $t \rightarrow x$; on the right side, $t \rightarrow x$ first, then $n \rightarrow \infty$.

We shall now show by means of several examples that limit processes cannot in general be interchanged without affecting the result. Afterward, we shall prove that under certain conditions the order in which limit operations are carried out is immaterial.

Our first example, and the simplest one, concerns a “double sequence.”

7.2 Example For $m = 1, 2, 3, \dots$, $n = 1, 2, 3, \dots$, let

$$s_{m,n} = \frac{m}{m+n}.$$

Then, for every fixed n ,

$$\lim_{m \rightarrow \infty} s_{m,n} = 1,$$

so that

$$(4) \quad \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} s_{m,n} = 1.$$

On the other hand, for every fixed m ,

$$\lim_{n \rightarrow \infty} s_{m,n} = 0,$$

so that

$$(5) \quad \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} s_{m,n} = 0.$$

7.3 Example Let

$$f_n(x) = \frac{x^2}{(1+x^2)^n} \quad (x \text{ real}; n = 0, 1, 2, \dots),$$

and consider

$$(6) \quad f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}.$$

Since $f_n(0) = 0$, we have $f(0) = 0$. For $x \neq 0$, the last series in (6) is a convergent geometric series with sum $1 + x^2$ (Theorem 3.26). Hence

$$(7) \quad f(x) = \begin{cases} 0 & (x = 0), \\ 1 + x^2 & (x \neq 0), \end{cases}$$

so that a convergent series of continuous functions may have a discontinuous sum.

7.4 Example For $m = 1, 2, 3, \dots$, put

$$f_m(x) = \lim_{n \rightarrow \infty} (\cos m! \pi x)^{2n}.$$

When $m!x$ is an integer, $f_m(x) = 1$. For all other values of x , $f_m(x) = 0$. Now let

$$f(x) = \lim_{m \rightarrow \infty} f_m(x).$$

For irrational x , $f_m(x) = 0$ for every m ; hence $f(x) = 0$. For rational x , say $x = p/q$, where p and q are integers, we see that $m!x$ is an integer if $m \geq q$, so that $f(x) = 1$. Hence

$$(8) \quad \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} (\cos m! \pi x)^{2n} = \begin{cases} 0 & (x \text{ irrational}), \\ 1 & (x \text{ rational}). \end{cases}$$

We have thus obtained an everywhere discontinuous limit function, which is not Riemann-integrable (Exercise 4, Chap. 6).

7.5 Example Let

$$(9) \quad f_n(x) = \frac{\sin nx}{\sqrt{n}} \quad (x \text{ real, } n = 1, 2, 3, \dots),$$

and

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) = 0.$$

Then $f'(x) = 0$, and

$$f'_n(x) = \sqrt{n} \cos nx,$$

so that $\{f'_n\}$ does not converge to f' . For instance,

$$f'_n(0) = \sqrt{n} \rightarrow +\infty$$

as $n \rightarrow \infty$, whereas $f'(0) = 0$.**7.6 Example** Let

$$(10) \quad f_n(x) = n^2 x(1 - x^2)^n \quad (0 \leq x \leq 1, n = 1, 2, 3, \dots).$$

For $0 < x \leq 1$, we have

$$\lim_{n \rightarrow \infty} f_n(x) = 0,$$

by Theorem 3.20(d). Since $f_n(0) = 0$, we see that

$$(11) \quad \lim_{n \rightarrow \infty} f_n(x) = 0 \quad (0 \leq x \leq 1).$$

A simple calculation shows that

$$\int_0^1 x(1 - x^2)^n dx = \frac{1}{2n + 2}.$$

Thus, in spite of (11),

$$\int_0^1 f_n(x) dx = \frac{n^2}{2n + 2} \rightarrow +\infty$$

as $n \rightarrow \infty$.If, in (10), we replace n^2 by n , (11) still holds, but we now have

$$\lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx = \lim_{n \rightarrow \infty} \frac{n}{2n + 2} = \frac{1}{2},$$

whereas

$$\int_0^1 \left[\lim_{n \rightarrow \infty} f_n(x) \right] dx = 0.$$

Thus the limit of the integral need not be equal to the integral of the limit, even if both are finite.

After these examples, which show what can go wrong if limit processes are interchanged carelessly, we now define a new mode of convergence, stronger than pointwise convergence as defined in Definition 7.1, which will enable us to arrive at positive results.

UNIFORM CONVERGENCE

7.7 Definition We say that a sequence of functions $\{f_n\}$, $n = 1, 2, 3, \dots$, converges *uniformly* on E to a function f if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies

$$(12) \quad |f_n(x) - f(x)| \leq \varepsilon$$

for all $x \in E$.

It is clear that every uniformly convergent sequence is pointwise convergent. Quite explicitly, the difference between the two concepts is this: If $\{f_n\}$ converges pointwise on E , then there exists a function f such that, for every $\varepsilon > 0$, and for every $x \in E$, there is an integer N , depending on ε and on x , such that (12) holds if $n \geq N$; if $\{f_n\}$ converges uniformly on E , it is possible, for each $\varepsilon > 0$, to find *one* integer N which will do for *all* $x \in E$.

We say that the series $\sum f_n(x)$ converges uniformly on E if the sequence $\{s_n\}$ of partial sums defined by

$$\sum_{i=1}^n f_i(x) = s_n(x)$$

converges uniformly on E .

The Cauchy criterion for uniform convergence is as follows.

7.8 Theorem *The sequence of functions $\{f_n\}$, defined on E , converges uniformly on E if and only if for every $\varepsilon > 0$ there exists an integer N such that $m \geq N$, $n \geq N$, $x \in E$ implies*

$$(13) \quad |f_n(x) - f_m(x)| \leq \varepsilon.$$

Proof Suppose $\{f_n\}$ converges uniformly on E , and let f be the limit function. Then there is an integer N such that $n \geq N$, $x \in E$ implies

$$|f_n(x) - f(x)| \leq \frac{\varepsilon}{2},$$

so that

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \leq \varepsilon$$

if $n \geq N$, $m \geq N$, $x \in E$.

Conversely, suppose the Cauchy condition holds. By Theorem 3.11, the sequence $\{f_n(x)\}$ converges, for every x , to a limit which we may call $f(x)$. Thus the sequence $\{f_n\}$ converges on E , to f . We have to prove that the convergence is uniform.

Let $\varepsilon > 0$ be given, and choose N such that (13) holds. Fix n , and let $m \rightarrow \infty$ in (13). Since $f_m(x) \rightarrow f(x)$ as $m \rightarrow \infty$, this gives

$$(14) \quad |f_n(x) - f(x)| \leq \varepsilon$$

for every $n \geq N$ and every $x \in E$, which completes the proof.

The following criterion is sometimes useful.

7.9 Theorem *Suppose*

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad (x \in E).$$

Put

$$M_n = \sup_{x \in E} |f_n(x) - f(x)|.$$

Then $f_n \rightarrow f$ uniformly on E if and only if $M_n \rightarrow 0$ as $n \rightarrow \infty$.

Since this is an immediate consequence of Definition 7.7, we omit the details of the proof.

For series, there is a very convenient test for uniform convergence, due to Weierstrass.

7.10 Theorem *Suppose $\{f_n\}$ is a sequence of functions defined on E , and suppose*

$$|f_n(x)| \leq M_n \quad (x \in E, n = 1, 2, 3, \dots).$$

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Note that the converse is not asserted (and is, in fact, not true).

Proof If $\sum M_n$ converges, then, for arbitrary $\varepsilon > 0$,

$$\left| \sum_{i=n}^m f_i(x) \right| \leq \sum_{i=n}^m M_i \leq \varepsilon \quad (x \in E),$$

provided m and n are large enough. Uniform convergence now follows from Theorem 7.8.

UNIFORM CONVERGENCE AND CONTINUITY

7.11 Theorem Suppose $f_n \rightarrow f$ uniformly on a set E in a metric space. Let x be a limit point of E , and suppose that

$$(15) \quad \lim_{t \rightarrow x} f_n(t) = A_n \quad (n = 1, 2, 3, \dots).$$

Then $\{A_n\}$ converges, and

$$(16) \quad \lim_{t \rightarrow x} f(t) = \lim_{n \rightarrow \infty} A_n.$$

In other words, the conclusion is that

$$(17) \quad \lim_{t \rightarrow x} \lim_{n \rightarrow \infty} f_n(t) = \lim_{n \rightarrow \infty} \lim_{t \rightarrow x} f_n(t).$$

Proof Let $\varepsilon > 0$ be given. By the uniform convergence of $\{f_n\}$, there exists N such that $n \geq N, m \geq N, t \in E$ imply

$$(18) \quad |f_n(t) - f_m(t)| \leq \varepsilon.$$

Letting $t \rightarrow x$ in (18), we obtain

$$|A_n - A_m| \leq \varepsilon$$

for $n \geq N, m \geq N$, so that $\{A_n\}$ is a Cauchy sequence and therefore converges, say to A .

Next,

$$(19) \quad |f(t) - A| \leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A|.$$

We first choose n such that

$$(20) \quad |f(t) - f_n(t)| \leq \frac{\varepsilon}{3}$$

for all $t \in E$ (this is possible by the uniform convergence), and such that

$$(21) \quad |A_n - A| \leq \frac{\varepsilon}{3}.$$

Then, for this n , we choose a neighborhood V of x such that

$$(22) \quad |f_n(t) - A_n| \leq \frac{\varepsilon}{3}$$

if $t \in V \cap E, t \neq x$.

Substituting the inequalities (20) to (22) into (19), we see that

$$|f(t) - A| \leq \varepsilon,$$

provided $t \in V \cap E, t \neq x$. This is equivalent to (16).

7.12 Theorem *If $\{f_n\}$ is a sequence of continuous functions on E , and if $f_n \rightarrow f$ uniformly on E , then f is continuous on E .*

This very important result is an immediate corollary of Theorem 7.11.

The converse is not true; that is, a sequence of continuous functions may converge to a continuous function, although the convergence is not uniform. Example 7.6 is of this kind (to see this, apply Theorem 7.9). But there is a case in which we can assert the converse.

7.13 Theorem *Suppose K is compact, and*

- (a) $\{f_n\}$ is a sequence of continuous functions on K ,
- (b) $\{f_n\}$ converges pointwise to a continuous function f on K ,
- (c) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K$, $n = 1, 2, 3, \dots$

Then $f_n \rightarrow f$ uniformly on K .

Proof Put $g_n = f_n - f$. Then g_n is continuous, $g_n \rightarrow 0$ pointwise, and $g_n \geq g_{n+1}$. We have to prove that $g_n \rightarrow 0$ uniformly on K .

Let $\varepsilon > 0$ be given. Let K_n be the set of all $x \in K$ with $g_n(x) \geq \varepsilon$. Since g_n is continuous, K_n is closed (Theorem 4.8), hence compact (Theorem 2.35). Since $g_n \geq g_{n+1}$, we have $K_n \supset K_{n+1}$. Fix $x \in K$. Since $g_n(x) \rightarrow 0$, we see that $x \notin K_n$ if n is sufficiently large. Thus $x \notin \bigcap K_n$. In other words, $\bigcap K_n$ is empty. Hence K_N is empty for some N (Theorem 2.36). It follows that $0 \leq g_n(x) < \varepsilon$ for all $x \in K$ and for all $n \geq N$. This proves the theorem.

Let us note that compactness is really needed here. For instance, if

$$f_n(x) = \frac{1}{nx + 1} \quad (0 < x < 1; n = 1, 2, 3, \dots)$$

then $f_n(x) \rightarrow 0$ monotonically in $(0, 1)$, but the convergence is not uniform.

7.14 Definition If X is a metric space, $\mathcal{C}(X)$ will denote the set of all complex-valued, continuous, bounded functions with domain X .

[Note that boundedness is redundant if X is compact (Theorem 4.15). Thus $\mathcal{C}(X)$ consists of all complex continuous functions on X if X is compact.]

We associate with each $f \in \mathcal{C}(X)$ its *supremum norm*

$$\|f\| = \sup_{x \in X} |f(x)|.$$

Since f is assumed to be bounded, $\|f\| < \infty$. It is obvious that $\|f\| = 0$ only if $f(x) = 0$ for every $x \in X$, that is, only if $f = 0$. If $h = f + g$, then

$$|h(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|$$

for all $x \in X$; hence

$$\|f + g\| \leq \|f\| + \|g\|.$$

If we define the distance between $f \in \mathcal{C}(X)$ and $g \in \mathcal{C}(X)$ to be $\|f - g\|$, it follows that Axioms 2.15 for a metric are satisfied.

We have thus made $\mathcal{C}(X)$ into a metric space.

Theorem 7.9 can be rephrased as follows:

A sequence $\{f_n\}$ converges to f with respect to the metric of $\mathcal{C}(X)$ if and only if $f_n \rightarrow f$ uniformly on X .

Accordingly, closed subsets of $\mathcal{C}(X)$ are sometimes called *uniformly closed*, the closure of a set $\mathcal{A} \subset \mathcal{C}(X)$ is called its *uniform closure*, and so on.

7.15 Theorem *The above metric makes $\mathcal{C}(X)$ into a complete metric space.*

Proof Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{C}(X)$. This means that to each $\varepsilon > 0$ corresponds an N such that $\|f_n - f_m\| < \varepsilon$ if $n \geq N$ and $m \geq N$. It follows (by Theorem 7.8) that there is a function f with domain X to which $\{f_n\}$ converges uniformly. By Theorem 7.12, f is continuous. Moreover, f is bounded, since there is an n such that $|f(x) - f_n(x)| < 1$ for all $x \in X$, and f_n is bounded.

Thus $f \in \mathcal{C}(X)$, and since $f_n \rightarrow f$ uniformly on X , we have $\|f - f_n\| \rightarrow 0$ as $n \rightarrow \infty$.

UNIFORM CONVERGENCE AND INTEGRATION

7.16 Theorem *Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$, for $n = 1, 2, 3, \dots$, and suppose $f_n \rightarrow f$ uniformly on $[a, b]$. Then $f \in \mathcal{R}(\alpha)$ on $[a, b]$, and*

$$(23) \quad \int_a^b f \, d\alpha = \lim_{n \rightarrow \infty} \int_a^b f_n \, d\alpha.$$

(The existence of the limit is part of the conclusion.)

Proof It suffices to prove this for real f_n . Put

$$(24) \quad \varepsilon_n = \sup |f_n(x) - f(x)|,$$

the supremum being taken over $a \leq x \leq b$. Then

$$f_n - \varepsilon_n \leq f \leq f_n + \varepsilon_n,$$

so that the upper and lower integrals of f (see Definition 6.2) satisfy

$$(25) \quad \int_a^b (f_n - \varepsilon_n) \, d\alpha \leq \int f \, d\alpha \leq \int f \, d\alpha \leq \int_a^b (f_n + \varepsilon_n) \, d\alpha.$$

Hence

$$0 \leq \int f \, d\alpha - \int f \, d\alpha \leq 2\varepsilon_n[\alpha(b) - \alpha(a)].$$

Since $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$ (Theorem 7.9), the upper and lower integrals of f are equal.

Thus $f \in \mathcal{R}(\alpha)$. Another application of (25) now yields

$$(26) \quad \left| \int_a^b f \, d\alpha - \int_a^b f_n \, d\alpha \right| \leq \varepsilon_n [\alpha(b) - \alpha(a)].$$

This implies (23).

Corollary If $f_n \in \mathcal{R}(\alpha)$ on $[a, b]$ and if

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (a \leq x \leq b),$$

the series converging uniformly on $[a, b]$, then

$$\int_a^b f \, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n \, d\alpha.$$

In other words, the series may be integrated term by term.

UNIFORM CONVERGENCE AND DIFFERENTIATION

We have already seen, in Example 7.5, that uniform convergence of $\{f_n\}$ implies nothing about the sequence $\{f'_n\}$. Thus stronger hypotheses are required for the assertion that $f'_n \rightarrow f'$ if $f_n \rightarrow f$.

7.17 Theorem Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$, to a function f , and

$$(27) \quad f'(x) = \lim_{n \rightarrow \infty} f'_n(x) \quad (a \leq x \leq b).$$

Proof Let $\varepsilon > 0$ be given. Choose N such that $n \geq N, m \geq N$, implies

$$(28) \quad |f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{2}$$

and

$$(29) \quad |f'_n(t) - f'_m(t)| < \frac{\varepsilon}{2(b-a)} \quad (a \leq t \leq b).$$

If we apply the mean value theorem 5.19 to the function $f_n - f_m$, (29) shows that

$$(30) \quad |f_n(x) - f_m(x) - f_n(t) + f_m(t)| \leq \frac{|x - t|\varepsilon}{2(b-a)} \leq \frac{\varepsilon}{2}$$

for any x and t on $[a, b]$, if $n \geq N$, $m \geq N$. The inequality

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |f_n(x_0) - f_m(x_0)|$$

implies, by (28) and (30), that

$$|f_n(x) - f_m(x)| < \varepsilon \quad (a \leq x \leq b, n \geq N, m \geq N),$$

so that $\{f_n\}$ converges uniformly on $[a, b]$. Let

$$f(x) = \lim_{n \rightarrow \infty} f_n(x) \quad (a \leq x \leq b).$$

Let us now fix a point x on $[a, b]$ and define

$$(31) \quad \phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \quad \phi(t) = \frac{f(t) - f(x)}{t - x}$$

for $a \leq t \leq b$, $t \neq x$. Then

$$(32) \quad \lim_{t \rightarrow x} \phi_n(t) = f'_n(x) \quad (n = 1, 2, 3, \dots).$$

The first inequality in (30) shows that

$$|\phi_n(t) - \phi_m(t)| \leq \frac{\varepsilon}{2(b-a)} \quad (n \geq N, m \geq N),$$

so that $\{\phi_n\}$ converges uniformly, for $t \neq x$. Since $\{f_n\}$ converges to f , we conclude from (31) that

$$(33) \quad \lim_{n \rightarrow \infty} \phi_n(t) = \phi(t)$$

uniformly for $a \leq t \leq b$, $t \neq x$.

If we now apply Theorem 7.11 to $\{\phi_n\}$, (32) and (33) show that

$$\lim_{t \rightarrow x} \phi(t) = \lim_{n \rightarrow \infty} f'_n(x);$$

and this is (27), by the definition of $\phi(t)$.

Remark: If the continuity of the functions f'_n is assumed in addition to the above hypotheses, then a much shorter proof of (27) can be based on Theorem 7.16 and the fundamental theorem of calculus.

7.18 Theorem *There exists a real continuous function on the real line which is nowhere differentiable.*

Proof Define

$$(34) \quad \varphi(x) = |x| \quad (-1 \leq x \leq 1)$$

and extend the definition of $\varphi(x)$ to all real x by requiring that

$$(35) \quad \varphi(x+2) = \varphi(x).$$

Then, for all s and t ,

$$(36) \quad |\varphi(s) - \varphi(t)| \leq |s - t|.$$

In particular, φ is continuous on R^1 . Define

$$(37) \quad f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \varphi(4^n x).$$

Since $0 \leq \varphi \leq 1$, Theorem 7.10 shows that the series (37) converges uniformly on R^1 . By Theorem 7.12, f is continuous on R^1 .

Now fix a real number x and a positive integer m . Put

$$(38) \quad \delta_m = \pm \frac{1}{2} \cdot 4^{-m}$$

where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x + \delta_m)$.

This can be done, since $4^m |\delta_m| = \frac{1}{2}$. Define

$$(39) \quad \gamma_n = \frac{\varphi(4^n(x + \delta_m)) - \varphi(4^n x)}{\delta_m}.$$

When $n > m$, then $4^n \delta_m$ is an even integer, so that $\gamma_n = 0$. When $0 \leq n \leq m$, (36) implies that $|\gamma_n| \leq 4^n$.

Since $|\gamma_m| = 4^m$, we conclude that

$$\begin{aligned} \left| \frac{f(x + \delta_m) - f(x)}{\delta_m} \right| &= \left| \sum_{n=0}^m \left(\frac{3}{4}\right)^n \gamma_n \right| \\ &\geq 3^m - \sum_{n=0}^{m-1} 3^n \\ &= \frac{1}{2}(3^m + 1). \end{aligned}$$

As $m \rightarrow \infty$, $\delta_m \rightarrow 0$. It follows that f is not differentiable at x .

EQUICONTINUOUS FAMILIES OF FUNCTIONS

In Theorem 3.6 we saw that every bounded sequence of complex numbers contains a convergent subsequence, and the question arises whether something similar is true for sequences of functions. To make the question more precise, we shall define two kinds of boundedness.

7.19 Definition Let $\{f_n\}$ be a sequence of functions defined on a set E .

We say that $\{f_n\}$ is *pointwise bounded* on E if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$, that is, if there exists a finite-valued function ϕ defined on E such that

$$|f_n(x)| < \phi(x) \quad (x \in E, n = 1, 2, 3, \dots).$$

We say that $\{f_n\}$ is *uniformly bounded* on E if there exists a number M such that

$$|f_n(x)| < M \quad (x \in E, n = 1, 2, 3, \dots).$$

Now if $\{f_n\}$ is pointwise bounded on E and E_1 is a countable subset of E , it is always possible to find a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E_1$. This can be done by the diagonal process which is used in the proof of Theorem 7.23.

However, even if $\{f_n\}$ is a uniformly bounded sequence of continuous functions on a compact set E , there need not exist a subsequence which converges pointwise on E . In the following example, this would be quite troublesome to prove with the equipment which we have at hand so far, but the proof is quite simple if we appeal to a theorem from Chap. 11.

7.20 Example Let

$$f_n(x) = \sin nx \quad (0 \leq x \leq 2\pi, n = 1, 2, 3, \dots).$$

Suppose there exists a sequence $\{n_k\}$ such that $\{\sin n_k x\}$ converges, for every $x \in [0, 2\pi]$. In that case we must have

$$\lim_{k \rightarrow \infty} (\sin n_k x - \sin n_{k+1} x) = 0 \quad (0 \leq x \leq 2\pi);$$

hence

$$(40) \quad \lim_{k \rightarrow \infty} (\sin n_k x - \sin n_{k+1} x)^2 = 0 \quad (0 \leq x \leq 2\pi).$$

By Lebesgue's theorem concerning integration of boundedly convergent sequences (Theorem 11.32), (40) implies

$$(41) \quad \lim_{k \rightarrow \infty} \int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 dx = 0.$$

But a simple calculation shows that

$$\int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 dx = 2\pi,$$

which contradicts (41).

Another question is whether every convergent sequence contains a uniformly convergent subsequence. Our next example will show that this need not be so, even if the sequence is uniformly bounded on a compact set. (Example 7.6 shows that a sequence of bounded functions may converge without being uniformly bounded; but it is trivial to see that uniform convergence of a sequence of bounded functions implies uniform boundedness.)

7.21 Example Let

$$f_n(x) = \frac{x^2}{x^2 + (1 - nx)^2} \quad (0 \leq x \leq 1, n = 1, 2, 3, \dots).$$

Then $|f_n(x)| \leq 1$, so that $\{f_n\}$ is uniformly bounded on $[0, 1]$. Also

$$\lim_{n \rightarrow \infty} f_n(x) = 0 \quad (0 \leq x \leq 1),$$

but

$$f_n\left(\frac{1}{n}\right) = 1 \quad (n = 1, 2, 3, \dots),$$

so that no subsequence can converge uniformly on $[0, 1]$.

The concept which is needed in this connection is that of equicontinuity; it is given in the following definition.

7.22 Definition A family \mathcal{F} of complex functions f defined on a set E in a metric space X is said to be *equicontinuous* on E if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon$$

whenever $d(x, y) < \delta$, $x \in E$, $y \in E$, and $f \in \mathcal{F}$. Here d denotes the metric of X .

It is clear that every member of an equicontinuous family is uniformly continuous.

The sequence of Example 7.21 is not equicontinuous.

Theorems 7.24 and 7.25 will show that there is a very close relation between equicontinuity, on the one hand, and uniform convergence of sequences of continuous functions, on the other. But first we describe a selection process which has nothing to do with continuity.

7.23 Theorem If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E , then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

Proof Let $\{x_i\}$, $i = 1, 2, 3, \dots$, be the points of E , arranged in a sequence. Since $\{f_n(x_1)\}$ is bounded, there exists a subsequence, which we shall denote by $\{f_{1,k}\}$, such that $\{f_{1,k}(x_1)\}$ converges as $k \rightarrow \infty$.

Let us now consider sequences S_1, S_2, S_3, \dots , which we represent by the array

$$\begin{array}{ccccccc} S_1: & f_{1,1} & f_{1,2} & f_{1,3} & f_{1,4} & \cdots \\ S_2: & f_{2,1} & f_{2,2} & f_{2,3} & f_{2,4} & \cdots \\ S_3: & f_{3,1} & f_{3,2} & f_{3,3} & f_{3,4} & \cdots \\ & \dots & \dots & \dots & \dots & \dots \end{array}$$

and which have the following properties:

- (a) S_n is a subsequence of S_{n-1} , for $n = 2, 3, 4, \dots$
- (b) $\{f_{n,k}(x_n)\}$ converges, as $k \rightarrow \infty$ (the boundedness of $\{f_n(x_n)\}$ makes it possible to choose S_n in this way);
- (c) The order in which the functions appear is the same in each sequence; i.e., if one function precedes another in S_1 , they are in the same relation in every S_n , until one or the other is deleted. Hence, when going from one row in the above array to the next below, functions may move to the left but never to the right.

We now go down the diagonal of the array; i.e., we consider the sequence

$$S: f_{1,1} \ f_{2,2} \ f_{3,3} \ f_{4,4} \ \cdots$$

By (c), the sequence S (except possibly its first $n - 1$ terms) is a subsequence of S_n , for $n = 1, 2, 3, \dots$. Hence (b) implies that $\{f_{n,n}(x_i)\}$ converges, as $n \rightarrow \infty$, for every $x_i \in E$.

7.24 Theorem *If K is a compact metric space, if $f_n \in \mathcal{C}(K)$ for $n = 1, 2, 3, \dots$, and if $\{f_n\}$ converges uniformly on K , then $\{f_n\}$ is equicontinuous on K .*

Proof Let $\varepsilon > 0$ be given. Since $\{f_n\}$ converges uniformly, there is an integer N such that

$$(42) \quad \|f_n - f_N\| < \varepsilon \quad (n > N).$$

(See Definition 7.14.) Since continuous functions are uniformly continuous on compact sets, there is a $\delta > 0$ such that

$$(43) \quad |f_i(x) - f_i(y)| < \varepsilon$$

if $1 \leq i \leq N$ and $d(x, y) < \delta$.

If $n > N$ and $d(x, y) < \delta$, it follows that

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| < 3\varepsilon.$$

In conjunction with (43), this proves the theorem.

7.25 Theorem *If K is compact, if $f_n \in \mathcal{C}(K)$ for $n = 1, 2, 3, \dots$, and if $\{f_n\}$ is pointwise bounded and equicontinuous on K , then*

- (a) $\{f_n\}$ is uniformly bounded on K ,
- (b) $\{f_n\}$ contains a uniformly convergent subsequence.

Proof

(a) Let $\varepsilon > 0$ be given and choose $\delta > 0$, in accordance with Definition 7.22, so that

$$(44) \quad |f_n(x) - f_n(y)| < \varepsilon$$

for all n , provided that $d(x, y) < \delta$.

Since K is compact, there are finitely many points p_1, \dots, p_r in K such that to every $x \in K$ corresponds at least one p_i with $d(x, p_i) < \delta$. Since $\{f_n\}$ is pointwise bounded, there exist $M_i < \infty$ such that $|f_n(p_i)| < M_i$ for all n . If $M = \max(M_1, \dots, M_r)$, then $|f_n(x)| < M + \varepsilon$ for every $x \in K$. This proves (a).

(b) Let E be a countable dense subset of K . (For the existence of such a set E , see Exercise 25, Chap. 2.) Theorem 7.23 shows that $\{f_n\}$ has a subsequence $\{f_{n_i}\}$ such that $\{f_{n_i}(x)\}$ converges for every $x \in E$.

Put $f_{n_i} = g_i$, to simplify the notation. We shall prove that $\{g_i\}$ converges uniformly on K .

Let $\varepsilon > 0$, and pick $\delta > 0$ as in the beginning of this proof. Let $V(x, \delta)$ be the set of all $y \in K$ with $d(x, y) < \delta$. Since E is dense in K , and K is compact, there are finitely many points x_1, \dots, x_m in E such that

$$(45) \quad K \subset V(x_1, \delta) \cup \dots \cup V(x_m, \delta).$$

Since $\{g_i(x)\}$ converges for every $x \in E$, there is an integer N such that

$$(46) \quad |g_i(x_s) - g_j(x_s)| < \varepsilon$$

whenever $i \geq N, j \geq N, 1 \leq s \leq m$.

If $x \in K$, (45) shows that $x \in V(x_s, \delta)$ for some s , so that

$$|g_i(x) - g_i(x_s)| < \varepsilon$$

for every i . If $i \geq N$ and $j \geq N$, it follows from (46) that

$$\begin{aligned} |g_i(x) - g_j(x)| &\leq |g_i(x) - g_i(x_s)| + |g_i(x_s) - g_j(x_s)| + |g_j(x_s) - g_j(x)| \\ &< 3\varepsilon. \end{aligned}$$

This completes the proof.

THE STONE-WEIERSTRASS THEOREM

7.26 Theorem *If f is a continuous complex function on $[a, b]$, there exists a sequence of polynomials P_n such that*

$$\lim_{n \rightarrow \infty} P_n(x) = f(x)$$

uniformly on $[a, b]$. If f is real, the P_n may be taken real.

This is the form in which the theorem was originally discovered by Weierstrass.

Proof We may assume, without loss of generality, that $[a, b] = [0, 1]$. We may also assume that $f(0) = f(1) = 0$. For if the theorem is proved for this case, consider

$$g(x) = f(x) - f(0) - x[f(1) - f(0)] \quad (0 \leq x \leq 1).$$

Here $g(0) = g(1) = 0$, and if g can be obtained as the limit of a uniformly convergent sequence of polynomials, it is clear that the same is true for f , since $f - g$ is a polynomial.

Furthermore, we define $f(x)$ to be zero for x outside $[0, 1]$. Then f is uniformly continuous on the whole line.

We put

$$(47) \quad Q_n(x) = c_n(1 - x^2)^n \quad (n = 1, 2, 3, \dots),$$

where c_n is chosen so that

$$(48) \quad \int_{-1}^1 Q_n(x) dx = 1 \quad (n = 1, 2, 3, \dots).$$

We need some information about the order of magnitude of c_n . Since

$$\begin{aligned} \int_{-1}^1 (1 - x^2)^n dx &= 2 \int_0^1 (1 - x^2)^n dx \geq 2 \int_0^{1/\sqrt{n}} (1 - x^2)^n dx \\ &\geq 2 \int_0^{1/\sqrt{n}} (1 - nx^2) dx \\ &= \frac{4}{3\sqrt{n}} \\ &> \frac{1}{\sqrt{n}}, \end{aligned}$$

it follows from (48) that

$$(49) \quad c_n < \sqrt{n}.$$

The inequality $(1 - x^2)^n \geq 1 - nx^2$ which we used above is easily shown to be true by considering the function

$$(1 - x^2)^n - 1 + nx^2$$

which is zero at $x = 0$ and whose derivative is positive in $(0, 1)$.

For any $\delta > 0$, (49) implies

$$(50) \quad Q_n(x) \leq \sqrt{n} (1 - \delta^2)^n \quad (\delta \leq |x| \leq 1),$$

so that $Q_n \rightarrow 0$ uniformly in $\delta \leq |x| \leq 1$.

Now set

$$(51) \quad P_n(x) = \int_{-1}^1 f(x+t) Q_n(t) dt \quad (0 \leq x \leq 1).$$

Our assumptions about f show, by a simple change of variable, that

$$P_n(x) = \int_{-x}^{1-x} f(x+t) Q_n(t) dt = \int_0^1 f(t) Q_n(t-x) dt,$$

and the last integral is clearly a polynomial in x . Thus $\{P_n\}$ is a sequence of polynomials, which are real if f is real.

Given $\varepsilon > 0$, we choose $\delta > 0$ such that $|y - x| < \delta$ implies

$$|f(y) - f(x)| < \frac{\varepsilon}{2}.$$

Let $M = \sup |f(x)|$. Using (48), (50), and the fact that $Q_n(x) \geq 0$, we see that for $0 \leq x \leq 1$,

$$\begin{aligned} |P_n(x) - f(x)| &= \left| \int_{-1}^1 [f(x+t) - f(x)] Q_n(t) dt \right| \\ &\leq \int_{-1}^1 |f(x+t) - f(x)| Q_n(t) dt \\ &\leq 2M \int_{-1}^{-\delta} Q_n(t) dt + \frac{\varepsilon}{2} \int_{-\delta}^{\delta} Q_n(t) dt + 2M \int_{\delta}^1 Q_n(t) dt \\ &\leq 4M \sqrt{n} (1 - \delta^2)^n + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

for all large enough n , which proves the theorem.

It is instructive to sketch the graphs of Q_n for a few values of n ; also, note that we needed uniform continuity of f to deduce uniform convergence of $\{P_n\}$.

In the proof of Theorem 7.32 we shall not need the full strength of Theorem 7.26, but only the following special case, which we state as a corollary.

7.27 Corollary *For every interval $[-a, a]$ there is a sequence of real polynomials P_n such that $P_n(0) = 0$ and such that*

$$\lim_{n \rightarrow \infty} P_n(x) = |x|$$

uniformly on $[-a, a]$.

Proof By Theorem 7.26, there exists a sequence $\{P_n^*\}$ of real polynomials which converges to $|x|$ uniformly on $[-a, a]$. In particular, $P_n^*(0) \rightarrow 0$ as $n \rightarrow \infty$. The polynomials

$$P_n(x) = P_n^*(x) - P_n^*(0) \quad (n = 1, 2, 3, \dots)$$

have desired properties.

We shall now isolate those properties of the polynomials which make the Weierstrass theorem possible.

7.28 Definition A family \mathcal{A} of complex functions defined on a set E is said to be an *algebra* if (i) $f + g \in \mathcal{A}$, (ii) $fg \in \mathcal{A}$, and (iii) $cf \in \mathcal{A}$ for all $f \in \mathcal{A}$, $g \in \mathcal{A}$ and for all complex constants c , that is, if \mathcal{A} is closed under addition, multiplication, and scalar multiplication. We shall also have to consider algebras of real functions; in this case, (iii) is of course only required to hold for all real c .

If \mathcal{A} has the property that $f \in \mathcal{A}$ whenever $f_n \in \mathcal{A}$ ($n = 1, 2, 3, \dots$) and $f_n \rightarrow f$ uniformly on E , then \mathcal{A} is said to be *uniformly closed*.

Let \mathcal{B} be the set of all functions which are limits of uniformly convergent sequences of members of \mathcal{A} . Then \mathcal{B} is called the *uniform closure* of \mathcal{A} . (See Definition 7.14.)

For example, the set of all polynomials is an algebra, and the Weierstrass theorem may be stated by saying that the set of continuous functions on $[a, b]$ is the uniform closure of the set of polynomials on $[a, b]$.

7.29 Theorem *Let \mathcal{B} be the uniform closure of an algebra \mathcal{A} of bounded functions. Then \mathcal{B} is a uniformly closed algebra.*

Proof If $f \in \mathcal{B}$ and $g \in \mathcal{B}$, there exist uniformly convergent sequences $\{f_n\}, \{g_n\}$ such that $f_n \rightarrow f, g_n \rightarrow g$ and $f_n \in \mathcal{A}, g_n \in \mathcal{A}$. Since we are dealing with bounded functions, it is easy to show that

$$f_n + g_n \rightarrow f + g, \quad f_n g_n \rightarrow fg, \quad cf_n \rightarrow cf,$$

where c is any constant, the convergence being uniform in each case.

Hence $f + g \in \mathcal{B}, fg \in \mathcal{B}$, and $cf \in \mathcal{B}$, so that \mathcal{B} is an algebra.

By Theorem 2.27, \mathcal{B} is (uniformly) closed.

7.30 Definition Let \mathcal{A} be a family of functions on a set E . Then \mathcal{A} is said to *separate points* on E if to every pair of distinct points $x_1, x_2 \in E$ there corresponds a function $f \in \mathcal{A}$ such that $f(x_1) \neq f(x_2)$.

If to each $x \in E$ there corresponds a function $g \in \mathcal{A}$ such that $g(x) \neq 0$, we say that \mathcal{A} *vanishes at no point of E* .

The algebra of all polynomials in one variable clearly has these properties on R^1 . An example of an algebra which does not separate points is the set of all even polynomials, say on $[-1, 1]$, since $f(-x) = f(x)$ for every even function f .

The following theorem will illustrate these concepts further.

7.31 Theorem Suppose \mathcal{A} is an algebra of functions on a set E , \mathcal{A} separates points on E , and \mathcal{A} vanishes at no point of E . Suppose x_1, x_2 are distinct points of E , and c_1, c_2 are constants (real if \mathcal{A} is a real algebra). Then \mathcal{A} contains a function f such that

$$f(x_1) = c_1, \quad f(x_2) = c_2.$$

Proof The assumptions show that \mathcal{A} contains functions g, h , and k such that

$$g(x_1) \neq g(x_2), \quad h(x_1) \neq 0, \quad k(x_2) \neq 0.$$

Put

$$u = gk - g(x_1)k, \quad v = gh - g(x_2)h.$$

Then $u \in \mathcal{A}, v \in \mathcal{A}, u(x_1) = v(x_2) = 0, u(x_2) \neq 0$, and $v(x_1) \neq 0$. Therefore

$$f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$$

has the desired properties.

We now have all the material needed for Stone's generalization of the Weierstrass theorem.

7.32 Theorem Let \mathcal{A} be an algebra of real continuous functions on a compact set K . If \mathcal{A} separates points on K and if \mathcal{A} vanishes at no point of K , then the uniform closure \mathcal{B} of \mathcal{A} consists of all real continuous functions on K .

We shall divide the proof into four steps.

STEP 1 If $f \in \mathcal{B}$, then $|f| \in \mathcal{B}$.

Proof Let

$$(52) \quad a = \sup |f(x)| \quad (x \in K)$$

and let $\varepsilon > 0$ be given. By Corollary 7.27 there exist real numbers c_1, \dots, c_n such that

$$(53) \quad \left| \sum_{i=1}^n c_i y^i - |y| \right| < \varepsilon \quad (-a \leq y \leq a).$$

Since \mathcal{B} is an algebra, the function

$$g = \sum_{i=1}^n c_i f^i$$

is a member of \mathcal{B} . By (52) and (53), we have

$$|g(x) - |f(x)|| < \varepsilon \quad (x \in K).$$

Since \mathcal{B} is uniformly closed, this shows that $|f| \in \mathcal{B}$.

STEP 2 If $f \in \mathcal{B}$ and $g \in \mathcal{B}$, then $\max(f, g) \in \mathcal{B}$ and $\min(f, g) \in \mathcal{B}$.

By $\max(f, g)$ we mean the function h defined by

$$h(x) = \begin{cases} f(x) & \text{if } f(x) \geq g(x), \\ g(x) & \text{if } f(x) < g(x), \end{cases}$$

and $\min(f, g)$ is defined likewise.

Proof Step 2 follows from step 1 and the identities

$$\begin{aligned} \max(f, g) &= \frac{f+g}{2} + \frac{|f-g|}{2}, \\ \min(f, g) &= \frac{f+g}{2} - \frac{|f-g|}{2}. \end{aligned}$$

By iteration, the result can of course be extended to any finite set of functions: If $f_1, \dots, f_n \in \mathcal{B}$, then $\max(f_1, \dots, f_n) \in \mathcal{B}$, and

$$\min(f_1, \dots, f_n) \in \mathcal{B}.$$

STEP 3 Given a real function f , continuous on K , a point $x \in K$, and $\varepsilon > 0$, there exists a function $g_x \in \mathcal{B}$ such that $g_x(x) = f(x)$ and

$$(54) \quad g_x(t) > f(t) - \varepsilon \quad (t \in K).$$

Proof Since $\mathcal{A} \subset \mathcal{B}$ and \mathcal{A} satisfies the hypotheses of Theorem 7.31 so does \mathcal{B} . Hence, for every $y \in K$, we can find a function $h_y \in \mathcal{B}$ such that

$$(55) \quad h_y(x) = f(x), \quad h_y(y) = f(y).$$

By the continuity of h_y , there exists an open set J_y , containing y , such that

$$(56) \quad h_y(t) > f(t) - \varepsilon \quad (t \in J_y).$$

Since K is compact, there is a finite set of points y_1, \dots, y_n such that

$$(57) \quad K \subset J_{y_1} \cup \dots \cup J_{y_n}.$$

Put

$$g_x = \max (h_{y_1}, \dots, h_{y_n}).$$

By step 2, $g_x \in \mathcal{B}$, and the relations (55) to (57) show that g_x has the other required properties.

STEP 4 *Given a real function f , continuous on K , and $\varepsilon > 0$, there exists a function $h \in \mathcal{B}$ such that*

$$(58) \quad |h(x) - f(x)| < \varepsilon \quad (x \in K).$$

Since \mathcal{B} is uniformly closed, this statement is equivalent to the conclusion of the theorem.

Proof Let us consider the functions g_x , for each $x \in K$, constructed in step 3. By the continuity of g_x , there exist open sets V_x containing x , such that

$$(59) \quad g_x(t) < f(t) + \varepsilon \quad (t \in V_x).$$

Since K is compact, there exists a finite set of points x_1, \dots, x_m such that

$$(60) \quad K \subset V_{x_1} \cup \dots \cup V_{x_m}.$$

Put

$$h = \min (g_{x_1}, \dots, g_{x_m}).$$

By step 2, $h \in \mathcal{B}$, and (54) implies

$$(61) \quad h(t) > f(t) - \varepsilon \quad (t \in K),$$

whereas (59) and (60) imply

$$(62) \quad h(t) < f(t) + \varepsilon \quad (t \in K).$$

Finally, (58) follows from (61) and (62).

Theorem 7.32 does not hold for complex algebras. A counterexample is given in Exercise 21. However, the conclusion of the theorem does hold, even for complex algebras, if an extra condition is imposed on \mathcal{A} , namely, that \mathcal{A} be *self-adjoint*. This means that for every $f \in \mathcal{A}$ its complex conjugate \bar{f} must also belong to \mathcal{A} ; \bar{f} is defined by $\bar{f}(x) = \overline{f(x)}$.

7.33 Theorem Suppose \mathcal{A} is a self-adjoint algebra of complex continuous functions on a compact set K , \mathcal{A} separates points on K , and \mathcal{A} vanishes at no point of K . Then the uniform closure \mathcal{B} of \mathcal{A} consists of all complex continuous functions on K . In other words, \mathcal{A} is dense $\mathcal{C}(K)$.

Proof Let \mathcal{A}_R be the set of all real functions on K which belong to \mathcal{A} .

If $f \in \mathcal{A}$ and $f = u + iv$, with u, v real, then $2u = f + \bar{f}$, and since \mathcal{A} is self-adjoint, we see that $u \in \mathcal{A}_R$. If $x_1 \neq x_2$, there exists $f \in \mathcal{A}$ such that $f(x_1) = 1, f(x_2) = 0$; hence $0 = u(x_2) \neq u(x_1) = 1$, which shows that \mathcal{A}_R separates points on K . If $x \in K$, then $g(x) \neq 0$ for some $g \in \mathcal{A}$, and there is a complex number λ such that $\lambda g(x) > 0$; if $f = \lambda g, f = u + iv$, it follows that $u(x) > 0$; hence \mathcal{A}_R vanishes at no point of K .

Thus \mathcal{A}_R satisfies the hypotheses of Theorem 7.32. It follows that every real continuous function on K lies in the uniform closure of \mathcal{A}_R , hence lies in \mathcal{B} . If f is a complex continuous function on $K, f = u + iv$, then $u \in \mathcal{B}, v \in \mathcal{B}$, hence $f \in \mathcal{B}$. This completes the proof.

EXERCISES

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.
2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E , prove that $\{f_n + g_n\}$ converges uniformly on E . If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E .
3. Construct sequences $\{f_n\}, \{g_n\}$ which converge uniformly on some set E , but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).
4. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

5. Let

$$f_n(x) = \begin{cases} 0 & \left(x < \frac{1}{n+1}\right), \\ \sin^2 \frac{\pi}{x} & \left(\frac{1}{n+1} \leq x \leq \frac{1}{n}\right), \\ 0 & \left(\frac{1}{n} < x\right). \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series Σf_n to show that absolute convergence, even for all x , does not imply uniform convergence.

6. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x .

7. For $n = 1, 2, 3, \dots$, x real, put

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f , and that the equation

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

8. If

$$I(x) = \begin{cases} 0 & (x \leq 0), \\ 1 & (x > 0), \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b) , and if $\Sigma |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \quad (a \leq x \leq b)$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

9. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E . Prove that

$$\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \rightarrow x$, and $x \in E$. Is the converse of this true?

10. Letting $\{x\}$ denote the fractional part of the real number x (see Exercise 16, Chap. 4, for the definition), consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{\{nx\}}{n^2} \quad (x \text{ real}).$$

Find all discontinuities of f , and show that they form a countable dense set. Show that f is nevertheless Riemann-integrable on every bounded interval.

11. Suppose $\{f_n\}, \{g_n\}$ are defined on E , and
- (a) $\sum f_n$ has uniformly bounded partial sums;
 - (b) $g_n \rightarrow 0$ uniformly on E ;
 - (c) $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$.

Prove that $\sum f_n g_n$ converges uniformly on E . *Hint*: Compare with Theorem 3.42.

12. Suppose g and $f_n (n = 1, 2, 3, \dots)$ are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g$, $f_n \rightarrow f$ uniformly on every compact subset of $(0, \infty)$, and

$$\int_0^{\infty} g(x) dx < \infty.$$

Prove that

$$\lim_{n \rightarrow \infty} \int_0^{\infty} f_n(x) dx = \int_0^{\infty} f(x) dx.$$

(See Exercises 7 and 8 of Chap. 6 for the relevant definitions.)

This is a rather weak form of Lebesgue's dominated convergence theorem (Theorem 11.32). Even in the context of the Riemann integral, uniform convergence can be replaced by pointwise convergence if it is assumed that $f \in \mathcal{R}$. (See the articles by F. Cunningham in *Math. Mag.*, vol. 40, 1967, pp. 179–186, and by H. Kestelman in *Amer. Math. Monthly*, vol. 77, 1970, pp. 182–187.)

13. Assume that $\{f_n\}$ is a sequence of monotonically increasing functions on R^1 with $0 \leq f_n(x) \leq 1$ for all x and all n .
- (a) Prove that there is a function f and a sequence $\{n_k\}$ such that

$$f(x) = \lim_{k \rightarrow \infty} f_{n_k}(x)$$

for every $x \in R^1$. (The existence of such a pointwise convergent subsequence is usually called *Helly's selection theorem*.)

- (b) If, moreover, f is continuous, prove that $f_{n_k} \rightarrow f$ uniformly on compact sets.

Hint: (i) Some subsequence $\{f_{n_i}\}$ converges at all rational points r , say, to $f(r)$. (ii) Define $f(x)$, for any $x \in R^1$, to be $\sup f(r)$, the sup being taken over all $r \leq x$. (iii) Show that $f_{n_i}(x) \rightarrow f(x)$ at every x at which f is continuous. (This is where monotonicity is strongly used.) (iv) A subsequence of $\{f_{n_i}\}$ converges at every point of discontinuity of f since there are at most countably many such points. This proves (a). To prove (b), modify your proof of (iii) appropriately.

14. Let f be a continuous real function on R^1 with the following properties:
 $0 \leq f(t) \leq 1$, $f(t+2) = f(t)$ for every t , and

$$f(t) = \begin{cases} 0 & (0 \leq t \leq \frac{1}{3}) \\ 1 & (\frac{2}{3} \leq t \leq 1). \end{cases}$$

Put $\Phi(t) = (x(t), y(t))$, where

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t), \quad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t).$$

Prove that Φ is *continuous* and that Φ maps $I = [0, 1]$ onto the unit square $I^2 \subset R^2$.
 If fact, show that Φ maps the Cantor set onto I^2 .

Hint: Each $(x_0, y_0) \in I^2$ has the form

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}, \quad y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n}$$

where each a_i is 0 or 1. If

$$t_0 = \sum_{i=1}^{\infty} 3^{-i-1} (2a_i)$$

show that $f(3^k t_0) = a_k$, and hence that $x(t_0) = x_0$, $y(t_0) = y_0$.

(This simple example of a so-called “space-filling curve” is due to I. J. Schoenberg, *Bull. A.M.S.*, vol. 44, 1938, pp. 519.)

15. Suppose f is a real continuous function on R^1 , $f_n(t) = f(nt)$ for $n = 1, 2, 3, \dots$, and $\{f_n\}$ is equicontinuous on $[0, 1]$. What conclusion can you draw about f ?
16. Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K , and $\{f_n\}$ converges pointwise on K . Prove that $\{f_n\}$ converges uniformly on K .
17. Define the notions of uniform convergence and equicontinuity for mappings into any metric space. Show that Theorems 7.9 and 7.12 are valid for mappings into any metric space, that Theorems 7.8 and 7.11 are valid for mappings into any complete metric space, and that Theorems 7.10, 7.16, 7.17, 7.24, and 7.25 hold for vector-valued functions, that is, for mappings into any R^k .
18. Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put

$$F_n(x) = \int_a^x f_n(t) dt \quad (a \leq x \leq b).$$

Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

19. Let K be a compact metric space, let S be a subset of $\mathcal{C}(K)$. Prove that S is compact (with respect to the metric defined in Section 7.14) if and only if S is uniformly closed, pointwise bounded, and equicontinuous. (If S is not equicontinuous, then S contains a sequence which has no equicontinuous subsequence, hence has no subsequence that converges uniformly on K .)

20. If f is continuous on $[0, 1]$ and if

$$\int_0^1 f(x)x^n dx = 0 \quad (n = 0, 1, 2, \dots),$$

prove that $f(x) = 0$ on $[0, 1]$. *Hint:* The integral of the product of f with any polynomial is zero. Use the Weierstrass theorem to show that $\int_0^1 f^2(x) dx = 0$.

21. Let K be the unit circle in the complex plane (i.e., the set of all z with $|z| = 1$), and let \mathcal{A} be the algebra of all functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^N c_n e^{in\theta} \quad (\theta \text{ real}).$$

Then \mathcal{A} separates points on K and \mathcal{A} vanishes at no point of K , but nevertheless there are continuous functions on K which are not in the uniform closure of \mathcal{A} . *Hint:* For every $f \in \mathcal{A}$

$$\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} d\theta = 0,$$

and this is also true for every f in the closure of \mathcal{A} .

22. Assume $f \in \mathcal{R}(\alpha)$ on $[a, b]$, and prove that there are polynomials P_n such that

$$\lim_{n \rightarrow \infty} \int_a^b |f - P_n|^2 d\alpha = 0.$$

(Compare with Exercise 12, Chap. 6.)

23. Put $P_0 = 0$, and define, for $n = 0, 1, 2, \dots$,

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

Prove that

$$\lim_{n \rightarrow \infty} P_n(x) = |x|,$$

uniformly on $[-1, 1]$.

(This makes it possible to prove the Stone-Weierstrass theorem without first proving Theorem 7.26.)

Hint: Use the identity

$$|x| - P_{n+1}(x) = [|x| - P_n(x)] \left[1 - \frac{|x| + P_n(x)}{2} \right]$$

to prove that $0 \leq P_n(x) \leq P_{n+1}(x) \leq |x|$ if $|x| \leq 1$, and that

$$|x| - P_n(x) \leq |x| \left(1 - \frac{|x|}{2} \right)^n < \frac{2}{n+1}$$

if $|x| \leq 1$.

24. Let X be a metric space, with metric d . Fix a point $a \in X$. Assign to each $p \in X$ the function f_p defined by

$$f_p(x) = d(x, p) - d(x, a) \quad (x \in X).$$

Prove that $|f_p(x)| \leq d(a, p)$ for all $x \in X$, and that therefore $f_p \in \mathcal{C}(X)$.
Prove that

$$\|f_p - f_q\| = d(p, q)$$

for all $p, q \in X$.

If $\Phi(p) = f_p$ it follows that Φ is an *isometry* (a distance-preserving mapping) of X onto $\Phi(X) \subset \mathcal{C}(X)$.

Let Y be the closure of $\Phi(X)$ in $\mathcal{C}(X)$. Show that Y is complete.

Conclusion: X is isometric to a dense subset of a complete metric space Y . (Exercise 24, Chap. 3 contains a different proof of this.)

25. Suppose ϕ is a continuous bounded real function in the strip defined by $0 \leq x \leq 1, -\infty < y < \infty$. Prove that the initial-value problem

$$y' = \phi(x, y), \quad y(0) = c$$

has a solution. (Note that the hypotheses of this existence theorem are less stringent than those of the corresponding uniqueness theorem; see Exercise 27, Chap. 5.)

Hint: Fix n . For $i = 0, \dots, n$, put $x_i = i/n$. Let f_n be a continuous function on $[0, 1]$ such that $f_n(0) = c$,

$$f'_n(t) = \phi(x_i, f_n(x_i)) \quad \text{if } x_i < t < x_{i+1},$$

and put

$$\Delta_n(t) = f'_n(t) - \phi(t, f_n(t)),$$

except at the points x_i , where $\Delta_n(t) = 0$. Then

$$f_n(x) = c + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)] dt.$$

Choose $M < \infty$ so that $|\phi| \leq M$. Verify the following assertions.

- (a) $|f'_n| \leq M$, $|\Delta_n| \leq 2M$, $\Delta_n \in \mathcal{R}$, and $|f_n| \leq |c| + M = M_1$, say, on $[0, 1]$, for all n .
(b) $\{f_n\}$ is equicontinuous on $[0, 1]$, since $|f'_n| \leq M$.
(c) Some $\{f_{n_k}\}$ converges to some f , uniformly on $[0, 1]$.
(d) Since ϕ is uniformly continuous on the rectangle $0 \leq x \leq 1, |y| \leq M_1$,

$$\phi(t, f_{n_k}(t)) \rightarrow \phi(t, f(t))$$

uniformly on $[0, 1]$.

- (e) $\Delta_n(t) \rightarrow 0$ uniformly on $[0, 1]$, since

$$\Delta_n(t) = \phi(x_i, f_n(x_i)) - \phi(t, f_n(t))$$

in (x_i, x_{i+1}) .

(f) Hence

$$f(x) = c + \int_0^x \phi(t, f(t)) \, dt.$$

This f is a solution of the given problem.

26. Prove an analogous existence theorem for the initial-value problem

$$\mathbf{y}' = \Phi(x, \mathbf{y}), \quad \mathbf{y}(0) = \mathbf{c},$$

where now $\mathbf{c} \in R^k$, $\mathbf{y} \in R^k$, and Φ is a continuous bounded mapping of the part of R^{k+1} defined by $0 \leq x \leq 1$, $\mathbf{y} \in R^k$ into R^k . (Compare Exercise 28, Chap. 5.) *Hint:* Use the vector-valued version of Theorem 7.25.

8

SOME SPECIAL FUNCTIONS

POWER SERIES

In this section we shall derive some properties of functions which are represented by power series, i.e., functions of the form

$$(1) \quad f(x) = \sum_{n=0}^{\infty} c_n x^n$$

or, more generally,

$$(2) \quad f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n.$$

These are called *analytic functions*.

We shall restrict ourselves to real values of x . Instead of circles of convergence (see Theorem 3.39) we shall therefore encounter intervals of convergence.

If (1) converges for all x in $(-R, R)$, for some $R > 0$ (R may be $+\infty$), we say that f is expanded in a power series about the point $x = 0$. Similarly, if (2) converges for $|x - a| < R$, f is said to be expanded in a power series about the point $x = a$. As a matter of convenience, we shall often take $a = 0$ without any loss of generality.

8.1 Theorem *Suppose the series*

$$(3) \quad \sum_{n=0}^{\infty} c_n x^n$$

converges for $|x| < R$, and define

$$(4) \quad f(x) = \sum_{n=0}^{\infty} c_n x^n \quad (|x| < R).$$

Then (3) converges uniformly on $[-R + \varepsilon, R - \varepsilon]$, no matter which $\varepsilon > 0$ is chosen. The function f is continuous and differentiable in $(-R, R)$, and

$$(5) \quad f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad (|x| < R).$$

Proof Let $\varepsilon > 0$ be given. For $|x| \leq R - \varepsilon$, we have

$$|c_n x^n| \leq |c_n (R - \varepsilon)^n|;$$

and since

$$\sum c_n (R - \varepsilon)^n$$

converges absolutely (every power series converges absolutely in the interior of its interval of convergence, by the root test), Theorem 7.10 shows the uniform convergence of (3) on $[-R + \varepsilon, R - \varepsilon]$.

Since $\sqrt[n]{n} \rightarrow 1$ as $n \rightarrow \infty$, we have

$$\limsup_{n \rightarrow \infty} \sqrt[n]{n |c_n|} = \limsup_{n \rightarrow \infty} \sqrt[n]{|c_n|},$$

so that the series (4) and (5) have the same interval of convergence.

Since (5) is a power series, it converges uniformly in $[-R + \varepsilon, R - \varepsilon]$, for every $\varepsilon > 0$, and we can apply Theorem 7.17 (for series instead of sequences). It follows that (5) holds if $|x| \leq R - \varepsilon$.

But, given any x such that $|x| < R$, we can find an $\varepsilon > 0$ such that $|x| < R - \varepsilon$. This shows that (5) holds for $|x| < R$.

Continuity of f follows from the existence of f' (Theorem 5.2).

Corollary *Under the hypotheses of Theorem 8.1, f has derivatives of all orders in $(-R, R)$, which are given by*

$$(6) \quad f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) c_n x^{n-k}.$$

In particular,

$$(7) \quad f^{(k)}(0) = k! c_k \quad (k = 0, 1, 2, \dots).$$

(Here $f^{(0)}$ means f , and $f^{(k)}$ is the k th derivative of f , for $k = 1, 2, 3, \dots$).

Proof Equation (6) follows if we apply Theorem 8.1 successively to f , f' , f'' , Putting $x = 0$ in (6), we obtain (7).

Formula (7) is very interesting. It shows, on the one hand, that the coefficients of the power series development of f are determined by the values of f and of its derivatives at a single point. On the other hand, if the coefficients are given, the values of the derivatives of f at the center of the interval of convergence can be read off immediately from the power series.

Note, however, that although a function f may have derivatives of all orders, the series $\sum c_n x^n$, where c_n is computed by (7), need not converge to $f(x)$ for any $x \neq 0$. In this case, f cannot be expanded in a power series about $x = 0$. For if we had $f(x) = \sum a_n x^n$, we should have

$$n!a_n = f^{(n)}(0);$$

hence $a_n = c_n$. An example of this situation is given in Exercise 1.

If the series (3) converges at an endpoint, say at $x = R$, then f is continuous not only in $(-R, R)$, but also at $x = R$. This follows from Abel's theorem (for simplicity of notation, we take $R = 1$):

8.2 Theorem Suppose $\sum c_n$ converges. Put

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \quad (-1 < x < 1).$$

Then

$$(8) \quad \lim_{x \rightarrow 1} f(x) = \sum_{n=0}^{\infty} c_n.$$

Proof Let $s_n = c_0 + \cdots + c_n$, $s_{-1} = 0$. Then

$$\sum_{n=0}^m c_n x^n = \sum_{n=0}^m (s_n - s_{n-1})x^n = (1-x) \sum_{n=0}^{m-1} s_n x^n + s_m x^m.$$

For $|x| < 1$, we let $m \rightarrow \infty$ and obtain

$$(9) \quad f(x) = (1-x) \sum_{n=0}^{\infty} s_n x^n.$$

Suppose $s = \lim_{n \rightarrow \infty} s_n$. Let $\varepsilon > 0$ be given. Choose N so that $n > N$ implies

$$|s - s_n| < \frac{\varepsilon}{2}.$$

Then, since

$$(1 - x) \sum_{n=0}^{\infty} x^n = 1 \quad (|x| < 1),$$

we obtain from (9)

$$|f(x) - s| = \left| (1 - x) \sum_{n=0}^{\infty} (s_n - s)x^n \right| \leq (1 - x) \sum_{n=0}^N |s_n - s| |x|^n + \frac{\varepsilon}{2} \leq \varepsilon$$

if $x > 1 - \delta$, for some suitably chosen $\delta > 0$. This implies (8).

As an application, let us prove Theorem 3.51, which asserts: *If Σa_n , Σb_n , Σc_n , converge to A , B , C , and if $c_n = a_0 b_n + \cdots + a_n b_0$, then $C = AB$.* We let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad g(x) = \sum_{n=0}^{\infty} b_n x^n, \quad h(x) = \sum_{n=0}^{\infty} c_n x^n,$$

for $0 \leq x \leq 1$. For $x < 1$, these series converge absolutely and hence may be multiplied according to Definition 3.48; when the multiplication is carried out, we see that

$$(10) \quad f(x) \cdot g(x) = h(x) \quad (0 \leq x < 1).$$

By Theorem 8.2,

$$(11) \quad f(x) \rightarrow A, \quad g(x) \rightarrow B, \quad h(x) \rightarrow C$$

as $x \rightarrow 1$. Equations (10) and (11) imply $AB = C$.

We now require a theorem concerning an inversion in the order of summation. (See Exercises 2 and 3.)

8.3 Theorem *Given a double sequence $\{a_{ij}\}$, $i = 1, 2, 3, \dots$, $j = 1, 2, 3, \dots$, suppose that*

$$(12) \quad \sum_{j=1}^{\infty} |a_{ij}| = b_i \quad (i = 1, 2, 3, \dots)$$

and Σb_i converges. Then

$$(13) \quad \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

Proof We could establish (13) by a direct procedure similar to (although more involved than) the one used in Theorem 3.55. However, the following method seems more interesting.

Let E be a countable set, consisting of the points x_0, x_1, x_2, \dots , and suppose $x_n \rightarrow x_0$ as $n \rightarrow \infty$. Define

$$(14) \quad f_i(x_0) = \sum_{j=1}^{\infty} a_{ij} \quad (i = 1, 2, 3, \dots),$$

$$(15) \quad f_i(x_n) = \sum_{j=1}^n a_{ij} \quad (i, n = 1, 2, 3, \dots),$$

$$(16) \quad g(x) = \sum_{i=1}^{\infty} f_i(x) \quad (x \in E).$$

Now, (14) and (15), together with (12), show that each f_i is continuous at x_0 . Since $|f_i(x)| \leq b_i$ for $x \in E$, (16) converges uniformly, so that g is continuous at x_0 (Theorem 7.11). It follows that

$$\begin{aligned} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} &= \sum_{i=1}^{\infty} f_i(x_0) = g(x_0) = \lim_{n \rightarrow \infty} g(x_n) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} f_i(x_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^{\infty} \sum_{j=1}^n a_{ij} \\ &= \lim_{n \rightarrow \infty} \sum_{j=1}^n \sum_{i=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}. \end{aligned}$$

8.4 Theorem Suppose

$$f(x) = \sum_{n=0}^{\infty} c_n x^n,$$

the series converging in $|x| < R$. If $-R < a < R$, then f can be expanded in a power series about the point $x = a$ which converges in $|x - a| < R - |a|$, and

$$(17) \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n \quad (|x - a| < R - |a|).$$

This is an extension of Theorem 5.15 and is also known as *Taylor's theorem*.

Proof We have

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} c_n [(x - a) + a]^n \\ &= \sum_{n=0}^{\infty} c_n \sum_{m=0}^n \binom{n}{m} a^{n-m} (x - a)^m \\ &= \sum_{m=0}^{\infty} \left[\sum_{n=m}^{\infty} \binom{n}{m} c_n a^{n-m} \right] (x - a)^m. \end{aligned}$$

This is the desired expansion about the point $x = a$. To prove its validity, we have to justify the change which was made in the order of summation. Theorem 8.3 shows that this is permissible if

$$(18) \quad \sum_{n=0}^{\infty} \sum_{m=0}^n \left| c_n \binom{n}{m} a^{n-m} (x-a)^m \right|$$

converges. But (18) is the same as

$$(19) \quad \sum_{n=0}^{\infty} |c_n| \cdot (|x-a| + |a|)^n,$$

and (19) converges if $|x-a| + |a| < R$.

Finally, the form of the coefficients in (17) follows from (7).

It should be noted that (17) may actually converge in a larger interval than the one given by $|x-a| < R - |a|$.

If two power series converge to the same function in $(-R, R)$, (7) shows that the two series must be identical, i.e., they must have the same coefficients. It is interesting that the same conclusion can be deduced from much weaker hypotheses:

8.5 Theorem *Suppose the series $\sum a_n x^n$ and $\sum b_n x^n$ converge in the segment $S = (-R, R)$. Let E be the set of all $x \in S$ at which*

$$(20) \quad \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n.$$

If E has a limit point in S , then $a_n = b_n$ for $n = 0, 1, 2, \dots$. Hence (20) holds for all $x \in S$.

Proof Put $c_n = a_n - b_n$ and

$$(21) \quad f(x) = \sum_{n=0}^{\infty} c_n x^n \quad (x \in S).$$

Then $f(x) = 0$ on E .

Let A be the set of all limit points of E in S , and let B consist of all other points of S . It is clear from the definition of "limit point" that B is open. Suppose we can prove that A is open. Then A and B are disjoint open sets. Hence they are separated (Definition 2.45). Since $S = A \cup B$, and S is connected, one of A and B must be empty. By hypothesis, A is not empty. Hence B is empty, and $A = S$. Since f is continuous in S , $A \subset E$. Thus $E = S$, and (7) shows that $c_n = 0$ for $n = 0, 1, 2, \dots$, which is the desired conclusion.

Thus we have to prove that A is open. If $x_0 \in A$, Theorem 8.4 shows that

$$(22) \quad f(x) = \sum_{n=0}^{\infty} d_n(x - x_0)^n \quad (|x - x_0| < R - |x_0|).$$

We claim that $d_n = 0$ for all n . Otherwise, let k be the smallest non-negative integer such that $d_k \neq 0$. Then

$$(23) \quad f(x) = (x - x_0)^k g(x) \quad (|x - x_0| < R - |x_0|),$$

where

$$(24) \quad g(x) = \sum_{m=0}^{\infty} d_{k+m}(x - x_0)^m.$$

Since g is continuous at x_0 and

$$g(x_0) = d_k \neq 0,$$

there exists a $\delta > 0$ such that $g(x) \neq 0$ if $|x - x_0| < \delta$. It follows from (23) that $f(x) \neq 0$ if $0 < |x - x_0| < \delta$. But this contradicts the fact that x_0 is a limit point of E .

Thus $d_n = 0$ for all n , so that $f(x) = 0$ for all x for which (22) holds, i.e., in a neighborhood of x_0 . This shows that A is open, and completes the proof.

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

We define

$$(25) \quad E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

The ratio test shows that this series converges for every complex z . Applying Theorem 3.50 on multiplication of absolutely convergent series, we obtain

$$\begin{aligned} E(z)E(w) &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{w^m}{m!} = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{z^k w^{n-k}}{k!(n-k)!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} z^k w^{n-k} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}, \end{aligned}$$

which gives us the important addition formula

$$(26) \quad E(z+w) = E(z)E(w) \quad (z, w \text{ complex}).$$

One consequence is that

$$(27) \quad E(z)E(-z) = E(z-z) = E(0) = 1 \quad (z \text{ complex}).$$

This shows that $E(z) \neq 0$ for all z . By (25), $E(x) > 0$ if $x > 0$; hence (27) shows that $E(x) > 0$ for all real x . By (25), $E(x) \rightarrow +\infty$ as $x \rightarrow +\infty$; hence (27) shows that $E(x) \rightarrow 0$ as $x \rightarrow -\infty$ along the real axis. By (25), $0 < x < y$ implies that $E(x) < E(y)$; by (27), it follows that $E(-y) < E(-x)$; hence E is strictly increasing on the whole real axis.

The addition formula also shows that

$$(28) \quad \lim_{h \rightarrow 0} \frac{E(z+h) - E(z)}{h} = E(z) \lim_{h \rightarrow 0} \frac{E(h) - 1}{h} = E(z);$$

the last equality follows directly from (25).

Iteration of (26) gives

$$(29) \quad E(z_1 + \cdots + z_n) = E(z_1) \cdots E(z_n).$$

Let us take $z_1 = \cdots = z_n = 1$. Since $E(1) = e$, where e is the number defined in Definition 3.30, we obtain

$$(30) \quad E(n) = e^n \quad (n = 1, 2, 3, \dots).$$

If $p = n/m$, where n, m are positive integers, then

$$(31) \quad [E(p)]^m = E(mp) = E(n) = e^n,$$

so that

$$(32) \quad E(p) = e^p \quad (p > 0, p \text{ rational}).$$

It follows from (27) that $E(-p) = e^{-p}$ if p is positive and rational. Thus (32) holds for all rational p .

In Exercise 6, Chap. 1, we suggested the definition

$$(33) \quad x^y = \sup x^p,$$

where the sup is taken over all rational p such that $p < y$, for any real y , and $x > 1$. If we thus define, for any real x ,

$$(34) \quad e^x = \sup e^p \quad (p < x, p \text{ rational}),$$

the continuity and monotonicity properties of E , together with (32), show that

$$(35) \quad E(x) = e^x$$

for all real x . Equation (35) explains why E is called the exponential function.

The notation $\exp(x)$ is often used in place of e^x , especially when x is a complicated expression.

Actually one may very well use (35) instead of (34) as the definition of e^x ; (35) is a much more convenient starting point for the investigation of the properties of e^x . We shall see presently that (33) may also be replaced by a more convenient definition [see (43)].

We now revert to the customary notation, e^x , in place of $E(x)$, and summarize what we have proved so far.

8.6 Theorem *Let e^x be defined on R^1 by (35) and (25). Then*

- (a) e^x is continuous and differentiable for all x ;
- (b) $(e^x)' = e^x$;
- (c) e^x is a strictly increasing function of x , and $e^x > 0$;
- (d) $e^{x+y} = e^x e^y$;
- (e) $e^x \rightarrow +\infty$ as $x \rightarrow +\infty$, $e^x \rightarrow 0$ as $x \rightarrow -\infty$;
- (f) $\lim_{x \rightarrow +\infty} x^n e^{-x} = 0$, for every n .

Proof We have already proved (a) to (e); (25) shows that

$$e^x > \frac{x^{n+1}}{(n+1)!}$$

for $x > 0$, so that

$$x^n e^{-x} < \frac{(n+1)!}{x},$$

and (f) follows. Part (f) shows that e^x tends to $+\infty$ “faster” than any power of x , as $x \rightarrow +\infty$.

Since E is strictly increasing and differentiable on R^1 , it has an inverse function L which is also strictly increasing and differentiable and whose domain is $E(R^1)$, that is, the set of all positive numbers. L is defined by

$$(36) \quad E(L(y)) = y \quad (y > 0),$$

or, equivalently, by

$$(37) \quad L(E(x)) = x \quad (x \text{ real}).$$

Differentiating (37), we get (compare Theorem 5.5)

$$L'(E(x)) \cdot E'(x) = 1.$$

Writing $y = E(x)$, this gives us

$$(38) \quad L'(y) = \frac{1}{y} \quad (y > 0).$$

Taking $x = 0$ in (37), we see that $L(1) = 0$. Hence (38) implies

$$(39) \quad L(y) = \int_1^y \frac{dx}{x}.$$

Quite frequently, (39) is taken as the starting point of the theory of the logarithm and the exponential function. Writing $u = E(x)$, $v = E(y)$, (26) gives

$$L(uv) = L(E(x) \cdot E(y)) = L(E(x + y)) = x + y,$$

so that

$$(40) \quad L(uv) = L(u) + L(v) \quad (u > 0, v > 0).$$

This shows that L has the familiar property which makes logarithms useful tools for computation. The customary notation for $L(x)$ is of course $\log x$.

As to the behavior of $\log x$ as $x \rightarrow +\infty$ and as $x \rightarrow 0$, Theorem 8.6(e) shows that

$$\begin{aligned} \log x &\rightarrow +\infty & \text{as } x &\rightarrow +\infty, \\ \log x &\rightarrow -\infty & \text{as } x &\rightarrow 0. \end{aligned}$$

It is easily seen that

$$(41) \quad x^n = E(nL(x))$$

if $x > 0$ and n is an integer. Similarly, if m is a positive integer, we have

$$(42) \quad x^{1/m} = E\left(\frac{1}{m} L(x)\right),$$

since each term of (42), when raised to the m th power, yields the corresponding term of (36). Combining (41) and (42), we obtain

$$(43) \quad x^\alpha = E(\alpha L(x)) = e^{\alpha \log x}$$

for any rational α .

We now define x^α , for any real α and any $x > 0$, by (43). The continuity and monotonicity of E and L show that this definition leads to the same result as the previously suggested one. The facts stated in Exercise 6 of Chap. 1, are trivial consequences of (43).

If we differentiate (43), we obtain, by Theorem 5.5,

$$(44) \quad (x^\alpha)' = E(\alpha L(x)) \cdot \frac{\alpha}{x} = \alpha x^{\alpha-1}.$$

Note that we have previously used (44) only for integral values of α , in which case (44) follows easily from Theorem 5.3(b). To prove (44) directly from the definition of the derivative, if x^α is defined by (33) and α is irrational, is quite troublesome.

The well-known integration formula for x^α follows from (44) if $\alpha \neq -1$, and from (38) if $\alpha = -1$. We wish to demonstrate one more property of $\log x$, namely,

$$(45) \quad \lim_{x \rightarrow +\infty} x^{-\alpha} \log x = 0$$

for every $\alpha > 0$. That is, $\log x \rightarrow +\infty$ "slower" than any positive power of x , as $x \rightarrow +\infty$.

For if $0 < \varepsilon < \alpha$, and $x > 1$, then

$$\begin{aligned} x^{-\alpha} \log x &= x^{-\alpha} \int_1^x t^{-1} dt < x^{-\alpha} \int_1^x t^{\varepsilon-1} dt \\ &= x^{-\alpha} \cdot \frac{x^{\varepsilon} - 1}{\varepsilon} < \frac{x^{\varepsilon-\alpha}}{\varepsilon}, \end{aligned}$$

and (45) follows. We could also have used Theorem 8.6(f) to derive (45).

THE TRIGONOMETRIC FUNCTIONS

Let us define

$$(46) \quad C(x) = \frac{1}{2} [E(ix) + E(-ix)], \quad S(x) = \frac{1}{2i} [E(ix) - E(-ix)].$$

We shall show that $C(x)$ and $S(x)$ coincide with the functions $\cos x$ and $\sin x$, whose definition is usually based on geometric considerations. By (25), $E(\bar{z}) = \overline{E(z)}$. Hence (46) shows that $C(x)$ and $S(x)$ are real for real x . Also,

$$(47) \quad E(ix) = C(x) + iS(x).$$

Thus $C(x)$ and $S(x)$ are the real and imaginary parts, respectively, of $E(ix)$, if x is real. By (27),

$$|E(ix)|^2 = E(ix)\overline{E(ix)} = E(ix)E(-ix) = 1,$$

so that

$$(48) \quad |E(ix)| = 1 \quad (x \text{ real}).$$

From (46) we can read off that $C(0) = 1$, $S(0) = 0$, and (28) shows that

$$(49) \quad C'(x) = -S(x), \quad S'(x) = C(x).$$

We assert that there exist positive numbers x such that $C(x) = 0$. For suppose this is not so. Since $C(0) = 1$, it then follows that $C(x) > 0$ for all $x > 0$, hence $S'(x) > 0$, by (49), hence S is strictly increasing; and since $S(0) = 0$, we have $S(x) > 0$ if $x > 0$. Hence if $0 < x < y$, we have

$$(50) \quad S(x)(y-x) < \int_x^y S(t) dt = C(x) - C(y) \leq 2.$$

The last inequality follows from (48) and (47). Since $S(x) > 0$, (50) cannot be true for large y , and we have a contradiction.

Let x_0 be the smallest positive number such that $C(x_0) = 0$. This exists, since the set of zeros of a continuous function is closed, and $C(0) \neq 0$. We define the number π by

$$(51) \quad \pi = 2x_0.$$

Then $C(\pi/2) = 0$, and (48) shows that $S(\pi/2) = \pm 1$. Since $C(x) > 0$ in $(0, \pi/2)$, S is increasing in $(0, \pi/2)$; hence $S(\pi/2) = 1$. Thus

$$E\left(\frac{\pi i}{2}\right) = i,$$

and the addition formula gives

$$(52) \quad E(\pi i) = -1, \quad E(2\pi i) = 1;$$

hence

$$(53) \quad E(z + 2\pi i) = E(z) \quad (z \text{ complex}).$$

8.7 Theorem

- (a) *The function E is periodic, with period $2\pi i$.*
- (b) *The functions C and S are periodic, with period 2π .*
- (c) *If $0 < t < 2\pi$, then $E(it) \neq 1$.*
- (d) *If z is a complex number with $|z| = 1$, there is a unique t in $[0, 2\pi)$ such that $E(it) = z$.*

Proof By (53), (a) holds; and (b) follows from (a) and (46).

Suppose $0 < t < \pi/2$ and $E(it) = x + iy$, with x, y real. Our preceding work shows that $0 < x < 1$, $0 < y < 1$. Note that

$$E(4it) = (x + iy)^4 = x^4 - 6x^2y^2 + y^4 + 4ixy(x^2 - y^2).$$

If $E(4it)$ is real, it follows that $x^2 - y^2 = 0$; since $x^2 + y^2 = 1$, by (48), we have $x^2 = y^2 = \frac{1}{2}$, hence $E(4it) = -1$. This proves (c).

If $0 \leq t_1 < t_2 < 2\pi$, then

$$E(it_2)[E(it_1)]^{-1} = E(it_2 - it_1) \neq 1,$$

by (c). This establishes the uniqueness assertion in (d).

To prove the existence assertion in (d), fix z so that $|z| = 1$. Write $z = x + iy$, with x and y real. Suppose first that $x \geq 0$ and $y \geq 0$. On $[0, \pi/2]$, C decreases from 1 to 0. Hence $C(t) = x$ for some $t \in [0, \pi/2]$. Since $C^2 + S^2 = 1$ and $S \geq 0$ on $[0, \pi/2]$, it follows that $z = E(it)$.

If $x < 0$ and $y \geq 0$, the preceding conditions are satisfied by $-iz$. Hence $-iz = E(it)$ for some $t \in [0, \pi/2]$, and since $i = E(\pi i/2)$, we obtain $z = E(i(t + \pi/2))$. Finally, if $y < 0$, the preceding two cases show that

$-z = E(it)$ for some $t \in (0, \pi)$. Hence $z = -E(it) = E(i(t + \pi))$.

This proves (d), and hence the theorem.

It follows from (d) and (48) that the curve γ defined by

$$(54) \quad \gamma(t) = E(it) \quad (0 \leq t \leq 2\pi)$$

is a simple closed curve whose range is the unit circle in the plane. Since $\gamma'(t) = iE(it)$, the length of γ is

$$\int_0^{2\pi} |\gamma'(t)| dt = 2\pi,$$

by Theorem 6.27. This is of course the expected result for the circumference of a circle of radius 1. It shows that π , defined by (51), has the usual geometric significance.

In the same way we see that the point $\gamma(t)$ describes a circular arc of length t_0 as t increases from 0 to t_0 . Consideration of the triangle whose vertices are

$$z_1 = 0, \quad z_2 = \gamma(t_0), \quad z_3 = C(t_0)$$

shows that $C(t)$ and $S(t)$ are indeed identical with $\cos t$ and $\sin t$, if the latter are defined in the usual way as ratios of the sides of a right triangle.

It should be stressed that we derived the basic properties of the trigonometric functions from (46) and (25), without any appeal to the geometric notion of angle. There are other nongeometric approaches to these functions. The papers by W. F. Eberlein (*Amer. Math. Monthly*, vol. 74, 1967, pp. 1223–1225) and by G. B. Robison (*Math. Mag.*, vol. 41, 1968, pp. 66–70) deal with these topics.

THE ALGEBRAIC COMPLETENESS OF THE COMPLEX FIELD

We are now in a position to give a simple proof of the fact that the complex field is algebraically complete, that is to say, that every nonconstant polynomial with complex coefficients has a complex root.

8.8 Theorem Suppose a_0, \dots, a_n are complex numbers, $n \geq 1$, $a_n \neq 0$,

$$P(z) = \sum_0^n a_k z^k.$$

Then $P(z) = 0$ for some complex number z .

Proof Without loss of generality, assume $a_n = 1$. Put

$$(55) \quad \mu = \inf |P(z)| \quad (z \text{ complex})$$

If $|z| = R$, then

$$(56) \quad |P(z)| \geq R^n [1 - |a_{n-1}| R^{-1} - \dots - |a_0| R^{-n}].$$

The right side of (56) tends to ∞ as $R \rightarrow \infty$. Hence there exists R_0 such that $|P(z)| > \mu$ if $|z| > R_0$. Since $|P|$ is continuous on the closed disc with center at 0 and radius R_0 , Theorem 4.16 shows that $|P(z_0)| = \mu$ for some z_0 .

We claim that $\mu = 0$.

If not, put $Q(z) = P(z + z_0)/P(z_0)$. Then Q is a nonconstant polynomial, $Q(0) = 1$, and $|Q(z)| \geq 1$ for all z . There is a smallest integer k , $1 \leq k \leq n$, such that

$$(57) \quad Q(z) = 1 + b_k z^k + \cdots + b_n z^n, \quad b_k \neq 0.$$

By Theorem 8.7(d) there is a real θ such that

$$(58) \quad e^{ik\theta} b_k = -|b_k|.$$

If $r > 0$ and $r^k |b_k| < 1$, (58) implies

$$|1 + b_k r^k e^{ik\theta}| = 1 - r^k |b_k|,$$

so that

$$|Q(re^{i\theta})| \leq 1 - r^k \{|b_k| - r|b_{k+1}| - \cdots - r^{n-k}|b_n|\}.$$

For sufficiently small r , the expression in braces is positive; hence $|Q(re^{i\theta})| < 1$, a contradiction.

Thus $\mu = 0$, that is, $P(z_0) = 0$.

Exercise 27 contains a more general result.

FOURIER SERIES

8.9 Definition A *trigonometric polynomial* is a finite sum of the form

$$(59) \quad f(x) = a_0 + \sum_{n=1}^N (a_n \cos nx + b_n \sin nx) \quad (x \text{ real}),$$

where $a_0, \dots, a_N, b_1, \dots, b_N$ are complex numbers. On account of the identities (46), (59) can also be written in the form

$$(60) \quad f(x) = \sum_{-N}^N c_n e^{inx} \quad (x \text{ real}),$$

which is more convenient for most purposes. It is clear that every trigonometric polynomial is periodic, with period 2π .

If n is a nonzero integer, e^{inx} is the derivative of e^{inx}/in , which also has period 2π . Hence

$$(61) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} dx = \begin{cases} 1 & (\text{if } n = 0), \\ 0 & (\text{if } n = \pm 1, \pm 2, \dots). \end{cases}$$

Let us multiply (60) by e^{-imx} , where m is an integer; if we integrate the product, (61) shows that

$$(62) \quad c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-imx} dx$$

for $|m| \leq N$. If $|m| > N$, the integral in (62) is 0.

The following observation can be read off from (60) and (62): The trigonometric polynomial f , given by (60), is *real* if and only if $c_{-n} = \overline{c_n}$ for $n = 0, \dots, N$.

In agreement with (60), we define a *trigonometric series* to be a series of the form

$$(63) \quad \sum_{-\infty}^{\infty} c_n e^{inx} \quad (x \text{ real});$$

the N th partial sum of (63) is defined to be the right side of (60).

If f is an integrable function on $[-\pi, \pi]$, the numbers c_m defined by (62) for all integers m are called the *Fourier coefficients* of f , and the series (63) formed with these coefficients is called the *Fourier series* of f .

The natural question which now arises is whether the Fourier series of f converges to f , or, more generally, whether f is determined by its Fourier series. That is to say, if we know the Fourier coefficients of a function, can we find the function, and if so, how?

The study of such series, and, in particular, the problem of representing a given function by a trigonometric series, originated in physical problems such as the theory of oscillations and the theory of heat conduction (Fourier's "Théorie analytique de la chaleur" was published in 1822). The many difficult and delicate problems which arose during this study caused a thorough revision and reformulation of the whole theory of functions of a real variable. Among many prominent names, those of Riemann, Cantor, and Lebesgue are intimately connected with this field, which nowadays, with all its generalizations and ramifications, may well be said to occupy a central position in the whole of analysis.

We shall be content to derive some basic theorems which are easily accessible by the methods developed in the preceding chapters. For more thorough investigations, the Lebesgue integral is a natural and indispensable tool.

We shall first study more general systems of functions which share a property analogous to (61).

8.10 Definition Let $\{\phi_n\}$ ($n = 1, 2, 3, \dots$) be a sequence of complex functions on $[a, b]$, such that

$$(64) \quad \int_a^b \phi_n(x) \overline{\phi_m(x)} dx = 0 \quad (n \neq m).$$

Then $\{\phi_n\}$ is said to be an *orthogonal system of functions* on $[a, b]$. If, in addition,

$$(65) \quad \int_a^b |\phi_n(x)|^2 dx = 1$$

for all n , $\{\phi_n\}$ is said to be *orthonormal*.

For example, the functions $(2\pi)^{-\frac{1}{2}} e^{inx}$ form an orthonormal system on $[-\pi, \pi]$. So do the real functions

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots$$

If $\{\phi_n\}$ is orthonormal on $[a, b]$ and if

$$(66) \quad c_n = \int_a^b f(t) \overline{\phi_n(t)} dt \quad (n = 1, 2, 3, \dots),$$

we call c_n the n th Fourier coefficient of f relative to $\{\phi_n\}$. We write

$$(67) \quad f(x) \sim \sum_1^\infty c_n \phi_n(x)$$

and call this series the Fourier series of f (relative to $\{\phi_n\}$).

Note that the symbol \sim used in (67) implies nothing about the convergence of the series; it merely says that the coefficients are given by (66).

The following theorems show that the partial sums of the Fourier series of f have a certain minimum property. We shall assume here and in the rest of this chapter that $f \in \mathscr{R}$, although this hypothesis can be weakened.

8.11 Theorem *Let $\{\phi_n\}$ be orthonormal on $[a, b]$. Let*

$$(68) \quad s_n(x) = \sum_{m=1}^n c_m \phi_m(x)$$

be the n th partial sum of the Fourier series of f , and suppose

$$(69) \quad t_n(x) = \sum_{m=1}^n \gamma_m \phi_m(x).$$

Then

$$(70) \quad \int_a^b |f - s_n|^2 dx \leq \int_a^b |f - t_n|^2 dx,$$

and equality holds if and only if

$$(71) \quad \gamma_m = c_m \quad (m = 1, \dots, n).$$

That is to say, among all functions t_n , s_n gives the best possible mean square approximation to f .

Proof Let \int denote the integral over $[a, b]$, Σ the sum from 1 to n . Then

$$\int f \bar{t}_n = \int f \sum \bar{\gamma}_m \bar{\phi}_m = \sum c_m \bar{\gamma}_m$$

by the definition of $\{c_m\}$,

$$\int |t_n|^2 = \int t_n \bar{t}_n = \int \sum \gamma_m \phi_m \sum \bar{\gamma}_k \bar{\phi}_k = \sum |\gamma_m|^2$$

since $\{\phi_m\}$ is orthonormal, and so

$$\begin{aligned} \int |f - t_n|^2 &= \int |f|^2 - \int f \bar{t}_n - \int \bar{f} t_n + \int |t_n|^2 \\ &= \int |f|^2 - \sum c_m \bar{\gamma}_m - \sum \bar{c}_m \gamma_m + \sum |\gamma_m|^2 \\ &= \int |f|^2 - \sum |c_m|^2 + \sum |\gamma_m - c_m|^2, \end{aligned}$$

which is evidently minimized if and only if $\gamma_m = c_m$.

Putting $\gamma_m = c_m$ in this calculation, we obtain

$$(72) \quad \int_a^b |s_n(x)|^2 dx = \sum_1^n |c_m|^2 \leq \int_a^b |f(x)|^2 dx,$$

since $\int |f - t_n|^2 \geq 0$.

8.12 Theorem If $\{\phi_n\}$ is orthonormal on $[a, b]$, and if

$$f(x) \sim \sum_{n=1}^{\infty} c_n \phi_n(x),$$

then

$$(73) \quad \sum_{n=1}^{\infty} |c_n|^2 \leq \int_a^b |f(x)|^2 dx.$$

In particular,

$$(74) \quad \lim_{n \rightarrow \infty} c_n = 0.$$

Proof Letting $n \rightarrow \infty$ in (72), we obtain (73), the so-called “Bessel inequality.”

8.13 Trigonometric series From now on we shall deal only with the trigonometric system. We shall consider functions f that have period 2π and that are Riemann-integrable on $[-\pi, \pi]$ (and hence on every bounded interval). The Fourier series of f is then the series (63) whose coefficients c_n are given by the integrals (62), and

$$(75) \quad s_N(x) = s_N(f; x) = \sum_{-N}^N c_n e^{inx}$$

is the N th partial sum of the Fourier series of f . The inequality (72) now takes the form

$$(76) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} |s_N(x)|^2 dx = \sum_{n=-N}^N |c_n|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx.$$

In order to obtain an expression for s_N that is more manageable than (75) we introduce the *Dirichlet kernel*

$$(77) \quad D_N(x) = \sum_{n=-N}^N e^{inx} = \frac{\sin(N + \frac{1}{2})x}{\sin(x/2)}.$$

The first of these equalities is the definition of $D_N(x)$. The second follows if both sides of the identity

$$(e^{ix} - 1)D_N(x) = e^{i(N+1)x} - e^{-iNx}$$

are multiplied by $e^{-ix/2}$.

By (62) and (75), we have

$$\begin{aligned} s_N(f; x) &= \sum_{n=-N}^N \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt e^{inx} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \sum_{n=-N}^N e^{in(x-t)} dt, \end{aligned}$$

so that

$$(78) \quad s_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_N(x-t) dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) D_N(t) dt.$$

The periodicity of all functions involved shows that it is immaterial over which interval we integrate, as long as its length is 2π . This shows that the two integrals in (78) are equal.

We shall prove just one theorem about the pointwise convergence of Fourier series.

8.14 Theorem *If, for some x , there are constants $\delta > 0$ and $M < \infty$ such that*

$$(79) \quad |f(x+t) - f(x)| \leq M|t|$$

for all $t \in (-\delta, \delta)$, then

$$(80) \quad \lim_{N \rightarrow \infty} s_N(f; x) = f(x).$$

Proof Define

$$(81) \quad g(t) = \frac{f(x-t) - f(x)}{\sin(t/2)}$$

for $0 < |t| \leq \pi$, and put $g(0) = 0$. By the definition (77),

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_N(x) dx = 1.$$

Hence (78) shows that

$$\begin{aligned} s_N(f; x) - f(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) \sin \left(N + \frac{1}{2} \right) t dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[g(t) \cos \frac{t}{2} \right] \sin Nt dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[g(t) \sin \frac{t}{2} \right] \cos Nt dt. \end{aligned}$$

By (79) and (81), $g(t) \cos (t/2)$ and $g(t) \sin (t/2)$ are bounded. The last two integrals thus tend to 0 as $N \rightarrow \infty$, by (74). This proves (80).

Corollary *If $f(x) = 0$ for all x in some segment J , then $\lim s_N(f; x) = 0$ for every $x \in J$.*

Here is another formulation of this corollary:

If $f(t) = g(t)$ for all t in some neighborhood of x , then

$$s_N(f; x) - s_N(g; x) = s_N(f - g; x) \rightarrow 0 \text{ as } N \rightarrow \infty.$$

This is usually called the *localization theorem*. It shows that the behavior of the sequence $\{s_N(f; x)\}$, as far as convergence is concerned, depends only on the values of f in some (arbitrarily small) neighborhood of x . Two Fourier series may thus have the same behavior in one interval, but may behave in entirely different ways in some other interval. We have here a very striking contrast between Fourier series and power series (Theorem 8.5).

We conclude with two other approximation theorems.

8.15 Theorem *If f is continuous (with period 2π) and if $\varepsilon > 0$, then there is a trigonometric polynomial P such that*

$$|P(x) - f(x)| < \varepsilon$$

for all real x .

Proof If we identify x and $x + 2\pi$, we may regard the 2π -periodic functions on R^1 as functions on the unit circle T , by means of the mapping $x \rightarrow e^{ix}$. The trigonometric polynomials, i.e., the functions of the form (60), form a self-adjoint algebra \mathcal{A} , which separates points on T , and which vanishes at no point of T . Since T is compact, Theorem 7.33 tells us that \mathcal{A} is dense in $\mathcal{C}(T)$. This is exactly what the theorem asserts.

A more precise form of this theorem appears in Exercise 15.

8.16 Parseval's theorem Suppose f and g are Riemann-integrable functions with period 2π , and

$$(82) \quad f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}, \quad g(x) \sim \sum_{-\infty}^{\infty} \gamma_n e^{inx}.$$

Then

$$(83) \quad \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - s_N(f; x)|^2 dx = 0,$$

$$(84) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \overline{g(x)} dx = \sum_{-\infty}^{\infty} c_n \bar{\gamma}_n,$$

$$(85) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{-\infty}^{\infty} |c_n|^2.$$

Proof Let us use the notation

$$(86) \quad \|h\|_2 = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |h(x)|^2 dx \right\}^{1/2}.$$

Let $\varepsilon > 0$ be given. Since $f \in \mathcal{R}$ and $f(\pi) = f(-\pi)$, the construction described in Exercise 12 of Chap. 6 yields a continuous 2π -periodic function h with

$$(87) \quad \|f - h\|_2 < \varepsilon.$$

By Theorem 8.15, there is a trigonometric polynomial P such that $|h(x) - P(x)| < \varepsilon$ for all x . Hence $\|h - P\|_2 < \varepsilon$. If P has degree N_0 , Theorem 8.11 shows that

$$(88) \quad \|h - s_N(h)\|_2 \leq \|h - P\|_2 < \varepsilon$$

for all $N \geq N_0$. By (72), with $h - f$ in place of f ,

$$(89) \quad \|s_N(h) - s_N(f)\|_2 = \|s_N(h - f)\|_2 \leq \|h - f\|_2 < \varepsilon.$$

Now the triangle inequality (Exercise 11, Chap. 6), combined with (87), (88), and (89), shows that

$$(90) \quad \|f - s_N(f)\|_2 < 3\varepsilon \quad (N \geq N_0).$$

This proves (83). Next,

$$(91) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} s_N(f) \bar{g} dx = \sum_{-N}^N c_n \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} \overline{g(x)} dx = \sum_{-N}^N c_n \bar{\gamma}_n,$$

and the Schwarz inequality shows that

$$(92) \quad \left| \int f \bar{g} - \int s_N(f) \bar{g} \right| \leq \int |f - s_N(f)| |g| \leq \left\{ \int |f - s_N|^2 \int |g|^2 \right\}^{1/2},$$

which tends to 0, as $N \rightarrow \infty$, by (83). Comparison of (91) and (92) gives (84). Finally, (85) is the special case $g = f$ of (84).

A more general version of Theorem 8.16 appears in Chap. 11.

THE GAMMA FUNCTION

This function is closely related to factorials and crops up in many unexpected places in analysis. Its origin, history, and development are very well described in an interesting article by P. J. Davis (*Amer. Math. Monthly*, vol. 66, 1959, pp. 849–869). Artin's book (cited in the Bibliography) is another good elementary introduction.

Our presentation will be very condensed, with only a few comments after each theorem. This section may thus be regarded as a large exercise, and as an opportunity to apply some of the material that has been presented so far.

8.17 Definition For $0 < x < \infty$,

$$(93) \quad \Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt.$$

The integral converges for these x . (When $x < 1$, both 0 and ∞ have to be looked at.)

8.18 Theorem

(a) *The functional equation*

$$\Gamma(x+1) = x\Gamma(x)$$

holds if $0 < x < \infty$.

(b) $\Gamma(n+1) = n!$ for $n = 1, 2, 3, \dots$

(c) $\log \Gamma$ is convex on $(0, \infty)$.

Proof An integration by parts proves (a). Since $\Gamma(1) = 1$, (a) implies (b), by induction. If $1 < p < \infty$ and $(1/p) + (1/q) = 1$, apply Hölder's inequality (Exercise 10, Chap. 6) to (93), and obtain

$$\Gamma\left(\frac{x}{p} + \frac{y}{q}\right) \leq \Gamma(x)^{1/p} \Gamma(y)^{1/q}.$$

This is equivalent to (c).

It is a rather surprising fact, discovered by Bohr and Møllerup, that these three properties characterize Γ completely.

8.19 Theorem *If f is a positive function on $(0, \infty)$ such that*

(a) $f(x+1) = xf(x)$,

(b) $f(1) = 1$,

(c) $\log f$ is convex,

then $f(x) = \Gamma(x)$.

Proof Since Γ satisfies (a), (b), and (c), it is enough to prove that $f(x)$ is uniquely determined by (a), (b), (c), for all $x > 0$. By (a), it is enough to do this for $x \in (0, 1)$.

Put $\varphi = \log f$. Then

$$(94) \quad \varphi(x+1) = \varphi(x) + \log x \quad (0 < x < \infty),$$

$\varphi(1) = 0$, and φ is convex. Suppose $0 < x < 1$, and n is a positive integer. By (94), $\varphi(n+1) = \log(n!)$. Consider the difference quotients of φ on the intervals $[n, n+1]$, $[n+1, n+1+x]$, $[n+1, n+2]$. Since φ is convex

$$\log n \leq \frac{\varphi(n+1+x) - \varphi(n+1)}{x} \leq \log(n+1).$$

Repeated application of (94) gives

$$\varphi(n+1+x) = \varphi(x) + \log [x(x+1) \cdots (x+n)].$$

Thus

$$0 \leq \varphi(x) - \log \left[\frac{n!n^x}{x(x+1) \cdots (x+n)} \right] \leq x \log \left(1 + \frac{1}{n} \right).$$

The last expression tends to 0 as $n \rightarrow \infty$. Hence $\varphi(x)$ is determined, and the proof is complete.

As a by-product we obtain the relation

$$(95) \quad \Gamma(x) = \lim_{n \rightarrow \infty} \frac{n!n^x}{x(x+1) \cdots (x+n)}$$

at least when $0 < x < 1$; from this one can deduce that (95) holds for all $x > 0$, since $\Gamma(x+1) = x\Gamma(x)$.

8.20 Theorem *If $x > 0$ and $y > 0$, then*

$$(96) \quad \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

This integral is the so-called *beta function* $B(x, y)$.

Proof Note that $B(1, y) = 1/y$, that $\log B(x, y)$ is a convex function of x , for each fixed y , by Hölder's inequality, as in Theorem 8.18, and that

$$(97) \quad B(x+1, y) = \frac{x}{x+y} B(x, y).$$

To prove (97), perform an integration by parts on

$$B(x+1, y) = \int_0^1 \left(\frac{t}{1-t} \right)^x (1-t)^{x+y-1} dt.$$

These three properties of $B(x, y)$ show, for each y , that Theorem 8.19 applies to the function f defined by

$$f(x) = \frac{\Gamma(x+y)}{\Gamma(y)} B(x, y).$$

Hence $f(x) = \Gamma(x)$.

8.21 Some consequences The substitution $t = \sin^2 \theta$ turns (96) into

$$(98) \quad 2 \int_0^{\pi/2} (\sin \theta)^{2x-1} (\cos \theta)^{2y-1} d\theta = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The special case $x = y = \frac{1}{2}$ gives

$$(99) \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

The substitution $t = s^2$ turns (93) into

$$(100) \quad \Gamma(x) = 2 \int_0^\infty s^{2x-1} e^{-s^2} ds \quad (0 < x < \infty).$$

The special case $x = \frac{1}{2}$ gives

$$(101) \quad \int_{-\infty}^\infty e^{-s^2} ds = \sqrt{\pi}.$$

By (99), the identity

$$(102) \quad \Gamma(x) = \frac{2^{x-1}}{\sqrt{\pi}} \Gamma\left(\frac{x}{2}\right) \Gamma\left(\frac{x+1}{2}\right)$$

follows directly from Theorem 8.19.

8.22 Stirling's formula This provides a simple approximate expression for $\Gamma(x+1)$ when x is large (hence for $n!$ when n is large). The formula is

$$(103) \quad \lim_{x \rightarrow \infty} \frac{\Gamma(x+1)}{(x/e)^x \sqrt{2\pi x}} = 1.$$

Here is a proof. Put $t = x(1 + u)$ in (93). This gives

$$(104) \quad \Gamma(x + 1) = x^{x+1} e^{-x} \int_{-1}^{\infty} [(1 + u)e^{-u}]^x du.$$

Determine $h(u)$ so that $h(0) = 1$ and

$$(105) \quad (1 + u)e^{-u} = \exp \left[-\frac{u^2}{2} h(u) \right]$$

if $-1 < u < \infty$, $u \neq 0$. Then

$$(106) \quad h(u) = \frac{2}{u^2} [u - \log(1 + u)].$$

It follows that h is continuous, and that $h(u)$ decreases monotonically from ∞ to 0 as u increases from -1 to ∞ .

The substitution $u = s\sqrt{2/x}$ turns (104) into

$$(107) \quad \Gamma(x + 1) = x^x e^{-x} \sqrt{2x} \int_{-\infty}^{\infty} \psi_x(s) ds$$

where

$$\psi_x(s) = \begin{cases} \exp[-s^2 h(s\sqrt{2/x})] & (-\sqrt{x/2} < s < \infty), \\ 0 & (s \leq -\sqrt{x/2}). \end{cases}$$

Note the following facts about $\psi_x(s)$:

- (a) For every s , $\psi_x(s) \rightarrow e^{-s^2}$ as $x \rightarrow \infty$.
- (b) The convergence in (a) is uniform on $[-A, A]$, for every $A < \infty$.
- (c) When $s < 0$, then $0 < \psi_x(s) < e^{-s^2}$.
- (d) When $s > 0$ and $x > 1$, then $0 < \psi_x(s) < \psi_1(s)$.
- (e) $\int_0^{\infty} \psi_1(s) ds < \infty$.

The convergence theorem stated in Exercise 12 of Chap. 7 can therefore be applied to the integral (107), and shows that this integral converges to $\sqrt{\pi}$ as $x \rightarrow \infty$, by (101). This proves (103).

A more detailed version of this proof may be found in R. C. Buck's "Advanced Calculus," pp. 216–218. For two other, entirely different, proofs, see W. Feller's article in *Amer. Math. Monthly*, vol. 74, 1967, pp. 1223–1225 (with a correction in vol. 75, 1968, p. 518) and pp. 20–24 of Artin's book.

Exercise 20 gives a simpler proof of a less precise result.

EXERCISES

1. Define

$$f(x) = \begin{cases} e^{-1/x^2} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \dots$

2. Let a_{ij} be the number in the i th row and j th column of the array

$$\begin{array}{cccccc} -1 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 0 & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

so that

$$a_{ij} = \begin{cases} 0 & (i < j), \\ -1 & (i = j), \\ 2^{j-i} & (i > j). \end{cases}$$

Prove that

$$\sum_i \sum_j a_{ij} = -2, \quad \sum_j \sum_i a_{ij} = 0.$$

3. Prove that

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$$

if $a_{ij} \geq 0$ for all i and j (the case $+\infty = +\infty$ may occur).

4. Prove the following limit relations:

$$(a) \lim_{x \rightarrow 0} \frac{b^x - 1}{x} = \log b \quad (b > 0).$$

$$(b) \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1.$$

$$(c) \lim_{x \rightarrow 0} (1+x)^{1/x} = e.$$

$$(d) \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

5. Find the following limits

$$(a) \lim_{x \rightarrow 0} \frac{e - (1+x)^{1/x}}{x}.$$

$$(b) \lim_{n \rightarrow \infty} \frac{n}{\log n} [n^{1/n} - 1].$$

$$(c) \lim_{x \rightarrow 0} \frac{\tan x - x}{x(1 - \cos x)}.$$

$$(d) \lim_{x \rightarrow 0} \frac{x - \sin x}{\tan x - x}.$$

6. Suppose $f(x)f(y) = f(x+y)$ for all real x and y .

(a) Assuming that f is differentiable and not zero, prove that

$$f(x) = e^{cx}$$

where c is a constant.

(b) Prove the same thing, assuming only that f is continuous.

7. If $0 < x < \frac{\pi}{2}$, prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

8. For $n = 0, 1, 2, \dots$, and x real, prove that

$$|\sin nx| \leq n |\sin x|.$$

Note that this inequality may be false for other values of n . For instance,

$$|\sin \frac{1}{2}\pi| > \frac{1}{2} |\sin \pi|.$$

9. (a) Put $s_N = 1 + (\frac{1}{2}) + \dots + (1/N)$. Prove that

$$\lim_{N \rightarrow \infty} (s_N - \log N)$$

exists. (The limit, often denoted by γ , is called Euler's constant. Its numerical value is $0.5772\dots$. It is not known whether γ is rational or not.)

(b) Roughly how large must m be so that $N = 10^m$ satisfies $s_N > 100$?

10. Prove that $\sum 1/p$ diverges; the sum extends over all primes.

(This shows that the primes form a fairly substantial subset of the positive integers.)

Hint: Given N , let p_1, \dots, p_k be those primes that divide at least one integer $\leq N$. Then

$$\begin{aligned}\sum_{n=1}^N \frac{1}{n} &\leq \prod_{j=1}^k \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots\right) \\ &= \prod_{j=1}^k \left(1 - \frac{1}{p_j}\right)^{-1} \\ &\leq \exp \sum_{j=1}^k \frac{2}{p_j}.\end{aligned}$$

The last inequality holds because

$$(1-x)^{-1} \leq e^{2x}$$

if $0 \leq x \leq \frac{1}{2}$.

(There are many proofs of this result. See, for instance, the article by I. Niven in *Amer. Math. Monthly*, vol. 78, 1971, pp. 272–273, and the one by R. Bellman in *Amer. Math. Monthly*, vol. 50, 1943, pp. 318–319.)

11. Suppose $f \in \mathcal{R}$ on $[0, A]$ for all $A < \infty$, and $f(x) \rightarrow 1$ as $x \rightarrow +\infty$. Prove that

$$\lim_{t \rightarrow 0} t \int_0^{\infty} e^{-tx} f(x) dx = 1 \quad (t > 0).$$

12. Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \leq \delta$, $f(x) = 0$ if $\delta < |x| \leq \pi$, and $f(x + 2\pi) = f(x)$ for all x .

- (a) Compute the Fourier coefficients of f .
(b) Conclude that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \quad (0 < \delta < \pi).$$

- (c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2 \delta} = \frac{\pi - \delta}{2}.$$

- (d) Let $\delta \rightarrow 0$ and prove that

$$\int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}.$$

- (e) Put $\delta = \pi/2$ in (c). What do you get?

13. Put $f(x) = x$ if $0 \leq x < 2\pi$, and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

14. If $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$, prove that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(A recent article by E. L. Stark contains many references to series of the form $\sum n^{-s}$, where s is a positive integer. See *Math. Mag.*, vol. 47, 1974, pp. 197–202.)

15. With D_n as defined in (77), put

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^N D_n(x).$$

Prove that

$$K_N(x) = \frac{1}{N+1} \cdot \frac{1 - \cos (N+1)x}{1 - \cos x}$$

and that

$$(a) K_N \geq 0,$$

$$(b) \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x) dx = 1,$$

$$(c) K_N(x) \leq \frac{1}{N+1} \cdot \frac{2}{1 - \cos \delta} \quad \text{if } 0 < \delta \leq |x| \leq \pi.$$

If $s_N = s_N(f; x)$ is the N th partial sum of the Fourier series of f , consider the arithmetic means

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}.$$

Prove that

$$\sigma_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t) dt,$$

and hence prove Fejér's theorem:

If f is continuous, with period 2π , then $\sigma_N(f; x) \rightarrow f(x)$ uniformly on $[-\pi, \pi]$.

Hint: Use properties (a), (b), (c) to proceed as in Theorem 7.26.

16. Prove a pointwise version of Fejér's theorem:

If $f \in \mathcal{R}$ and $f(x+)$, $f(x-)$ exist for some x , then

$$\lim_{N \rightarrow \infty} \sigma_N(f; x) = \frac{1}{2}[f(x+) + f(x-)].$$

17. Assume f is bounded and monotonic on $[-\pi, \pi]$, with Fourier coefficients c_n , as given by (62).

(a) Use Exercise 17 of Chap. 6 to prove that $\{nc_n\}$ is a bounded sequence.

(b) Combine (a) with Exercise 16 and with Exercise 14(e) of Chap. 3, to conclude that

$$\lim_{N \rightarrow \infty} s_N(f; x) = \frac{1}{2}[f(x+) + f(x-)]$$

for every x .

(c) Assume only that $f \in \mathcal{R}$ on $[-\pi, \pi]$ and that f is monotonic in some segment $(\alpha, \beta) \subset [-\pi, \pi]$. Prove that the conclusion of (b) holds for every $x \in (\alpha, \beta)$.

(This is an application of the localization theorem.)

18. Define

$$\begin{aligned} f(x) &= x^3 - \sin^2 x \tan x \\ g(x) &= 2x^2 - \sin^2 x - x \tan x. \end{aligned}$$

Find out, for each of these two functions, whether it is positive or negative for all $x \in (0, \pi/2)$, or whether it changes sign. Prove your answer.

19. Suppose f is a continuous function on R^1 , $f(x + 2\pi) = f(x)$, and α/π is irrational. Prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x + n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

for every x . *Hint:* Do it first for $f(x) = e^{ikx}$.

20. The following simple computation yields a good approximation to Stirling's formula.

For $m = 1, 2, 3, \dots$, define

$$f(x) = (m + 1 - x) \log m + (x - m) \log(m + 1)$$

if $m \leq x \leq m + 1$, and define

$$g(x) = \frac{x}{m} - 1 + \log m$$

if $m - \frac{1}{2} \leq x < m + \frac{1}{2}$. Draw the graphs of f and g . Note that $f(x) \leq \log x \leq g(x)$ if $x \geq 1$ and that

$$\int_1^n f(x) dx = \log(n!) - \frac{1}{2} \log n > -\frac{1}{2} + \int_1^n g(x) dx.$$

Integrate $\log x$ over $[1, n]$. Conclude that

$$\frac{7}{8} < \log(n!) - (n + \frac{1}{2}) \log n + n < 1$$

for $n = 2, 3, 4, \dots$ (Note: $\log \sqrt{2\pi} \sim 0.918\dots$) Thus

$$e^{7/8} < \frac{n!}{(n/e)^n \sqrt{n}} < e.$$

21. Let

$$L_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| dt \quad (n = 1, 2, 3, \dots).$$

Prove that there exists a constant $C > 0$ such that

$$L_n > C \log n \quad (n = 1, 2, 3, \dots),$$

or, more precisely, that the sequence

$$\left\{ L_n - \frac{4}{\pi^2} \log n \right\}$$

is bounded.

22. If α is real and $-1 < x < 1$, prove Newton's binomial theorem

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n.$$

Hint: Denote the right side by $f(x)$. Prove that the series converges. Prove that

$$(1+x)f'(x) = \alpha f(x)$$

and solve this differential equation.

Show also that

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n! \Gamma(\alpha)} x^n$$

if $-1 < x < 1$ and $\alpha > 0$.

23. Let γ be a continuously differentiable *closed* curve in the complex plane, with parameter interval $[a, b]$, and assume that $\gamma(t) \neq 0$ for every $t \in [a, b]$. Define the *index* of γ to be

$$\text{Ind}(\gamma) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt.$$

Prove that $\text{Ind}(\gamma)$ is always an integer.

Hint: There exists φ on $[a, b]$ with $\varphi' = \gamma'/\gamma$, $\varphi(a) = 0$. Hence $\gamma \exp(-\varphi)$ is constant. Since $\gamma(a) = \gamma(b)$ it follows that $\exp \varphi(b) = \exp \varphi(a) = 1$. Note that $\varphi(b) = 2\pi i \text{Ind}(\gamma)$.

Compute $\text{Ind}(\gamma)$ when $\gamma(t) = e^{int}$, $a = 0$, $b = 2\pi$.

Explain why $\text{Ind}(\gamma)$ is often called the *winding number* of γ around 0.

24. Let γ be as in Exercise 23, and assume in addition that the range of γ does not intersect the negative real axis. Prove that $\text{Ind}(\gamma) = 0$. *Hint:* For $0 \leq c < \infty$, $\text{Ind}(\gamma + c)$ is a continuous integer-valued function of c . Also, $\text{Ind}(\gamma + c) \rightarrow 0$ as $c \rightarrow \infty$.

25. Suppose γ_1 and γ_2 are curves as in Exercise 23, and

$$|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)| \quad (a \leq t \leq b).$$

Prove that $\text{Ind}(\gamma_1) = \text{Ind}(\gamma_2)$.

Hint: Put $\gamma = \gamma_2/\gamma_1$. Then $|1 - \gamma| < 1$, hence $\text{Ind}(\gamma) = 0$, by Exercise 24.

Also,

$$\frac{\gamma'}{\gamma} = \frac{\gamma_2'}{\gamma_2} - \frac{\gamma_1'}{\gamma_1}.$$

26. Let γ be a *closed* curve in the complex plane (not necessarily differentiable) with parameter interval $[0, 2\pi]$, such that $\gamma(t) \neq 0$ for every $t \in [0, 2\pi]$.

Choose $\delta > 0$ so that $|\gamma(t)| > \delta$ for all $t \in [0, 2\pi]$. If P_1 and P_2 are trigonometric polynomials such that $|P_j(t) - \gamma(t)| < \delta/4$ for all $t \in [0, 2\pi]$ (their existence is assured by Theorem 8.15), prove that

$$\text{Ind}(P_1) = \text{Ind}(P_2)$$

by applying Exercise 25.

Define this common value to be $\text{Ind}(\gamma)$.

Prove that the statements of Exercises 24 and 25 hold without any differentiability assumption.

27. Let f be a continuous complex function defined in the complex plane. Suppose there is a positive integer n and a complex number $c \neq 0$ such that

$$\lim_{|z| \rightarrow \infty} z^{-n} f(z) = c.$$

Prove that $f(z) = 0$ for at least one complex number z .

Note that this is a generalization of Theorem 8.8.

Hint: Assume $f(z) \neq 0$ for all z , define

$$\gamma_r(t) = f(re^{it})$$

for $0 \leq r < \infty$, $0 \leq t \leq 2\pi$, and prove the following statements about the curves γ_r :

(a) $\text{Ind}(\gamma_0) = 0$.

(b) $\text{Ind}(\gamma_r) = n$ for all sufficiently large r .

(c) $\text{Ind}(\gamma_r)$ is a continuous function of r , on $[0, \infty)$.

[In (b) and (c), use the last part of Exercise 26.]

Show that (a), (b), and (c) are contradictory, since $n > 0$.

28. Let \bar{D} be the closed unit disc in the complex plane. (Thus $z \in \bar{D}$ if and only if $|z| \leq 1$.) Let g be a continuous mapping of \bar{D} into the unit circle T . (Thus, $|g(z)| = 1$ for every $z \in \bar{D}$.)

Prove that $g(z) = -z$ for at least one $z \in T$.

Hint: For $0 \leq r \leq 1$, $0 \leq t \leq 2\pi$, put

$$\gamma_r(t) = g(re^{it}),$$

and put $\psi(t) = e^{-it}\gamma_1(t)$. If $g(z) \neq -z$ for every $z \in T$, then $\psi(t) \neq -1$ for every $t \in [0, 2\pi]$. Hence $\text{Ind}(\psi) = 0$, by Exercises 24 and 26. It follows that $\text{Ind}(\gamma_1) = 1$. But $\text{Ind}(\gamma_0) = 0$. Derive a contradiction, as in Exercise 27.

29. Prove that every continuous mapping f of \bar{D} into \bar{D} has a fixed point in \bar{D} .

(This is the 2-dimensional case of Brouwer's fixed-point theorem.)

Hint: Assume $f(z) \neq z$ for every $z \in \bar{D}$. Associate to each $z \in \bar{D}$ the point $g(z) \in T$ which lies on the ray that starts at $f(z)$ and passes through z . Then g maps \bar{D} into T , $g(z) = z$ if $z \in T$, and g is continuous, because

$$g(z) = z - s(z)[f(z) - z],$$

where $s(z)$ is the unique nonnegative root of a certain quadratic equation whose coefficients are continuous functions of f and z . Apply Exercise 28.

30. Use Stirling's formula to prove that

$$\lim_{x \rightarrow \infty} \frac{\Gamma(x+c)}{x^c \Gamma(x)} = 1$$

for every real constant c .

31. In the proof of Theorem 7.26 it was shown that

$$\int_{-1}^1 (1-x^2)^n dx \geq \frac{4}{3\sqrt{n}}$$

for $n = 1, 2, 3, \dots$. Use Theorem 8.20 and Exercise 30 to show the more precise result

$$\lim_{n \rightarrow \infty} \sqrt{n} \int_{-1}^1 (1-x^2)^n dx = \sqrt{\pi}.$$

9

FUNCTIONS OF SEVERAL VARIABLES

LINEAR TRANSFORMATIONS

We begin this chapter with a discussion of sets of vectors in euclidean n -space R^n . The algebraic facts presented here extend without change to finite-dimensional vector spaces over any field of scalars. However, for our purposes it is quite sufficient to stay within the familiar framework provided by the euclidean spaces.

9.1 Definitions

- (a) A nonempty set $X \subset R^n$ is a *vector space* if $\mathbf{x} + \mathbf{y} \in X$ and $c\mathbf{x} \in X$ for all $\mathbf{x} \in X$, $\mathbf{y} \in X$, and for all scalars c .
- (b) If $\mathbf{x}_1, \dots, \mathbf{x}_k \in R^n$ and c_1, \dots, c_k are scalars, the vector

$$c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k$$

is called a *linear combination* of $\mathbf{x}_1, \dots, \mathbf{x}_k$. If $S \subset R^n$ and if E is the set of all linear combinations of elements of S , we say that S *spans* E , or that E is the *span* of S .

Observe that every span is a vector space.

(c) A set consisting of vectors $\mathbf{x}_1, \dots, \mathbf{x}_k$ (we shall use the notation $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ for such a set) is said to be *independent* if the relation $c_1\mathbf{x}_1 + \dots + c_k\mathbf{x}_k = \mathbf{0}$ implies that $c_1 = \dots = c_k = 0$. Otherwise $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is said to be *dependent*.

Observe that no independent set contains the null vector.

(d) If a vector space X contains an independent set of r vectors but contains no independent set of $r + 1$ vectors, we say that X has *dimension* r , and write: $\dim X = r$.

The set consisting of $\mathbf{0}$ alone is a vector space; its dimension is 0.

(e) An independent subset of a vector space X which spans X is called a *basis* of X .

Observe that if $B = \{\mathbf{x}_1, \dots, \mathbf{x}_r\}$ is a basis of X , then every $\mathbf{x} \in X$ has a unique representation of the form $\mathbf{x} = \sum c_j \mathbf{x}_j$. Such a representation exists since B spans X , and it is unique since B is independent. The numbers c_1, \dots, c_r are called the *coordinates* of \mathbf{x} with respect to the basis B .

The most familiar example of a basis is the set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, where \mathbf{e}_j is the vector in R^n whose j th coordinate is 1 and whose other coordinates are all 0. If $\mathbf{x} \in R^n$, $\mathbf{x} = (x_1, \dots, x_n)$, then $\mathbf{x} = \sum x_j \mathbf{e}_j$. We shall call

$$\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$$

the *standard basis* of R^n .

9.2 Theorem *Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then $\dim X \leq r$.*

Proof If this is false, there is a vector space X which contains an independent set $Q = \{\mathbf{y}_1, \dots, \mathbf{y}_{r+1}\}$ and which is spanned by a set S_0 consisting of r vectors.

Suppose $0 \leq i < r$, and suppose a set S_i has been constructed which spans X and which consists of all \mathbf{y}_j with $1 \leq j \leq i$ plus a certain collection of $r - i$ members of S_0 , say $\mathbf{x}_1, \dots, \mathbf{x}_{r-i}$. (In other words, S_i is obtained from S_0 by replacing i of its elements by members of Q , without altering the span.) Since S_i spans X , \mathbf{y}_{i+1} is in the span of S_i ; hence there are scalars $a_1, \dots, a_{i+1}, b_1, \dots, b_{r-i}$, with $a_{i+1} = 1$, such that

$$\sum_{j=1}^{i+1} a_j \mathbf{y}_j + \sum_{k=1}^{r-i} b_k \mathbf{x}_k = \mathbf{0}.$$

If all b_k 's were 0, the independence of Q would force all a_j 's to be 0, a contradiction. It follows that some $\mathbf{x}_k \in S_i$ is a linear combination of the other members of $T_i = S_i \cup \{\mathbf{y}_{i+1}\}$. Remove this \mathbf{x}_k from T_i and call the remaining set S_{i+1} . Then S_{i+1} spans the same set as T_i , namely X , so that S_{i+1} has the properties postulated for S_i with $i + 1$ in place of i .

Starting with S_0 , we thus construct sets S_1, \dots, S_r . The last of these consists of y_1, \dots, y_r , and our construction shows that it spans X . But Q is independent; hence y_{r+1} is not in the span of S_r . This contradiction establishes the theorem.

Corollary $\dim R^n = n$.

Proof Since $\{e_1, \dots, e_n\}$ spans R^n , the theorem shows that $\dim R^n \leq n$. Since $\{e_1, \dots, e_n\}$ is independent, $\dim R^n \geq n$.

9.3 Theorem Suppose X is a vector space, and $\dim X = n$.

- (a) A set E of n vectors in X spans X if and only if E is independent.
- (b) X has a basis, and every basis consists of n vectors.
- (c) If $1 \leq r \leq n$ and $\{y_1, \dots, y_r\}$ is an independent set in X , then X has a basis containing $\{y_1, \dots, y_r\}$.

Proof Suppose $E = \{x_1, \dots, x_n\}$. Since $\dim X = n$, the set $\{x_1, \dots, x_n, y\}$ is dependent, for every $y \in X$. If E is independent, it follows that y is in the span of E ; hence E spans X . Conversely, if E is dependent, one of its members can be removed without changing the span of E . Hence E cannot span X , by Theorem 9.2. This proves (a).

Since $\dim X = n$, X contains an independent set of n vectors, and (a) shows that every such set is a basis of X ; (b) now follows from 9.1(d) and 9.2.

To prove (c), let $\{x_1, \dots, x_n\}$ be a basis of X . The set

$$S = \{y_1, \dots, y_r, x_1, \dots, x_n\}$$

spans X and is dependent, since it contains more than n vectors. The argument used in the proof of Theorem 9.2 shows that one of the x_i 's is a linear combination of the other members of S . If we remove this x_i from S , the remaining set still spans X . This process can be repeated r times and leads to a basis of X which contains $\{y_1, \dots, y_r\}$, by (a).

9.4 Definitions A mapping A of a vector space X into a vector space Y is said to be a *linear transformation* if

$$A(x_1 + x_2) = Ax_1 + Ax_2, \quad A(cx) = cAx$$

for all $x, x_1, x_2 \in X$ and all scalars c . Note that one often writes Ax instead of $A(x)$ if A is linear.

Observe that $A0 = 0$ if A is linear. Observe also that a linear transformation A of X into Y is completely determined by its action on any basis: If

$\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is a basis of X , then every $\mathbf{x} \in X$ has a unique representation of the form

$$\mathbf{x} = \sum_{i=1}^n c_i \mathbf{x}_i,$$

and the linearity of A allows us to compute $A\mathbf{x}$ from the vectors $A\mathbf{x}_1, \dots, A\mathbf{x}_n$ and the coordinates c_1, \dots, c_n by the formula

$$A\mathbf{x} = \sum_{i=1}^n c_i A\mathbf{x}_i.$$

Linear transformations of X into X are often called *linear operators* on X . If A is a linear operator on X which (i) is one-to-one and (ii) maps X onto X , we say that A is *invertible*. In this case we can define an operator A^{-1} on X by requiring that $A^{-1}(A\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$. It is trivial to verify that we then also have $A(A^{-1}\mathbf{x}) = \mathbf{x}$, for all $\mathbf{x} \in X$, and that A^{-1} is linear.

An important fact about linear operators on finite-dimensional vector spaces is that each of the above conditions (i) and (ii) implies the other:

9.5 Theorem *A linear operator A on a finite-dimensional vector space X is one-to-one if and only if the range of A is all of X .*

Proof Let $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ be a basis of X . The linearity of A shows that its range $\mathcal{R}(A)$ is the span of the set $Q = \{A\mathbf{x}_1, \dots, A\mathbf{x}_n\}$. We therefore infer from Theorem 9.3(a) that $\mathcal{R}(A) = X$ if and only if Q is independent. We have to prove that this happens if and only if A is one-to-one.

Suppose A is one-to-one and $\sum c_i A\mathbf{x}_i = \mathbf{0}$. Then $A(\sum c_i \mathbf{x}_i) = \mathbf{0}$, hence $\sum c_i \mathbf{x}_i = \mathbf{0}$, hence $c_1 = \dots = c_n = 0$, and we conclude that Q is independent.

Conversely, suppose Q is independent and $A(\sum c_i \mathbf{x}_i) = \mathbf{0}$. Then $\sum c_i A\mathbf{x}_i = \mathbf{0}$, hence $c_1 = \dots = c_n = 0$, and we conclude: $A\mathbf{x} = \mathbf{0}$ only if $\mathbf{x} = \mathbf{0}$. If now $A\mathbf{x} = A\mathbf{y}$, then $A(\mathbf{x} - \mathbf{y}) = A\mathbf{x} - A\mathbf{y} = \mathbf{0}$, so that $\mathbf{x} - \mathbf{y} = \mathbf{0}$, and this says that A is one-to-one.

9.6 Definitions

(a) Let $L(X, Y)$ be the set of all linear transformations of the vector space X into the vector space Y . Instead of $L(X, X)$, we shall simply write $L(X)$. If $A_1, A_2 \in L(X, Y)$ and if c_1, c_2 are scalars, define $c_1 A_1 + c_2 A_2$ by

$$(c_1 A_1 + c_2 A_2)\mathbf{x} = c_1 A_1 \mathbf{x} + c_2 A_2 \mathbf{x} \quad (\mathbf{x} \in X).$$

It is then clear that $c_1 A_1 + c_2 A_2 \in L(X, Y)$.

(b) If X, Y, Z are vector spaces, and if $A \in L(X, Y)$ and $B \in L(Y, Z)$, we define their *product* BA to be the composition of A and B :

$$(BA)\mathbf{x} = B(A\mathbf{x}) \quad (\mathbf{x} \in X).$$

Then $BA \in L(X, Z)$.

Note that BA need not be the same as AB , even if $X = Y = Z$.

(c) For $A \in L(R^n, R^m)$, define the *norm* $\|A\|$ of A to be the sup of all numbers $|Ax|$, where x ranges over all vectors in R^n with $|x| \leq 1$.

Observe that the inequality

$$|Ax| \leq \|A\| |x|$$

holds for all $x \in R^n$. Also, if λ is such that $|Ax| \leq \lambda|x|$ for all $x \in R^n$, then $\|A\| \leq \lambda$.

9.7 Theorem

(a) If $A \in L(R^n, R^m)$, then $\|A\| < \infty$ and A is a uniformly continuous mapping of R^n into R^m .

(b) If $A, B \in L(R^n, R^m)$ and c is a scalar, then

$$\|A + B\| \leq \|A\| + \|B\|, \quad \|cA\| = |c| \|A\|.$$

With the distance between A and B defined as $\|A - B\|$, $L(R^n, R^m)$ is a metric space.

(c) If $A \in L(R^n, R^m)$ and $B \in L(R^m, R^k)$, then

$$\|BA\| \leq \|B\| \|A\|.$$

Proof

(a) Let $\{e_1, \dots, e_n\}$ be the standard basis in R^n and suppose $x = \sum c_i e_i$, $|x| \leq 1$, so that $|c_i| \leq 1$ for $i = 1, \dots, n$. Then

$$|Ax| = \left| \sum c_i A e_i \right| \leq \sum |c_i| |A e_i| \leq \sum |A e_i|$$

so that

$$\|A\| \leq \sum_{i=1}^n |A e_i| < \infty.$$

Since $|Ax - Ay| \leq \|A\| |x - y|$ if $x, y \in R^n$, we see that A is uniformly continuous.

(b) The inequality in (b) follows from

$$|(A + B)x| = |Ax + Bx| \leq |Ax| + |Bx| \leq (\|A\| + \|B\|) |x|.$$

The second part of (b) is proved in the same manner. If

$$A, B, C \in L(R^n, R^m),$$

we have the triangle inequality

$$\|A - C\| = \|(A - B) + (B - C)\| \leq \|A - B\| + \|B - C\|,$$

and it is easily verified that $\|A - B\|$ has the other properties of a metric (Definition 2.15).

(c) Finally, (c) follows from

$$|(BA)\mathbf{x}| = |B(A\mathbf{x})| \leq \|B\| |A\mathbf{x}| \leq \|B\| \|A\| |\mathbf{x}|.$$

Since we now have metrics in the spaces $L(R^n, R^m)$, the concepts of open set, continuity, etc., make sense for these spaces. Our next theorem utilizes these concepts.

9.8 Theorem *Let Ω be the set of all invertible linear operators on R^n .*

(a) *If $A \in \Omega$, $B \in L(R^n)$, and*

$$\|B - A\| \cdot \|A^{-1}\| < 1,$$

then $B \in \Omega$.

(b) *Ω is an open subset of $L(R^n)$, and the mapping $A \rightarrow A^{-1}$ is continuous on Ω .*

(This mapping is also obviously a 1-1 mapping of Ω onto Ω , which is its own inverse.)

Proof

(a) Put $\|A^{-1}\| = 1/\alpha$, put $\|B - A\| = \beta$. Then $\beta < \alpha$. For every $\mathbf{x} \in R^n$,

$$\begin{aligned} \alpha |\mathbf{x}| &= \alpha |A^{-1}A\mathbf{x}| \leq \alpha \|A^{-1}\| \cdot |A\mathbf{x}| \\ &= |A\mathbf{x}| \leq |(A - B)\mathbf{x}| + |B\mathbf{x}| \leq \beta |\mathbf{x}| + |B\mathbf{x}|, \end{aligned}$$

so that

$$(1) \quad (\alpha - \beta) |\mathbf{x}| \leq |B\mathbf{x}| \quad (\mathbf{x} \in R^n).$$

Since $\alpha - \beta > 0$, (1) shows that $B\mathbf{x} \neq 0$ if $\mathbf{x} \neq 0$. Hence B is 1-1. By Theorem 9.5, $B \in \Omega$. This holds for all B with $\|B - A\| < \alpha$. Thus we have (a) and the fact that Ω is open.

(b) Next, replace \mathbf{x} by $B^{-1}\mathbf{y}$ in (1). The resulting inequality

$$(2) \quad (\alpha - \beta) |B^{-1}\mathbf{y}| \leq |BB^{-1}\mathbf{y}| = |\mathbf{y}| \quad (\mathbf{y} \in R^n)$$

shows that $\|B^{-1}\| \leq (\alpha - \beta)^{-1}$. The identity

$$B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1},$$

combined with Theorem 9.7(c), implies therefore that

$$\|B^{-1} - A^{-1}\| \leq \|B^{-1}\| \|A - B\| \|A^{-1}\| \leq \frac{\beta}{\alpha(\alpha - \beta)}.$$

This establishes the continuity assertion made in (b), since $\beta \rightarrow 0$ as $B \rightarrow A$.

the independence of $\{z_1, \dots, z_p\}$ implies that

$$(5) \quad c_{kj} = \sum_i b_{ki} a_{ij} \quad (1 \leq k \leq p, 1 \leq j \leq n).$$

This shows how to compute the p by n matrix $[BA]$ from $[B]$ and $[A]$. If we define the product $[B][A]$ to be $[BA]$, then (5) describes the usual rule of matrix multiplication.

Finally, suppose $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ are standard bases of R^n and R^m , and A is given by (4). The Schwarz inequality shows that

$$|Ax|^2 = \sum_i \left(\sum_j a_{ij} c_j \right)^2 \leq \sum_i \left(\sum_j a_{ij}^2 \cdot \sum_j c_j^2 \right) = \sum_{i,j} a_{ij}^2 |x|^2.$$

Thus

$$(6) \quad \|A\| \leq \left\{ \sum_{i,j} a_{ij}^2 \right\}^{1/2}.$$

If we apply (6) to $B - A$ in place of A , where $A, B \in L(R^n, R^m)$, we see that if the matrix elements a_{ij} are continuous functions of a parameter, then the same is true of A . More precisely:

If S is a metric space, if a_{11}, \dots, a_{mn} are real continuous functions on S , and if, for each $p \in S$, A_p is the linear transformation of R^n into R^m whose matrix has entries $a_{ij}(p)$, then the mapping $p \rightarrow A_p$ is a continuous mapping of S into $L(R^n, R^m)$.

DIFFERENTIATION

9.10 Preliminaries In order to arrive at a definition of the derivative of a function whose domain is R^n (or an open subset of R^n), let us take another look at the familiar case $n = 1$, and let us see how to interpret the derivative in that case in a way which will naturally extend to $n > 1$.

If f is a real function with domain $(a, b) \subset R^1$ and if $x \in (a, b)$, then $f'(x)$ is usually defined to be the real number

$$(7) \quad \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h},$$

provided, of course, that this limit exists. Thus

$$(8) \quad f(x+h) - f(x) = f'(x)h + r(h)$$

where the “remainder” $r(h)$ is small, in the sense that

$$(9) \quad \lim_{h \rightarrow 0} \frac{r(h)}{h} = 0.$$

Note that (8) expresses the difference $f(x+h) - f(x)$ as the sum of the *linear function* that takes h to $f'(x)h$, plus a small remainder.

We can therefore regard the derivative of f at x , not as a real number, but as the linear operator on R^1 that takes h to $f'(x)h$.

[Observe that every real number α gives rise to a linear operator on R^1 ; the operator in question is simply multiplication by α . Conversely, every linear function that carries R^1 to R^1 is multiplication by some real number. It is this natural 1-1 correspondence between R^1 and $L(R^1)$ which motivates the preceding statements.]

Let us next consider a function \mathbf{f} that maps $(a, b) \subset R^1$ into R^m . In that case, $\mathbf{f}'(x)$ was defined to be that vector $\mathbf{y} \in R^m$ (if there is one) for which

$$(10) \quad \lim_{h \rightarrow 0} \left\{ \frac{\mathbf{f}(x+h) - \mathbf{f}(x)}{h} - \mathbf{y} \right\} = \mathbf{0}.$$

We can again rewrite this in the form

$$(11) \quad \mathbf{f}(x+h) - \mathbf{f}(x) = h\mathbf{y} + \mathbf{r}(h),$$

where $\mathbf{r}(h)/h \rightarrow \mathbf{0}$ as $h \rightarrow 0$. The main term on the right side of (11) is again a *linear function* of h . Every $\mathbf{y} \in R^m$ induces a linear transformation of R^1 into R^m , by associating to each $h \in R^1$ the vector $h\mathbf{y} \in R^m$. This identification of R^m with $L(R^1, R^m)$ allows us to regard $\mathbf{f}'(x)$ as a member of $L(R^1, R^m)$.

Thus, if \mathbf{f} is a differentiable mapping of $(a, b) \subset R^1$ into R^m , and if $x \in (a, b)$, then $\mathbf{f}'(x)$ is the linear transformation of R^1 into R^m that satisfies

$$(12) \quad \lim_{h \rightarrow 0} \frac{\mathbf{f}(x+h) - \mathbf{f}(x) - \mathbf{f}'(x)h}{h} = \mathbf{0},$$

or, equivalently,

$$(13) \quad \lim_{h \rightarrow 0} \frac{|\mathbf{f}(x+h) - \mathbf{f}(x) - \mathbf{f}'(x)h|}{|h|} = 0.$$

We are now ready for the case $n > 1$.

9.11 Definition Suppose E is an open set in R^n , \mathbf{f} maps E into R^m , and $\mathbf{x} \in E$. If there exists a linear transformation A of R^n into R^m such that

$$(14) \quad \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{|\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) - A\mathbf{h}|}{|\mathbf{h}|} = 0,$$

then we say that \mathbf{f} is *differentiable at \mathbf{x}* , and we write

$$(15) \quad \mathbf{f}'(\mathbf{x}) = A.$$

If \mathbf{f} is differentiable at every $\mathbf{x} \in E$, we say that \mathbf{f} is *differentiable in E* .

It is of course understood in (14) that $\mathbf{h} \in R^n$. If $|\mathbf{h}|$ is small enough, then $\mathbf{x} + \mathbf{h} \in E$, since E is open. Thus $\mathbf{f}(\mathbf{x} + \mathbf{h})$ is defined, $\mathbf{f}(\mathbf{x} + \mathbf{h}) \in R^m$, and since $A \in L(R^n, R^m)$, $A\mathbf{h} \in R^m$. Thus

$$\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) - A\mathbf{h} \in R^m.$$

The norm in the numerator of (14) is that of R^m . In the denominator we have the R^n -norm of \mathbf{h} .

There is an obvious uniqueness problem which has to be settled before we go any further.

9.12 Theorem Suppose E and \mathbf{f} are as in Definition 9.11, $\mathbf{x} \in E$, and (14) holds with $A = A_1$ and with $A = A_2$. Then $A_1 = A_2$.

Proof If $B = A_1 - A_2$, the inequality

$$|B\mathbf{h}| \leq |\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) - A_1\mathbf{h}| + |\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) - A_2\mathbf{h}|$$

shows that $|B\mathbf{h}|/|\mathbf{h}| \rightarrow 0$ as $\mathbf{h} \rightarrow \mathbf{0}$. For fixed $\mathbf{h} \neq \mathbf{0}$, it follows that

$$(16) \quad \frac{|B(t\mathbf{h})|}{|t\mathbf{h}|} \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

The linearity of B shows that the left side of (16) is independent of t . Thus $B\mathbf{h} = 0$ for every $\mathbf{h} \in R^n$. Hence $B = 0$.

9.13 Remarks

(a) The relation (14) can be rewritten in the form

$$(17) \quad \mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) = \mathbf{f}'(\mathbf{x})\mathbf{h} + \mathbf{r}(\mathbf{h})$$

where the remainder $\mathbf{r}(\mathbf{h})$ satisfies

$$(18) \quad \lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{|\mathbf{r}(\mathbf{h})|}{|\mathbf{h}|} = 0.$$

We may interpret (17), as in Sec. 9.10, by saying that for fixed \mathbf{x} and small \mathbf{h} , the left side of (17) is approximately equal to $\mathbf{f}'(\mathbf{x})\mathbf{h}$, that is, to the value of a linear transformation applied to \mathbf{h} .

(b) Suppose \mathbf{f} and E are as in Definition 9.11, and \mathbf{f} is differentiable in E . For every $\mathbf{x} \in E$, $\mathbf{f}'(\mathbf{x})$ is then a function, namely, a linear transformation of R^n into R^m . But \mathbf{f}' is also a function: \mathbf{f}' maps E into $L(R^n, R^m)$.

(c) A glance at (17) shows that \mathbf{f} is continuous at any point at which \mathbf{f} is differentiable.

(d) The derivative defined by (14) or (17) is often called the *differential* of \mathbf{f} at \mathbf{x} , or the *total derivative* of \mathbf{f} at \mathbf{x} , to distinguish it from the partial derivatives that will occur later.

9.14 Example We have defined derivatives of functions carrying R^n to R^m to be linear transformations of R^n into R^m . What is the derivative of such a linear transformation? The answer is very simple.

If $A \in L(R^n, R^m)$ and if $\mathbf{x} \in R^n$, then

$$(19) \quad A'(\mathbf{x}) = A.$$

Note that \mathbf{x} appears on the left side of (19), but not on the right. Both sides of (19) are members of $L(R^n, R^m)$, whereas $A\mathbf{x} \in R^m$.

The proof of (19) is a triviality, since

$$(20) \quad A(\mathbf{x} + \mathbf{h}) - A\mathbf{x} = A\mathbf{h},$$

by the linearity of A . With $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$, the numerator in (14) is thus 0 for every $\mathbf{h} \in R^n$. In (17), $\mathbf{r}(\mathbf{h}) = \mathbf{0}$.

We now extend the chain rule (Theorem 5.5) to the present situation.

9.15 Theorem Suppose E is an open set in R^n , \mathbf{f} maps E into R^m , \mathbf{f} is differentiable at $\mathbf{x}_0 \in E$, \mathbf{g} maps an open set containing $\mathbf{f}(E)$ into R^k , and \mathbf{g} is differentiable at $\mathbf{f}(\mathbf{x}_0)$. Then the mapping \mathbf{F} of E into R^k defined by

$$\mathbf{F}(\mathbf{x}) = \mathbf{g}(\mathbf{f}(\mathbf{x}))$$

is differentiable at \mathbf{x}_0 , and

$$(21) \quad \mathbf{F}'(\mathbf{x}_0) = \mathbf{g}'(\mathbf{f}(\mathbf{x}_0))\mathbf{f}'(\mathbf{x}_0).$$

On the right side of (21), we have the product of two linear transformations, as defined in Sec. 9.6.

Proof Put $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$, $A = \mathbf{f}'(\mathbf{x}_0)$, $B = \mathbf{g}'(\mathbf{y}_0)$, and define

$$\mathbf{u}(\mathbf{h}) = \mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - A\mathbf{h},$$

$$\mathbf{v}(\mathbf{k}) = \mathbf{g}(\mathbf{y}_0 + \mathbf{k}) - \mathbf{g}(\mathbf{y}_0) - B\mathbf{k},$$

for all $\mathbf{h} \in R^n$ and $\mathbf{k} \in R^m$ for which $\mathbf{f}(\mathbf{x}_0 + \mathbf{h})$ and $\mathbf{g}(\mathbf{y}_0 + \mathbf{k})$ are defined. Then

$$(22) \quad |\mathbf{u}(\mathbf{h})| = \varepsilon(\mathbf{h})|\mathbf{h}|, \quad |\mathbf{v}(\mathbf{k})| = \eta(\mathbf{k})|\mathbf{k}|,$$

where $\varepsilon(\mathbf{h}) \rightarrow 0$ as $\mathbf{h} \rightarrow \mathbf{0}$ and $\eta(\mathbf{k}) \rightarrow 0$ as $\mathbf{k} \rightarrow \mathbf{0}$.

Given \mathbf{h} , put $\mathbf{k} = \mathbf{f}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{f}(\mathbf{x}_0)$. Then

$$(23) \quad |\mathbf{k}| = |A\mathbf{h} + \mathbf{u}(\mathbf{h})| \leq [\|A\| + \varepsilon(\mathbf{h})] |\mathbf{h}|,$$

and

$$\begin{aligned} \mathbf{F}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{F}(\mathbf{x}_0) - B A \mathbf{h} &= \mathbf{g}(\mathbf{y}_0 + \mathbf{k}) - \mathbf{g}(\mathbf{y}_0) - B A \mathbf{h} \\ &= B(\mathbf{k} - A\mathbf{h}) + \mathbf{v}(\mathbf{k}) \\ &= B\mathbf{u}(\mathbf{h}) + \mathbf{v}(\mathbf{k}). \end{aligned}$$

Hence (22) and (23) imply, for $\mathbf{h} \neq \mathbf{0}$, that

$$\frac{|\mathbf{F}(\mathbf{x}_0 + \mathbf{h}) - \mathbf{F}(\mathbf{x}_0) - B\mathbf{A}\mathbf{h}|}{|\mathbf{h}|} \leq \|B\| \varepsilon(\mathbf{h}) + [\|A\| + \varepsilon(\mathbf{h})]\eta(\mathbf{k}).$$

Let $\mathbf{h} \rightarrow \mathbf{0}$. Then $\varepsilon(\mathbf{h}) \rightarrow 0$. Also, $\mathbf{k} \rightarrow \mathbf{0}$, by (23), so that $\eta(\mathbf{k}) \rightarrow 0$. It follows that $\mathbf{F}'(\mathbf{x}_0) = B\mathbf{A}$, which is what (21) asserts.

9.16 Partial derivatives We again consider a function \mathbf{f} that maps an open set $E \subset R^n$ into R^m . Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ be the standard bases of R^n and R^m . The *components of \mathbf{f}* are the real functions f_1, \dots, f_m defined by

$$(24) \quad \mathbf{f}(\mathbf{x}) = \sum_{i=1}^m f_i(\mathbf{x})\mathbf{u}_i \quad (\mathbf{x} \in E),$$

or, equivalently, by $f_i(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \cdot \mathbf{u}_i$, $1 \leq i \leq m$.

For $\mathbf{x} \in E$, $1 \leq i \leq m$, $1 \leq j \leq n$, we define

$$(25) \quad (D_j f_i)(\mathbf{x}) = \lim_{t \rightarrow 0} \frac{f_i(\mathbf{x} + t\mathbf{e}_j) - f_i(\mathbf{x})}{t},$$

provided the limit exists. Writing $f_i(x_1, \dots, x_n)$ in place of $f_i(\mathbf{x})$, we see that $D_j f_i$ is the derivative of f_i with respect to x_j , keeping the other variables fixed. The notation

$$(26) \quad \frac{\partial f_i}{\partial x_j}$$

is therefore often used in place of $D_j f_i$, and $D_j f_i$ is called a *partial derivative*.

In many cases where the existence of a derivative is sufficient when dealing with functions of one variable, continuity or at least boundedness of the partial derivatives is needed for functions of several variables. For example, the functions f and g described in Exercise 7, Chap. 4, are not continuous, although their partial derivatives exist at every point of R^2 . Even for continuous functions, the existence of all partial derivatives does not imply differentiability in the sense of Definition 9.11; see Exercises 6 and 14, and Theorem 9.21.

However, if \mathbf{f} is known to be differentiable at a point \mathbf{x} , then its partial derivatives exist at \mathbf{x} , and they determine the linear transformation $\mathbf{f}'(\mathbf{x})$ completely:

9.17 Theorem Suppose \mathbf{f} maps an open set $E \subset R^n$ into R^m , and \mathbf{f} is differentiable at a point $\mathbf{x} \in E$. Then the partial derivatives $(D_j f_i)(\mathbf{x})$ exist, and

$$(27) \quad \mathbf{f}'(\mathbf{x})\mathbf{e}_j = \sum_{i=1}^m (D_j f_i)(\mathbf{x})\mathbf{u}_i \quad (1 \leq j \leq n).$$

Here, as in Sec. 9.16, $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ are the standard bases of R^n and R^m .

Proof Fix j . Since \mathbf{f} is differentiable at \mathbf{x} ,

$$\mathbf{f}(\mathbf{x} + t\mathbf{e}_j) - \mathbf{f}(\mathbf{x}) = \mathbf{f}'(\mathbf{x})(t\mathbf{e}_j) + \mathbf{r}(t\mathbf{e}_j)$$

where $|\mathbf{r}(t\mathbf{e}_j)|/t \rightarrow 0$ as $t \rightarrow 0$. The linearity of $\mathbf{f}'(\mathbf{x})$ shows therefore that

$$(28) \quad \lim_{t \rightarrow 0} \frac{\mathbf{f}(\mathbf{x} + t\mathbf{e}_j) - \mathbf{f}(\mathbf{x})}{t} = \mathbf{f}'(\mathbf{x})\mathbf{e}_j.$$

If we now represent \mathbf{f} in terms of its components, as in (24), then (28) becomes

$$(29) \quad \lim_{t \rightarrow 0} \sum_{i=1}^m \frac{f_i(\mathbf{x} + t\mathbf{e}_j) - f_i(\mathbf{x})}{t} \mathbf{u}_i = \mathbf{f}'(\mathbf{x})\mathbf{e}_j.$$

It follows that each quotient in this sum has a limit, as $t \rightarrow 0$ (see Theorem 4.10), so that each $(D_j f_i)(\mathbf{x})$ exists, and then (27) follows from (29).

Here are some consequences of Theorem 9.17:

Let $[\mathbf{f}'(\mathbf{x})]$ be the matrix that represents $\mathbf{f}'(\mathbf{x})$ with respect to our standard bases, as in Sec. 9.9.

Then $\mathbf{f}'(\mathbf{x})\mathbf{e}_j$ is the j th column vector of $[\mathbf{f}'(\mathbf{x})]$, and (27) shows therefore that the number $(D_j f_i)(\mathbf{x})$ occupies the spot in the i th row and j th column of $[\mathbf{f}'(\mathbf{x})]$. Thus

$$[\mathbf{f}'(\mathbf{x})] = \begin{bmatrix} (D_1 f_1)(\mathbf{x}) & \cdots & (D_n f_1)(\mathbf{x}) \\ \cdots & \cdots & \cdots \\ (D_1 f_m)(\mathbf{x}) & \cdots & (D_n f_m)(\mathbf{x}) \end{bmatrix}.$$

If $\mathbf{h} = \sum h_j \mathbf{e}_j$ is any vector in R^n , then (27) implies that

$$(30) \quad \mathbf{f}'(\mathbf{x})\mathbf{h} = \sum_{i=1}^m \left\{ \sum_{j=1}^n (D_j f_i)(\mathbf{x}) h_j \right\} \mathbf{u}_i.$$

9.18 Example Let γ be a differentiable mapping of the segment $(a, b) \subset R^1$ into an open set $E \subset R^n$, in other words, γ is a differentiable curve in E . Let f be a real-valued differentiable function with domain E . Thus f is a differentiable mapping of E into R^1 . Define

$$(31) \quad g(t) = f(\gamma(t)) \quad (a < t < b).$$

The chain rule asserts then that

$$(32) \quad g'(t) = f'(\gamma(t))\gamma'(t) \quad (a < t < b).$$

Since $\gamma'(t) \in L(R^1, R^n)$ and $f'(\gamma(t)) \in L(R^n, R^1)$, (32) defines $g'(t)$ as a linear operator on R^1 . This agrees with the fact that g maps (a, b) into R^1 . However, $g'(t)$ can also be regarded as a real number. (This was discussed in Sec. 9.10.) This number can be computed in terms of the partial derivatives of f and the derivatives of the components of γ , as we shall now see.

With respect to the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of R^n , $[\gamma'(t)]$ is the n by 1 matrix (a “column matrix”) which has $\gamma'_i(t)$ in the i th row, where $\gamma_1, \dots, \gamma_n$ are the components of γ . For every $\mathbf{x} \in E$, $[f'(\mathbf{x})]$ is the 1 by n matrix (a “row matrix”) which has $(D_j f)(\mathbf{x})$ in the j th column. Hence $[g'(t)]$ is the 1 by 1 matrix whose only entry is the real number

$$(33) \quad g'(t) = \sum_{i=1}^n (D_i f)(\gamma(t)) \gamma'_i(t).$$

This is a frequently encountered special case of the chain rule. It can be rephrased in the following manner.

Associate with each $\mathbf{x} \in E$ a vector, the so-called “gradient” of f at \mathbf{x} , defined by

$$(34) \quad (\nabla f)(\mathbf{x}) = \sum_{i=1}^n (D_i f)(\mathbf{x}) \mathbf{e}_i.$$

Since

$$(35) \quad \gamma'(t) = \sum_{i=1}^n \gamma'_i(t) \mathbf{e}_i,$$

(33) can be written in the form

$$(36) \quad g'(t) = (\nabla f)(\gamma(t)) \cdot \gamma'(t),$$

the scalar product of the vectors $(\nabla f)(\gamma(t))$ and $\gamma'(t)$.

Let us now fix an $\mathbf{x} \in E$, let $\mathbf{u} \in R^n$ be a unit vector (that is, $|\mathbf{u}| = 1$), and specialize γ so that

$$(37) \quad \gamma(t) = \mathbf{x} + t\mathbf{u} \quad (-\infty < t < \infty).$$

Then $\gamma'(t) = \mathbf{u}$ for every t . Hence (36) shows that

$$(38) \quad g'(0) = (\nabla f)(\mathbf{x}) \cdot \mathbf{u}.$$

On the other hand, (37) shows that

$$g(t) - g(0) = f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x}).$$

Hence (38) gives

$$(39) \quad \lim_{t \rightarrow 0} \frac{f(\mathbf{x} + t\mathbf{u}) - f(\mathbf{x})}{t} = (\nabla f)(\mathbf{x}) \cdot \mathbf{u}.$$

The limit in (39) is usually called the *directional derivative* of f at \mathbf{x} , in the direction of the unit vector \mathbf{u} , and may be denoted by $(D_{\mathbf{u}}f)(\mathbf{x})$.

If f and \mathbf{x} are fixed, but \mathbf{u} varies, then (39) shows that $(D_{\mathbf{u}}f)(\mathbf{x})$ attains its maximum when \mathbf{u} is a positive scalar multiple of $(\nabla f)(\mathbf{x})$. [The case $(\nabla f)(\mathbf{x}) = \mathbf{0}$ should be excluded here.]

If $\mathbf{u} = \sum u_i \mathbf{e}_i$, then (39) shows that $(D_{\mathbf{u}}f)(\mathbf{x})$ can be expressed in terms of the partial derivatives of f at \mathbf{x} by the formula

$$(40) \quad (D_{\mathbf{u}}f)(\mathbf{x}) = \sum_{i=1}^n (D_i f)(\mathbf{x}) u_i.$$

Some of these ideas will play a role in the following theorem.

9.19 Theorem *Suppose \mathbf{f} maps a convex open set $E \subset R^n$ into R^m , \mathbf{f} is differentiable in E , and there is a real number M such that*

$$\|\mathbf{f}'(\mathbf{x})\| \leq M$$

for every $\mathbf{x} \in E$. Then

$$|\mathbf{f}(\mathbf{b}) - \mathbf{f}(\mathbf{a})| \leq M|\mathbf{b} - \mathbf{a}|$$

for all $\mathbf{a} \in E$, $\mathbf{b} \in E$.

Proof Fix $\mathbf{a} \in E$, $\mathbf{b} \in E$. Define

$$\gamma(t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

for all $t \in R^1$ such that $\gamma(t) \in E$. Since E is convex, $\gamma(t) \in E$ if $0 \leq t \leq 1$. Put

$$\mathbf{g}(t) = \mathbf{f}(\gamma(t)).$$

Then

$$\mathbf{g}'(t) = \mathbf{f}'(\gamma(t))\gamma'(t) = \mathbf{f}'(\gamma(t))(\mathbf{b} - \mathbf{a}),$$

so that

$$|\mathbf{g}'(t)| \leq \|\mathbf{f}'(\gamma(t))\| |\mathbf{b} - \mathbf{a}| \leq M|\mathbf{b} - \mathbf{a}|$$

for all $t \in [0, 1]$. By Theorem 5.19,

$$|\mathbf{g}(1) - \mathbf{g}(0)| \leq M|\mathbf{b} - \mathbf{a}|.$$

But $\mathbf{g}(0) = \mathbf{f}(\mathbf{a})$ and $\mathbf{g}(1) = \mathbf{f}(\mathbf{b})$. This completes the proof.

Corollary *If, in addition, $\mathbf{f}'(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in E$, then \mathbf{f} is constant.*

Proof To prove this, note that the hypotheses of the theorem hold now with $M = 0$.

9.20 Definition A differentiable mapping \mathbf{f} of an open set $E \subset R^n$ into R^m is said to be *continuously differentiable* in E if \mathbf{f}' is a continuous mapping of E into $L(R^n, R^m)$.

More explicitly, it is required that to every $\mathbf{x} \in E$ and to every $\varepsilon > 0$ corresponds a $\delta > 0$ such that

$$\|\mathbf{f}'(\mathbf{y}) - \mathbf{f}'(\mathbf{x})\| < \varepsilon$$

if $\mathbf{y} \in E$ and $|\mathbf{x} - \mathbf{y}| < \delta$.

If this is so, we also say that \mathbf{f} is a \mathcal{C}' -mapping, or that $\mathbf{f} \in \mathcal{C}'(E)$.

9.21 Theorem Suppose \mathbf{f} maps an open set $E \subset R^n$ into R^m . Then $\mathbf{f} \in \mathcal{C}'(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m$, $1 \leq j \leq n$.

Proof Assume first that $\mathbf{f} \in \mathcal{C}'(E)$. By (27),

$$(D_j f_i)(\mathbf{x}) = (\mathbf{f}'(\mathbf{x})\mathbf{e}_j) \cdot \mathbf{u}_i$$

for all i, j , and for all $\mathbf{x} \in E$. Hence

$$(D_j f_i)(\mathbf{y}) - (D_j f_i)(\mathbf{x}) = \{[\mathbf{f}'(\mathbf{y}) - \mathbf{f}'(\mathbf{x})]\mathbf{e}_j\} \cdot \mathbf{u}_i$$

and since $|\mathbf{u}_i| = |\mathbf{e}_j| = 1$, it follows that

$$\begin{aligned} |(D_j f_i)(\mathbf{y}) - (D_j f_i)(\mathbf{x})| &\leq |[\mathbf{f}'(\mathbf{y}) - \mathbf{f}'(\mathbf{x})]\mathbf{e}_j| \\ &\leq \|\mathbf{f}'(\mathbf{y}) - \mathbf{f}'(\mathbf{x})\|. \end{aligned}$$

Hence $D_j f_i$ is continuous.

For the converse, it suffices to consider the case $m = 1$. (Why?) Fix $\mathbf{x} \in E$ and $\varepsilon > 0$. Since E is open, there is an open ball $S \subset E$, with center at \mathbf{x} and radius r , and the continuity of the functions $D_j f$ shows that r can be chosen so that

$$(41) \quad |(D_j f)(\mathbf{y}) - (D_j f)(\mathbf{x})| < \frac{\varepsilon}{n} \quad (\mathbf{y} \in S, 1 \leq j \leq n).$$

Suppose $\mathbf{h} = \sum h_j \mathbf{e}_j$, $|\mathbf{h}| < r$, put $\mathbf{v}_0 = \mathbf{0}$, and $\mathbf{v}_k = h_1 \mathbf{e}_1 + \cdots + h_k \mathbf{e}_k$, for $1 \leq k \leq n$. Then

$$(42) \quad f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \sum_{j=1}^n [f(\mathbf{x} + \mathbf{v}_j) - f(\mathbf{x} + \mathbf{v}_{j-1})].$$

Since $|\mathbf{v}_k| < r$ for $1 \leq k \leq n$ and since S is convex, the segments with end points $\mathbf{x} + \mathbf{v}_{j-1}$ and $\mathbf{x} + \mathbf{v}_j$ lie in S . Since $\mathbf{v}_j = \mathbf{v}_{j-1} + h_j \mathbf{e}_j$, the mean value theorem (5.10) shows that the j th summand in (42) is equal to

$$h_j (D_j f)(\mathbf{x} + \mathbf{v}_{j-1} + \theta_j h_j \mathbf{e}_j)$$

for some $\theta_j \in (0, 1)$, and this differs from $h_j(D_j f)(\mathbf{x})$ by less than $|h_j|\varepsilon/n$, using (41). By (42), it follows that

$$\left| f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - \sum_{j=1}^n h_j(D_j f)(\mathbf{x}) \right| \leq \frac{1}{n} \sum_{j=1}^n |h_j|\varepsilon \leq |\mathbf{h}|\varepsilon$$

for all \mathbf{h} such that $|\mathbf{h}| < r$.

This says that f is differentiable at \mathbf{x} and that $f'(\mathbf{x})$ is the linear function which assigns the number $\sum h_j(D_j f)(\mathbf{x})$ to the vector $\mathbf{h} = \sum h_j \mathbf{e}_j$. The matrix $[f'(\mathbf{x})]$ consists of the row $(D_1 f)(\mathbf{x}), \dots, (D_n f)(\mathbf{x})$; and since $D_1 f, \dots, D_n f$ are continuous functions on E , the concluding remarks of Sec. 9.9 show that $f \in \mathcal{C}'(E)$.

THE CONTRACTION PRINCIPLE

We now interrupt our discussion of differentiation to insert a fixed point theorem that is valid in arbitrary complete metric spaces. It will be used in the proof of the inverse function theorem.

9.22 Definition Let X be a metric space, with metric d . If φ maps X into X and if there is a number $c < 1$ such that

$$(43) \quad d(\varphi(x), \varphi(y)) \leq c d(x, y)$$

for all $x, y \in X$, then φ is said to be a *contraction* of X into X .

9.23 Theorem If X is a complete metric space, and if φ is a contraction of X into X , then there exists one and only one $x \in X$ such that $\varphi(x) = x$.

In other words, φ has a unique fixed point. The uniqueness is a triviality, for if $\varphi(x) = x$ and $\varphi(y) = y$, then (43) gives $d(x, y) \leq c d(x, y)$, which can only happen when $d(x, y) = 0$.

The *existence* of a fixed point of φ is the essential part of the theorem. The proof actually furnishes a constructive method for locating the fixed point.

Proof Pick $x_0 \in X$ arbitrarily, and define $\{x_n\}$ recursively, by setting

$$(44) \quad x_{n+1} = \varphi(x_n) \quad (n = 0, 1, 2, \dots).$$

Choose $c < 1$ so that (43) holds. For $n \geq 1$ we then have

$$d(x_{n+1}, x_n) = d(\varphi(x_n), \varphi(x_{n-1})) \leq c d(x_n, x_{n-1}).$$

Hence induction gives

$$(45) \quad d(x_{n+1}, x_n) \leq c^n d(x_1, x_0) \quad (n = 0, 1, 2, \dots).$$

If $n < m$, it follows that

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=n+1}^m d(x_i, x_{i-1}) \\ &\leq (c^n + c^{n+1} + \cdots + c^{m-1}) d(x_1, x_0) \\ &\leq [(1 - c)^{-1} d(x_1, x_0)] c^n. \end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\lim x_n = x$ for some $x \in X$.

Since φ is a contraction, φ is continuous (in fact, uniformly continuous) on X . Hence

$$\varphi(x) = \lim_{n \rightarrow \infty} \varphi(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = x.$$

THE INVERSE FUNCTION THEOREM

The inverse function theorem states, roughly speaking, that a continuously differentiable mapping \mathbf{f} is invertible in a neighborhood of any point \mathbf{x} at which the linear transformation $\mathbf{f}'(\mathbf{x})$ is invertible:

9.24 Theorem Suppose \mathbf{f} is a \mathcal{C}' -mapping of an open set $E \subset R^n$ into R^n , $\mathbf{f}'(\mathbf{a})$ is invertible for some $\mathbf{a} \in E$, and $\mathbf{b} = \mathbf{f}(\mathbf{a})$. Then

- (a) there exist open sets U and V in R^n such that $\mathbf{a} \in U$, $\mathbf{b} \in V$, \mathbf{f} is one-to-one on U , and $\mathbf{f}(U) = V$;
- (b) if \mathbf{g} is the inverse of \mathbf{f} [which exists, by (a)], defined in V by

$$\mathbf{g}(\mathbf{f}(\mathbf{x})) = \mathbf{x} \quad (\mathbf{x} \in U),$$

then $\mathbf{g} \in \mathcal{C}'(V)$.

Writing the equation $\mathbf{y} = \mathbf{f}(\mathbf{x})$ in component form, we arrive at the following interpretation of the conclusion of the theorem: The system of n equations

$$y_i = f_i(x_1, \dots, x_n) \quad (1 \leq i \leq n)$$

can be solved for x_1, \dots, x_n in terms of y_1, \dots, y_n , if we restrict \mathbf{x} and \mathbf{y} to small enough neighborhoods of \mathbf{a} and \mathbf{b} ; the solutions are unique and continuously differentiable.

Proof

- (a) Put $\mathbf{f}'(\mathbf{a}) = A$, and choose λ so that

$$(46) \quad 2\lambda \|A^{-1}\| = 1.$$

Since \mathbf{f}' is continuous at \mathbf{a} , there is an open ball $U \subset E$, with center at \mathbf{a} , such that

$$(47) \quad \|\mathbf{f}'(\mathbf{x}) - A\| < \lambda \quad (\mathbf{x} \in U).$$

We associate to each $\mathbf{y} \in R^n$ a function φ , defined by

$$(48) \quad \varphi(\mathbf{x}) = \mathbf{x} + A^{-1}(\mathbf{y} - \mathbf{f}(\mathbf{x})) \quad (\mathbf{x} \in E).$$

Note that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$ if and only if \mathbf{x} is a fixed point of φ .

Since $\varphi'(\mathbf{x}) = I - A^{-1}\mathbf{f}'(\mathbf{x}) = A^{-1}(A - \mathbf{f}'(\mathbf{x}))$, (46) and (47) imply that

$$(49) \quad \|\varphi'(\mathbf{x})\| < \frac{1}{2} \quad (\mathbf{x} \in U).$$

Hence

$$(50) \quad |\varphi(\mathbf{x}_1) - \varphi(\mathbf{x}_2)| \leq \frac{1}{2}|\mathbf{x}_1 - \mathbf{x}_2| \quad (\mathbf{x}_1, \mathbf{x}_2 \in U),$$

by Theorem 9.19. It follows that φ has at most one fixed point in U , so that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$ for at most one $\mathbf{x} \in U$.

Thus \mathbf{f} is 1-1 in U .

Next, put $V = \mathbf{f}(U)$, and pick $\mathbf{y}_0 \in V$. Then $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$ for some $\mathbf{x}_0 \in U$. Let B be an open ball with center at \mathbf{x}_0 and radius $r > 0$, so small that its closure \bar{B} lies in U . We will show that $\mathbf{y} \in V$ whenever $|\mathbf{y} - \mathbf{y}_0| < \lambda r$. This proves, of course, that V is open.

Fix \mathbf{y} , $|\mathbf{y} - \mathbf{y}_0| < \lambda r$. With φ as in (48),

$$|\varphi(\mathbf{x}_0) - \mathbf{x}_0| = |A^{-1}(\mathbf{y} - \mathbf{y}_0)| < \|A^{-1}\|\lambda r = \frac{r}{2}.$$

If $\mathbf{x} \in \bar{B}$, it therefore follows from (50) that

$$\begin{aligned} |\varphi(\mathbf{x}) - \mathbf{x}_0| &\leq |\varphi(\mathbf{x}) - \varphi(\mathbf{x}_0)| + |\varphi(\mathbf{x}_0) - \mathbf{x}_0| \\ &< \frac{1}{2}|\mathbf{x} - \mathbf{x}_0| + \frac{r}{2} \leq r; \end{aligned}$$

hence $\varphi(\mathbf{x}) \in B$. Note that (50) holds if $\mathbf{x}_1 \in \bar{B}$, $\mathbf{x}_2 \in \bar{B}$.

Thus φ is a contraction of \bar{B} into \bar{B} . Being a closed subset of R^n , \bar{B} is complete. Theorem 9.23 implies therefore that φ has a fixed point $\mathbf{x} \in \bar{B}$. For this \mathbf{x} , $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. Thus $\mathbf{y} \in \mathbf{f}(\bar{B}) \subset \mathbf{f}(U) = V$.

This proves part (a) of the theorem.

(b) Pick $\mathbf{y} \in V$, $\mathbf{y} + \mathbf{k} \in V$. Then there exist $\mathbf{x} \in U$, $\mathbf{x} + \mathbf{h} \in U$, so that $\mathbf{y} = \mathbf{f}(\mathbf{x})$, $\mathbf{y} + \mathbf{k} = \mathbf{f}(\mathbf{x} + \mathbf{h})$. With φ as in (48),

$$\varphi(\mathbf{x} + \mathbf{h}) - \varphi(\mathbf{x}) = \mathbf{h} + A^{-1}[\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x} + \mathbf{h})] = \mathbf{h} - A^{-1}\mathbf{k}.$$

By (50), $|\mathbf{h} - A^{-1}\mathbf{k}| \leq \frac{1}{2}|\mathbf{h}|$. Hence $|A^{-1}\mathbf{k}| \geq \frac{1}{2}|\mathbf{h}|$, and

$$(51) \quad |\mathbf{h}| \leq 2\|A^{-1}\| |\mathbf{k}| = \lambda^{-1}|\mathbf{k}|.$$

By (46), (47), and Theorem 9.8, $\mathbf{f}'(\mathbf{x})$ has an inverse, say T . Since $\mathbf{g}(\mathbf{y} + \mathbf{k}) - \mathbf{g}(\mathbf{y}) - T\mathbf{k} = \mathbf{h} - T\mathbf{k} = -T[\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) - \mathbf{f}'(\mathbf{x})\mathbf{h}]$,

(51) implies

$$\frac{|\mathbf{g}(\mathbf{y} + \mathbf{k}) - \mathbf{g}(\mathbf{y}) - T\mathbf{k}|}{|\mathbf{k}|} \leq \frac{\|T\|}{\lambda} \cdot \frac{|\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) - \mathbf{f}'(\mathbf{x})\mathbf{h}|}{|\mathbf{h}|}.$$

As $\mathbf{k} \rightarrow \mathbf{0}$, (51) shows that $\mathbf{h} \rightarrow \mathbf{0}$. The right side of the last inequality thus tends to 0. Hence the same is true of the left. We have thus proved that $\mathbf{g}'(\mathbf{y}) = T$. But T was chosen to be the inverse of $\mathbf{f}'(\mathbf{x}) = \mathbf{f}'(\mathbf{g}(\mathbf{y}))$. Thus

$$(52) \quad \mathbf{g}'(\mathbf{y}) = \{\mathbf{f}'(\mathbf{g}(\mathbf{y}))\}^{-1} \quad (\mathbf{y} \in V).$$

Finally, note that \mathbf{g} is a continuous mapping of V onto U (since \mathbf{g} is differentiable), that \mathbf{f}' is a continuous mapping of U into the set Ω of all invertible elements of $L(R^n)$, and that inversion is a continuous mapping of Ω onto Ω , by Theorem 9.8. If we combine these facts with (52), we see that $\mathbf{g} \in \mathcal{C}'(V)$.

This completes the proof.

Remark. The full force of the assumption that $\mathbf{f} \in \mathcal{C}'(E)$ was only used in the last paragraph of the preceding proof. Everything else, down to Eq. (52), was derived from the existence of $\mathbf{f}'(\mathbf{x})$ for $\mathbf{x} \in E$, the invertibility of $\mathbf{f}'(\mathbf{a})$, and the continuity of \mathbf{f}' at just the point \mathbf{a} . In this connection, we refer to the article by A. Nijenhuis in *Amer. Math. Monthly*, vol. 81, 1974, pp. 969–980.

The following is an immediate consequence of part (a) of the inverse function theorem.

9.25 Theorem *If \mathbf{f} is a \mathcal{C}' -mapping of an open set $E \subset R^n$ into R^n and if $\mathbf{f}'(\mathbf{x})$ is invertible for every $\mathbf{x} \in E$, then $\mathbf{f}(W)$ is an open subset of R^n for every open set $W \subset E$.*

In other words, \mathbf{f} is an *open mapping* of E into R^n .

The hypotheses made in this theorem ensure that each point $\mathbf{x} \in E$ has a neighborhood in which \mathbf{f} is 1-1. This may be expressed by saying that \mathbf{f} is *locally* one-to-one in E . But \mathbf{f} need not be 1-1 in E under these circumstances. For an example, see Exercise 17.

THE IMPLICIT FUNCTION THEOREM

If f is a continuously differentiable real function in the plane, then the equation $f(x, y) = 0$ can be solved for y in terms of x in a neighborhood of any point

(a, b) at which $f(a, b) = 0$ and $\partial f/\partial y \neq 0$. Likewise, one can solve for x in terms of y near (a, b) if $\partial f/\partial x \neq 0$ at (a, b) . For a simple example which illustrates the need for assuming $\partial f/\partial y \neq 0$, consider $f(x, y) = x^2 + y^2 - 1$.

The preceding very informal statement is the simplest case (the case $m = n = 1$ of Theorem 9.28) of the so-called "implicit function theorem." Its proof makes strong use of the fact that continuously differentiable transformations behave locally very much like their derivatives. Accordingly, we first prove Theorem 9.27, the linear version of Theorem 9.28.

9.26 Notation If $\mathbf{x} = (x_1, \dots, x_n) \in R^n$ and $\mathbf{y} = (y_1, \dots, y_m) \in R^m$, let us write (\mathbf{x}, \mathbf{y}) for the point (or vector)

$$(x_1, \dots, x_n, y_1, \dots, y_m) \in R^{n+m}.$$

In what follows, the first entry in (\mathbf{x}, \mathbf{y}) or in a similar symbol will always be a vector in R^n , the second will be a vector in R^m .

Every $A \in L(R^{n+m}, R^n)$ can be split into two linear transformations A_x and A_y , defined by

$$(53) \quad A_x \mathbf{h} = A(\mathbf{h}, \mathbf{0}), \quad A_y \mathbf{k} = A(\mathbf{0}, \mathbf{k})$$

for any $\mathbf{h} \in R^n$, $\mathbf{k} \in R^m$. Then $A_x \in L(R^n, R^n)$, $A_y \in L(R^m, R^n)$, and

$$(54) \quad A(\mathbf{h}, \mathbf{k}) = A_x \mathbf{h} + A_y \mathbf{k}.$$

The linear version of the implicit function theorem is now almost obvious.

9.27 Theorem *If $A \in L(R^{n+m}, R^n)$ and if A_x is invertible, then there corresponds to every $\mathbf{k} \in R^m$ a unique $\mathbf{h} \in R^n$ such that $A(\mathbf{h}, \mathbf{k}) = \mathbf{0}$.*

This \mathbf{h} can be computed from \mathbf{k} by the formula

$$(55) \quad \mathbf{h} = -(A_x)^{-1} A_y \mathbf{k}.$$

Proof By (54), $A(\mathbf{h}, \mathbf{k}) = \mathbf{0}$ if and only if

$$A_x \mathbf{h} + A_y \mathbf{k} = \mathbf{0},$$

which is the same as (55) when A_x is invertible.

The conclusion of Theorem 9.27 is, in other words, that the equation $A(\mathbf{h}, \mathbf{k}) = \mathbf{0}$ can be solved (uniquely) for \mathbf{h} if \mathbf{k} is given, and that the solution \mathbf{h} is a linear function of \mathbf{k} . Those who have some acquaintance with linear algebra will recognize this as a very familiar statement about systems of linear equations.

9.28 Theorem *Let \mathbf{f} be a \mathcal{C}' -mapping of an open set $E \subset R^{n+m}$ into R^n , such that $\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$ for some point $(\mathbf{a}, \mathbf{b}) \in E$.*

Put $A = \mathbf{f}'(\mathbf{a}, \mathbf{b})$ and assume that A_x is invertible.

it follows that $F'(\mathbf{a}, \mathbf{b})$ is the linear operator on R^{n+m} that maps (\mathbf{h}, \mathbf{k}) to $(A(\mathbf{h}, \mathbf{k}), \mathbf{k})$. If this image vector is $\mathbf{0}$, then $A(\mathbf{h}, \mathbf{k}) = \mathbf{0}$ and $\mathbf{k} = \mathbf{0}$, hence $A(\mathbf{h}, \mathbf{0}) = \mathbf{0}$, and Theorem 9.27 implies that $\mathbf{h} = \mathbf{0}$. It follows that $F'(\mathbf{a}, \mathbf{b})$ is 1-1; hence it is invertible (Theorem 9.5).

The inverse function theorem can therefore be applied to F . It shows that there exist open sets U and V in R^{n+m} , with $(\mathbf{a}, \mathbf{b}) \in U$, $(\mathbf{0}, \mathbf{b}) \in V$, such that F is a 1-1 mapping of U onto V .

We let W be the set of all $\mathbf{y} \in R^m$ such that $(\mathbf{0}, \mathbf{y}) \in V$. Note that $\mathbf{b} \in W$.

It is clear that W is open since V is open.

If $\mathbf{y} \in W$, then $(\mathbf{0}, \mathbf{y}) = F(\mathbf{x}, \mathbf{y})$ for some $(\mathbf{x}, \mathbf{y}) \in U$. By (60), $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ for this \mathbf{x} .

Suppose, with the same \mathbf{y} , that $(\mathbf{x}', \mathbf{y}) \in U$ and $\mathbf{f}(\mathbf{x}', \mathbf{y}) = \mathbf{0}$. Then

$$F(\mathbf{x}', \mathbf{y}) = (\mathbf{f}(\mathbf{x}', \mathbf{y}), \mathbf{y}) = (\mathbf{f}(\mathbf{x}, \mathbf{y}), \mathbf{y}) = F(\mathbf{x}, \mathbf{y}).$$

Since F is 1-1 in U , it follows that $\mathbf{x}' = \mathbf{x}$.

This proves the first part of the theorem.

For the second part, define $\mathbf{g}(\mathbf{y})$, for $\mathbf{y} \in W$, so that $(\mathbf{g}(\mathbf{y}), \mathbf{y}) \in U$ and (57) holds. Then

$$(61) \quad F(\mathbf{g}(\mathbf{y}), \mathbf{y}) = (\mathbf{0}, \mathbf{y}) \quad (\mathbf{y} \in W).$$

If \mathbf{G} is the mapping of V onto U that inverts F , then $\mathbf{G} \in \mathcal{C}'$, by the inverse function theorem, and (61) gives

$$(62) \quad (\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{G}(\mathbf{0}, \mathbf{y}) \quad (\mathbf{y} \in W).$$

Since $\mathbf{G} \in \mathcal{C}'$, (62) shows that $\mathbf{g} \in \mathcal{C}'$.

Finally, to compute $\mathbf{g}'(\mathbf{b})$, put $(\mathbf{g}(\mathbf{y}), \mathbf{y}) = \Phi(\mathbf{y})$. Then

$$(63) \quad \Phi'(\mathbf{y})\mathbf{k} = (\mathbf{g}'(\mathbf{y})\mathbf{k}, \mathbf{k}) \quad (\mathbf{y} \in W, \mathbf{k} \in R^m).$$

By (57), $\mathbf{f}(\Phi(\mathbf{y})) = \mathbf{0}$ in W . The chain rule shows therefore that

$$\mathbf{f}'(\Phi(\mathbf{y}))\Phi'(\mathbf{y}) = \mathbf{0}.$$

When $\mathbf{y} = \mathbf{b}$, then $\Phi(\mathbf{y}) = (\mathbf{a}, \mathbf{b})$, and $\mathbf{f}'(\Phi(\mathbf{y})) = A$. Thus

$$(64) \quad A\Phi'(\mathbf{b}) = \mathbf{0}.$$

It now follows from (64), (63), and (54), that

$$A_x \mathbf{g}'(\mathbf{b})\mathbf{k} + A_y \mathbf{k} = A(\mathbf{g}'(\mathbf{b})\mathbf{k}, \mathbf{k}) = A\Phi'(\mathbf{b})\mathbf{k} = \mathbf{0}$$

for every $\mathbf{k} \in R^m$. Thus

$$(65) \quad A_x \mathbf{g}'(\mathbf{b}) + A_y = \mathbf{0}.$$

This is equivalent to (58), and completes the proof.

Note. In terms of the components of \mathbf{f} and \mathbf{g} , (65) becomes

$$\sum_{j=1}^n (D_j f_i)(\mathbf{a}, \mathbf{b})(D_k g_j)(\mathbf{b}) = -(D_{n+k} f_i)(\mathbf{a}, \mathbf{b})$$

or

$$\sum_{j=1}^n \left(\frac{\partial f_i}{\partial x_j} \right) \left(\frac{\partial g_j}{\partial y_k} \right) = - \left(\frac{\partial f_i}{\partial y_k} \right)$$

where $1 \leq i \leq n$, $1 \leq k \leq m$.

For each k , this is a system of n linear equations in which the derivatives $\partial g_j / \partial y_k$ ($1 \leq j \leq n$) are the unknowns.

9.29 Example Take $n = 2$, $m = 3$, and consider the mapping $\mathbf{f} = (f_1, f_2)$ of R^5 into R^2 given by

$$f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2 y_1 - 4y_2 + 3$$

$$f_2(x_1, x_2, y_1, y_2, y_3) = x_2 \cos x_1 - 6x_1 + 2y_1 - y_3.$$

If $\mathbf{a} = (0, 1)$ and $\mathbf{b} = (3, 2, 7)$, then $\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$.

With respect to the standard bases, the matrix of the transformation $A = \mathbf{f}'(\mathbf{a}, \mathbf{b})$ is

$$[A] = \begin{bmatrix} 2 & 3 & 1 & -4 & 0 \\ -6 & 1 & 2 & 0 & -1 \end{bmatrix}.$$

Hence

$$[A_x] = \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix}, \quad [A_y] = \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

We see that the column vectors of $[A_x]$ are independent. Hence A_x is invertible and the implicit function theorem asserts the existence of a \mathcal{C}' -mapping \mathbf{g} , defined in a neighborhood of $(3, 2, 7)$, such that $\mathbf{g}(3, 2, 7) = (0, 1)$ and $\mathbf{f}(\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{0}$.

We can use (58) to compute $\mathbf{g}'(3, 2, 7)$: Since

$$[(A_x)^{-1}] = [A_x]^{-1} = \frac{1}{20} \begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix}$$

(58) gives

$$[\mathbf{g}'(3, 2, 7)] = -\frac{1}{20} \begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{5} & -\frac{3}{20} \\ -\frac{1}{2} & \frac{6}{5} & \frac{1}{10} \end{bmatrix}.$$

In terms of partial derivatives, the conclusion is that

$$\begin{aligned} D_1 g_1 &= \frac{1}{4} & D_2 g_1 &= \frac{1}{5} & D_3 g_1 &= -\frac{3}{20} \\ D_1 g_2 &= -\frac{1}{2} & D_2 g_2 &= \frac{6}{5} & D_3 g_2 &= \frac{1}{10} \end{aligned}$$

at the point (3, 2, 7).

THE RANK THEOREM

Although this theorem is not as important as the inverse function theorem or the implicit function theorem, we include it as another interesting illustration of the general principle that the local behavior of a continuously differentiable mapping F near a point \mathbf{x} is similar to that of the linear transformation $F'(\mathbf{x})$.

Before stating it, we need a few more facts about linear transformations.

9.30 Definitions Suppose X and Y are vector spaces, and $A \in L(X, Y)$, as in Definition 9.6. The *null space* of A , $\mathcal{N}(A)$, is the set of all $\mathbf{x} \in X$ at which $A\mathbf{x} = \mathbf{0}$. It is clear that $\mathcal{N}(A)$ is a vector space in X .

Likewise, the *range* of A , $\mathcal{R}(A)$, is a vector space in Y .

The *rank* of A is defined to be the dimension of $\mathcal{R}(A)$.

For example, the invertible elements of $L(R^n)$ are precisely those whose rank is n . This follows from Theorem 9.5.

If $A \in L(X, Y)$ and A has rank 0, then $A\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in X$, hence $\mathcal{N}(A) = X$. In this connection, see Exercise 25.

9.31 Projections Let X be a vector space. An operator $P \in L(X)$ is said to be a *projection in X* if $P^2 = P$.

More explicitly, the requirement is that $P(P\mathbf{x}) = P\mathbf{x}$ for every $\mathbf{x} \in X$. In other words, P fixes every vector in its range $\mathcal{R}(P)$.

Here are some elementary properties of projections:

(a) *If P is a projection in X , then every $\mathbf{x} \in X$ has a unique representation of the form*

$$\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$$

where $\mathbf{x}_1 \in \mathcal{R}(P)$, $\mathbf{x}_2 \in \mathcal{N}(P)$.

To obtain the representation, put $\mathbf{x}_1 = P\mathbf{x}$, $\mathbf{x}_2 = \mathbf{x} - \mathbf{x}_1$. Then $P\mathbf{x}_2 = P\mathbf{x} - P\mathbf{x}_1 = P\mathbf{x} - P^2\mathbf{x} = \mathbf{0}$. As regards the uniqueness, apply P to the equation $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$. Since $\mathbf{x}_1 \in \mathcal{R}(P)$, $P\mathbf{x}_1 = \mathbf{x}_1$; since $P\mathbf{x}_2 = \mathbf{0}$, it follows that $\mathbf{x}_1 = P\mathbf{x}$.

(b) *If X is a finite-dimensional vector space and if X_1 is a vector space in X , then there is a projection P in X with $\mathcal{R}(P) = X_1$.*

If X_1 contains only $\mathbf{0}$, this is trivial: put $P\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in X$.

Assume $\dim X_1 = k > 0$. By Theorem 9.3, X has then a basis $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ such that $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$ is a basis of X_1 . Define

$$P(c_1\mathbf{u}_1 + \dots + c_n\mathbf{u}_n) = c_1\mathbf{u}_1 + \dots + c_k\mathbf{u}_k$$

for arbitrary scalars c_1, \dots, c_n .

Then $P\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in X_1$, and $X_1 = \mathcal{R}(P)$.

Note that $\{\mathbf{u}_{k+1}, \dots, \mathbf{u}_n\}$ is a basis of $\mathcal{N}(P)$. Note also that there are infinitely many projections in X , with range X_1 , if $0 < \dim X_1 < \dim X$.

9.32 Theorem Suppose m, n, r are nonnegative integers, $m \geq r, n \geq r$, \mathbf{F} is a \mathcal{C}' -mapping of an open set $E \subset R^n$ into R^m , and $\mathbf{F}'(\mathbf{x})$ has rank r for every $\mathbf{x} \in E$.

Fix $\mathbf{a} \in E$, put $A = \mathbf{F}'(\mathbf{a})$, let Y_1 be the range of A , and let P be a projection in R^m whose range is Y_1 . Let Y_2 be the null space of P .

Then there are open sets U and V in R^n , with $\mathbf{a} \in U, U \subset E$, and there is a 1-1 \mathcal{C}' -mapping \mathbf{H} of V onto U (whose inverse is also of class \mathcal{C}') such that

$$(66) \quad \mathbf{F}(\mathbf{H}(\mathbf{x})) = A\mathbf{x} + \varphi(A\mathbf{x}) \quad (\mathbf{x} \in V)$$

where φ is a \mathcal{C}' -mapping of the open set $A(V) \subset Y_1$ into Y_2 .

After the proof we shall give a more geometric description of the information that (66) contains.

Proof If $r = 0$, Theorem 9.19 shows that $\mathbf{F}(\mathbf{x})$ is constant in a neighborhood U of \mathbf{a} , and (66) holds trivially, with $V = U, \mathbf{H}(\mathbf{x}) = \mathbf{x}, \varphi(\mathbf{0}) = \mathbf{F}(\mathbf{a})$.

From now on we assume $r > 0$. Since $\dim Y_1 = r, Y_1$ has a basis $\{\mathbf{y}_1, \dots, \mathbf{y}_r\}$. Choose $\mathbf{z}_i \in R^n$ so that $A\mathbf{z}_i = \mathbf{y}_i$ ($1 \leq i \leq r$), and define a linear mapping S of Y_1 into R^n by setting

$$(67) \quad S(c_1\mathbf{y}_1 + \dots + c_r\mathbf{y}_r) = c_1\mathbf{z}_1 + \dots + c_r\mathbf{z}_r$$

for all scalars c_1, \dots, c_r .

Then $AS\mathbf{y}_i = A\mathbf{z}_i = \mathbf{y}_i$ for $1 \leq i \leq r$. Thus

$$(68) \quad AS\mathbf{y} = \mathbf{y} \quad (\mathbf{y} \in Y_1).$$

Define a mapping \mathbf{G} of E into R^n by setting

$$(69) \quad \mathbf{G}(\mathbf{x}) = \mathbf{x} + SP[\mathbf{F}(\mathbf{x}) - A\mathbf{x}] \quad (\mathbf{x} \in E).$$

Since $\mathbf{F}'(\mathbf{a}) = A$, differentiation of (69) shows that $\mathbf{G}'(\mathbf{a}) = I$, the identity operator on R^n . By the inverse function theorem, there are open sets U and V in R^n , with $\mathbf{a} \in U$, such that \mathbf{G} is a 1-1 mapping of U onto V whose inverse \mathbf{H} is also of class \mathcal{C}' . Moreover, by shrinking U and V , if necessary, we can arrange it so that V is convex and $\mathbf{H}'(\mathbf{x})$ is invertible for every $\mathbf{x} \in V$.

Note that $ASPA = A$, since $PA = A$ and (68) holds. Therefore (69) gives

$$(70) \quad AG(\mathbf{x}) = PF(\mathbf{x}) \quad (\mathbf{x} \in E).$$

In particular, (70) holds for $\mathbf{x} \in U$. If we replace \mathbf{x} by $\mathbf{H}(\mathbf{x})$, we obtain

$$(71) \quad PF(\mathbf{H}(\mathbf{x})) = A\mathbf{x} \quad (\mathbf{x} \in V).$$

Define

$$(72) \quad \psi(\mathbf{x}) = \mathbf{F}(\mathbf{H}(\mathbf{x})) - A\mathbf{x} \quad (\mathbf{x} \in V).$$

Since $PA = A$, (71) implies that $P\psi(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in V$. Thus ψ is a \mathcal{C}' -mapping of V into Y_2 .

Since V is open, it is clear that $A(V)$ is an open subset of its range $\mathcal{R}(A) = Y_1$.

To complete the proof, i.e., to go from (72) to (66), we have to show that there is a \mathcal{C}' -mapping φ of $A(V)$ into Y_2 which satisfies

$$(73) \quad \varphi(A\mathbf{x}) = \psi(\mathbf{x}) \quad (\mathbf{x} \in V).$$

As a step toward (73), we will first prove that

$$(74) \quad \psi(\mathbf{x}_1) = \psi(\mathbf{x}_2)$$

if $\mathbf{x}_1 \in V$, $\mathbf{x}_2 \in V$, $A\mathbf{x}_1 = A\mathbf{x}_2$.

Put $\Phi(\mathbf{x}) = \mathbf{F}(\mathbf{H}(\mathbf{x}))$, for $\mathbf{x} \in V$. Since $\mathbf{H}'(\mathbf{x})$ has rank n for every $\mathbf{x} \in V$, and $\mathbf{F}'(\mathbf{x})$ has rank r for every $\mathbf{x} \in U$, it follows that

$$(75) \quad \text{rank } \Phi'(\mathbf{x}) = \text{rank } \mathbf{F}'(\mathbf{H}(\mathbf{x}))\mathbf{H}'(\mathbf{x}) = r \quad (\mathbf{x} \in V).$$

Fix $\mathbf{x} \in V$. Let M be the range of $\Phi'(\mathbf{x})$. Then $M \subset R^m$, $\dim M = r$. By (71),

$$(76) \quad P\Phi'(\mathbf{x}) = A.$$

Thus P maps M onto $\mathcal{R}(A) = Y_1$. Since M and Y_1 have the same dimension, it follows that P (restricted to M) is 1-1.

Suppose now that $A\mathbf{h} = \mathbf{0}$. Then $P\Phi'(\mathbf{x})\mathbf{h} = \mathbf{0}$, by (76). But $\Phi'(\mathbf{x})\mathbf{h} \in M$, and P is 1-1 on M . Hence $\Phi'(\mathbf{x})\mathbf{h} = \mathbf{0}$. A look at (72) shows now that we have proved the following:

If $\mathbf{x} \in V$ and $A\mathbf{h} = \mathbf{0}$, then $\psi'(\mathbf{x})\mathbf{h} = \mathbf{0}$.

We can now prove (74). Suppose $\mathbf{x}_1 \in V$, $\mathbf{x}_2 \in V$, $A\mathbf{x}_1 = A\mathbf{x}_2$. Put $\mathbf{h} = \mathbf{x}_2 - \mathbf{x}_1$ and define

$$(77) \quad \mathbf{g}(t) = \psi(\mathbf{x}_1 + t\mathbf{h}) \quad (0 \leq t \leq 1).$$

The convexity of V shows that $\mathbf{x}_1 + t\mathbf{h} \in V$ for these t . Hence

$$(78) \quad \mathbf{g}'(t) = \psi'(\mathbf{x}_1 + t\mathbf{h})\mathbf{h} = \mathbf{0} \quad (0 \leq t \leq 1),$$

so that $\mathbf{g}(1) = \mathbf{g}(0)$. But $\mathbf{g}(1) = \psi(\mathbf{x}_2)$ and $\mathbf{g}(0) = \psi(\mathbf{x}_1)$. This proves (74).

By (74), $\psi(\mathbf{x})$ depends only on $A\mathbf{x}$, for $\mathbf{x} \in V$. Hence (73) defines φ unambiguously in $A(V)$. It only remains to be proved that $\varphi \in \mathcal{C}'$.

Fix $\mathbf{y}_0 \in A(V)$, fix $\mathbf{x}_0 \in V$ so that $A\mathbf{x}_0 = \mathbf{y}_0$. Since V is open, \mathbf{y}_0 has a neighborhood W in Y_1 such that the vector

$$(79) \quad \mathbf{x} = \mathbf{x}_0 + S(\mathbf{y} - \mathbf{y}_0)$$

lies in V for all $\mathbf{y} \in W$. By (68),

$$A\mathbf{x} = A\mathbf{x}_0 + \mathbf{y} - \mathbf{y}_0 = \mathbf{y}.$$

Thus (73) and (79) give

$$(80) \quad \varphi(\mathbf{y}) = \psi(\mathbf{x}_0 - S\mathbf{y}_0 + S\mathbf{y}) \quad (\mathbf{y} \in W).$$

This formula shows that $\varphi \in \mathcal{C}'$ in W , hence in $A(V)$, since \mathbf{y}_0 was chosen arbitrarily in $A(V)$.

The proof is now complete.

Here is what the theorem tells us about the geometry of the mapping \mathbf{F} .

If $\mathbf{y} \in \mathbf{F}(U)$ then $\mathbf{y} = \mathbf{F}(\mathbf{H}(\mathbf{x}))$ for some $\mathbf{x} \in V$, and (66) shows that $P\mathbf{y} = A\mathbf{x}$. Therefore

$$(81) \quad \mathbf{y} = P\mathbf{y} + \varphi(P\mathbf{y}) \quad (\mathbf{y} \in \mathbf{F}(U)).$$

This shows that \mathbf{y} is determined by its projection $P\mathbf{y}$, and that P , restricted to $\mathbf{F}(U)$, is a 1-1 mapping of $\mathbf{F}(U)$ onto $A(V)$. Thus $\mathbf{F}(U)$ is an “ r -dimensional surface” with precisely one point “over” each point of $A(V)$. We may also regard $\mathbf{F}(U)$ as the graph of φ .

If $\Phi(\mathbf{x}) = \mathbf{F}(\mathbf{H}(\mathbf{x}))$, as in the proof, then (66) shows that the level sets of Φ (these are the sets on which Φ attains a given value) are precisely the level sets of A in V . These are “flat” since they are intersections with V of translates of the vector space $\mathcal{N}(A)$. Note that $\dim \mathcal{N}(A) = n - r$ (Exercise 25).

The level sets of \mathbf{F} in U are the images under \mathbf{H} of the flat level sets of Φ in V . They are thus “ $(n - r)$ -dimensional surfaces” in U .

DETERMINANTS

Determinants are numbers associated to square matrices, and hence to the operators represented by such matrices. They are 0 if and only if the corresponding operator fails to be invertible. They can therefore be used to decide whether the hypotheses of some of the preceding theorems are satisfied. They will play an even more important role in Chap. 10.

9.33 Definition If (j_1, \dots, j_n) is an ordered n -tuple of integers, define

$$(82) \quad s(j_1, \dots, j_n) = \prod_{p < q} \operatorname{sgn}(j_q - j_p),$$

where $\operatorname{sgn} x = 1$ if $x > 0$, $\operatorname{sgn} x = -1$ if $x < 0$, $\operatorname{sgn} x = 0$ if $x = 0$. Then $s(j_1, \dots, j_n) = 1, -1$, or 0 , and it changes sign if any two of the j 's are interchanged.

Let $[A]$ be the matrix of a linear operator A on R^n , relative to the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$, with entries $a(i, j)$ in the i th row and j th column. The determinant of $[A]$ is defined to be the number

$$(83) \quad \det [A] = \sum s(j_1, \dots, j_n) a(1, j_1) a(2, j_2) \cdots a(n, j_n).$$

The sum in (83) extends over all ordered n -tuples of integers (j_1, \dots, j_n) with $1 \leq j_r \leq n$.

The column vectors \mathbf{x}_j of $[A]$ are

$$(84) \quad \mathbf{x}_j = \sum_{i=1}^n a(i, j) \mathbf{e}_i \quad (1 \leq j \leq n).$$

It will be convenient to think of $\det [A]$ as a function of the column vectors of $[A]$. If we write

$$\det (\mathbf{x}_1, \dots, \mathbf{x}_n) = \det [A],$$

\det is now a real function on the set of all ordered n -tuples of vectors in R^n .

9.34 Theorem

(a) If I is the identity operator on R^n , then

$$\det [I] = \det (\mathbf{e}_1, \dots, \mathbf{e}_n) = 1.$$

(b) \det is a linear function of each of the column vectors \mathbf{x}_j , if the others are held fixed.

(c) If $[A]_1$ is obtained from $[A]$ by interchanging two columns, then $\det [A]_1 = -\det [A]$.

(d) If $[A]$ has two equal columns, then $\det [A] = 0$.

Proof If $A = I$, then $a(i, i) = 1$ and $a(i, j) = 0$ for $i \neq j$. Hence

$$\det [I] = s(1, 2, \dots, n) = 1,$$

which proves (a). By (82), $s(j_1, \dots, j_n) = 0$ if any two of the j 's are equal. Each of the remaining $n!$ products in (83) contains exactly one factor from each column. This proves (b). Part (c) is an immediate consequence of the fact that $s(j_1, \dots, j_n)$ changes sign if any two of the j 's are interchanged, and (d) is a corollary of (c).

9.35 Theorem *If $[A]$ and $[B]$ are n by n matrices, then*

$$\det ([B][A]) = \det [B] \det [A].$$

Proof If $\mathbf{x}_1, \dots, \mathbf{x}_n$ are the columns of $[A]$, define

$$(85) \quad \Delta_B(\mathbf{x}_1, \dots, \mathbf{x}_n) = \Delta_B[A] = \det ([B][A]).$$

The columns of $[B][A]$ are the vectors $B\mathbf{x}_1, \dots, B\mathbf{x}_n$. Thus

$$(86) \quad \Delta_B(\mathbf{x}_1, \dots, \mathbf{x}_n) = \det (B\mathbf{x}_1, \dots, B\mathbf{x}_n).$$

By (86) and Theorem 9.34, Δ_B also has properties 9.34 (b) to (d). By (b) and (84),

$$\Delta_B[A] = \Delta_B \left(\sum_i a(i, 1) \mathbf{e}_i, \mathbf{x}_2, \dots, \mathbf{x}_n \right) = \sum_i a(i, 1) \Delta_B(\mathbf{e}_i, \mathbf{x}_2, \dots, \mathbf{x}_n).$$

Repeating this process with $\mathbf{x}_2, \dots, \mathbf{x}_n$, we obtain

$$(87) \quad \Delta_B[A] = \sum a(i_1, 1) a(i_2, 2) \cdots a(i_n, n) \Delta_B(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n}),$$

the sum being extended over all ordered n -tuples (i_1, \dots, i_n) with $1 \leq i_r \leq n$. By (c) and (d),

$$(88) \quad \Delta_B(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_n}) = t(i_1, \dots, i_n) \Delta_B(\mathbf{e}_1, \dots, \mathbf{e}_n),$$

where $t = 1, 0$, or -1 , and since $[B][I] = [B]$, (85) shows that

$$(89) \quad \Delta_B(\mathbf{e}_1, \dots, \mathbf{e}_n) = \det [B].$$

Substituting (89) and (88) into (87), we obtain

$$\det ([B][A]) = \left\{ \sum a(i_1, 1) \cdots a(i_n, n) t(i_1, \dots, i_n) \right\} \det [B],$$

for all n by n matrices $[A]$ and $[B]$. Taking $B = I$, we see that the above sum in braces is $\det [A]$. This proves the theorem.

9.36 Theorem *A linear operator A on R^n is invertible if and only if $\det [A] \neq 0$.*

Proof If A is invertible, Theorem 9.35 shows that

$$\det [A] \det [A^{-1}] = \det [AA^{-1}] = \det [I] = 1,$$

so that $\det [A] \neq 0$.

If A is not invertible, the columns $\mathbf{x}_1, \dots, \mathbf{x}_n$ of $[A]$ are dependent (Theorem 9.5); hence there is one, say, \mathbf{x}_k , such that

$$(90) \quad \mathbf{x}_k + \sum_{j \neq k} c_j \mathbf{x}_j = 0$$

for certain scalars c_j . By 9.34 (b) and (d), \mathbf{x}_k can be replaced by $\mathbf{x}_k + c_j \mathbf{x}_j$ without altering the determinant, if $j \neq k$. Repeating, we see that \mathbf{x}_k can

be replaced by the left side of (90), i.e., by $\mathbf{0}$, without altering the determinant. But a matrix which has $\mathbf{0}$ for one column has determinant 0. Hence $\det [A] = 0$.

9.37 Remark Suppose $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ and $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ are bases in R^n . Every linear operator A on R^n determines matrices $[A]$ and $[A]_U$, with entries a_{ij} and α_{ij} , given by

$$A\mathbf{e}_j = \sum_i a_{ij} \mathbf{e}_i, \quad A\mathbf{u}_j = \sum_i \alpha_{ij} \mathbf{u}_i.$$

If $\mathbf{u}_j = B\mathbf{e}_j = \sum_i b_{ij} \mathbf{e}_i$, then $A\mathbf{u}_j$ is equal to

$$\sum_k \alpha_{kj} B\mathbf{e}_k = \sum_k \alpha_{kj} \sum_i b_{ik} \mathbf{e}_i = \sum_i \left(\sum_k b_{ik} \alpha_{kj} \right) \mathbf{e}_i,$$

and also to

$$AB\mathbf{e}_j = A \sum_k b_{kj} \mathbf{e}_k = \sum_i \left(\sum_k a_{ik} b_{kj} \right) \mathbf{e}_i.$$

Thus $\sum_k b_{ik} \alpha_{kj} = \sum_k a_{ik} b_{kj}$, or

$$(91) \quad [B][A]_U = [A][B].$$

Since B is invertible, $\det [B] \neq 0$. Hence (91), combined with Theorem 9.35, shows that

$$(92) \quad \det [A]_U = \det [A].$$

The determinant of the matrix of a linear operator does therefore not depend on the basis which is used to construct the matrix. *It is thus meaningful to speak of the determinant of a linear operator, without having any basis in mind.*

9.38 Jacobians If \mathbf{f} maps an open set $E \subset R^n$ into R^n , and if \mathbf{f} is differentiable at a point $\mathbf{x} \in E$, the determinant of the linear operator $\mathbf{f}'(\mathbf{x})$ is called the *Jacobian of \mathbf{f} at \mathbf{x}* . In symbols,

$$(93) \quad J_{\mathbf{f}}(\mathbf{x}) = \det \mathbf{f}'(\mathbf{x}).$$

We shall also use the notation

$$(94) \quad \frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)}$$

for $J_{\mathbf{f}}(\mathbf{x})$, if $(y_1, \dots, y_n) = \mathbf{f}(x_1, \dots, x_n)$.

In terms of Jacobians, the crucial hypothesis in the inverse function theorem is that $J_{\mathbf{f}}(\mathbf{a}) \neq 0$ (compare Theorem 9.36). If the implicit function theorem is stated in terms of the functions (59), the assumption made there on A amounts to

$$\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} \neq 0.$$

DERIVATIVES OF HIGHER ORDER

9.39 Definition Suppose f is a real function defined in an open set $E \subset R^n$, with partial derivatives D_1f, \dots, D_nf . If the functions D_jf are themselves differentiable, then the *second-order partial derivatives* of f are defined by

$$D_{ij}f = D_i D_j f \quad (i, j = 1, \dots, n).$$

If all these functions $D_{ij}f$ are continuous in E , we say that f is of class \mathcal{C}'' in E , or that $f \in \mathcal{C}''(E)$.

A mapping \mathbf{f} of E into R^m is said to be of class \mathcal{C}'' if each component of \mathbf{f} is of class \mathcal{C}'' .

It can happen that $D_{ij}f \neq D_{ji}f$ at some point, although both derivatives exist (see Exercise 27). However, we shall see below that $D_{ij}f = D_{ji}f$ whenever these derivatives are continuous.

For simplicity (and without loss of generality) we state our next two theorems for real functions of two variables. The first one is a mean value theorem.

9.40 Theorem Suppose f is defined in an open set $E \subset R^2$, and D_1f and $D_{21}f$ exist at every point of E . Suppose $Q \subset E$ is a closed rectangle with sides parallel to the coordinate axes, having (a, b) and $(a+h, b+k)$ as opposite vertices ($h \neq 0, k \neq 0$). Put

$$\Delta(f, Q) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b).$$

Then there is a point (x, y) in the interior of Q such that

$$(95) \quad \Delta(f, Q) = hk(D_{21}f)(x, y).$$

Note the analogy between (95) and Theorem 5.10; the area of Q is hk .

Proof Put $u(t) = f(t, b+k) - f(t, b)$. Two applications of Theorem 5.10 show that there is an x between a and $a+h$, and that there is a y between b and $b+k$, such that

$$\begin{aligned} \Delta(f, Q) &= u(a+h) - u(a) \\ &= hu'(x) \\ &= h[(D_1f)(x, b+k) - (D_1f)(x, b)] \\ &= hk(D_{21}f)(x, y). \end{aligned}$$

9.41 Theorem Suppose f is defined in an open set $E \subset R^2$, suppose that D_1f , $D_{21}f$, and D_2f exist at every point of E , and $D_{21}f$ is continuous at some point $(a, b) \in E$.

Then $D_{12}f$ exists at (a, b) and

$$(96) \quad (D_{12}f)(a, b) = (D_{21}f)(a, b).$$

Corollary $D_{21}f = D_{12}f$ if $f \in \mathcal{C}''(E)$.

Proof Put $A = (D_{21}f)(a, b)$. Choose $\varepsilon > 0$. If Q is a rectangle as in Theorem 9.40, and if h and k are sufficiently small, we have

$$|A - (D_{21}f)(x, y)| < \varepsilon$$

for all $(x, y) \in Q$. Thus

$$\left| \frac{\Delta(f, Q)}{hk} - A \right| < \varepsilon,$$

by (95). Fix h , and let $k \rightarrow 0$. Since D_2f exists in E , the last inequality implies that

$$(97) \quad \left| \frac{(D_2f)(a+h, b) - (D_2f)(a, b)}{h} - A \right| \leq \varepsilon.$$

Since ε was arbitrary, and since (97) holds for all sufficiently small $h \neq 0$, it follows that $(D_{12}f)(a, b) = A$. This gives (96).

DIFFERENTIATION OF INTEGRALS

Suppose φ is a function of two variables which can be integrated with respect to one and which can be differentiated with respect to the other. Under what conditions will the result be the same if these two limit processes are carried out in the opposite order? To state the question more precisely: Under what conditions on φ can one prove that the equation

$$(98) \quad \frac{d}{dt} \int_a^b \varphi(x, t) dx = \int_a^b \frac{\partial \varphi}{\partial t}(x, t) dx$$

is true? (A counter example is furnished by Exercise 28.)

It will be convenient to use the notation

$$(99) \quad \varphi^t(x) = \varphi(x, t).$$

Thus φ^t is, for each t , a function of one variable.

9.42 Theorem Suppose

- (a) $\varphi(x, t)$ is defined for $a \leq x \leq b$, $c \leq t \leq d$;
- (b) α is an increasing function on $[a, b]$;

- (c) $\varphi^t \in \mathcal{R}(\alpha)$ for every $t \in [c, d]$;
 (d) $c < s < d$, and to every $\varepsilon > 0$ corresponds a $\delta > 0$ such that

$$|(D_2 \varphi)(x, t) - (D_2 \varphi)(x, s)| < \varepsilon$$

for all $x \in [a, b]$ and for all $t \in (s - \delta, s + \delta)$.

Define

$$(100) \quad f(t) = \int_a^b \varphi(x, t) d\alpha(x) \quad (c \leq t \leq d).$$

Then $(D_2 \varphi)^s \in \mathcal{R}(\alpha)$, $f'(s)$ exists, and

$$(101) \quad f'(s) = \int_a^b (D_2 \varphi)(x, s) d\alpha(x).$$

Note that (c) simply asserts the existence of the integrals (100) for all $t \in [c, d]$. Note also that (d) certainly holds whenever $D_2 \varphi$ is continuous on the rectangle on which φ is defined.

Proof Consider the difference quotients

$$\psi(x, t) = \frac{\varphi(x, t) - \varphi(x, s)}{t - s}$$

for $0 < |t - s| < \delta$. By Theorem 5.10 there corresponds to each (x, t) a number u between s and t such that

$$\psi(x, t) = (D_2 \varphi)(x, u).$$

Hence (d) implies that

$$(102) \quad |\psi(x, t) - (D_2 \varphi)(x, s)| < \varepsilon \quad (a \leq x \leq b, \quad 0 < |t - s| < \delta).$$

Note that

$$(103) \quad \frac{f(t) - f(s)}{t - s} = \int_a^b \psi(x, t) d\alpha(x).$$

By (102), $\psi^t \rightarrow (D_2 \varphi)^s$, uniformly on $[a, b]$, as $t \rightarrow s$. Since each $\psi^t \in \mathcal{R}(\alpha)$, the desired conclusion follows from (103) and Theorem 7.16.

9.43 Example One can of course prove analogues of Theorem 9.42 with $(-\infty, \infty)$ in place of $[a, b]$. Instead of doing this, let us simply look at an example. Define

$$(104) \quad f(t) = \int_{-\infty}^{\infty} e^{-x^2} \cos(xt) dx$$

and

$$(105) \quad g(t) = - \int_{-\infty}^{\infty} x e^{-x^2} \sin(xt) dx,$$

for $-\infty < t < \infty$. Both integrals exist (they converge absolutely) since the absolute values of the integrands are at most $\exp(-x^2)$ and $|x| \exp(-x^2)$, respectively.

Note that g is obtained from f by differentiating the integrand with respect to t . We claim that f is differentiable and that

$$(106) \quad f'(t) = g(t) \quad (-\infty < t < \infty).$$

To prove this, let us first examine the difference quotients of the cosine: if $\beta > 0$, then

$$(107) \quad \frac{\cos(\alpha + \beta) - \cos \alpha}{\beta} + \sin \alpha = \frac{1}{\beta} \int_{\alpha}^{\alpha + \beta} (\sin \alpha - \sin t) dt.$$

Since $|\sin \alpha - \sin t| \leq |t - \alpha|$, the right side of (107) is at most $\beta/2$ in absolute value; the case $\beta < 0$ is handled similarly. Thus

$$(108) \quad \left| \frac{\cos(\alpha + \beta) - \cos \alpha}{\beta} + \sin \alpha \right| \leq |\beta|$$

for all β (if the left side is interpreted to be 0 when $\beta = 0$).

Now fix t , and fix $h \neq 0$. Apply (108) with $\alpha = xt$, $\beta = xh$; it follows from (104) and (105) that

$$\left| \frac{f(t+h) - f(t)}{h} - g(t) \right| \leq |h| \int_{-\infty}^{\infty} x^2 e^{-x^2} dx.$$

When $h \rightarrow 0$, we thus obtain (106).

Let us go a step further: An integration by parts, applied to (104), shows that

$$(109) \quad f(t) = 2 \int_{-\infty}^{\infty} x e^{-x^2} \frac{\sin(xt)}{t} dx.$$

Thus $tf(t) = -2g(t)$, and (106) implies now that f satisfies the differential equation

$$(110) \quad 2f'(t) + tf(t) = 0.$$

If we solve this differential equation and use the fact that $f(0) = \sqrt{\pi}$ (see Sec. 8.21), we find that

$$(111) \quad f(t) = \sqrt{\pi} \exp\left(-\frac{t^2}{4}\right).$$

The integral (104) is thus explicitly determined.

EXERCISES

1. If S is a nonempty subset of a vector space X , prove (as asserted in Sec. 9.1) that the span of S is a vector space.
2. Prove (as asserted in Sec. 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible.
3. Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.
4. Prove (as asserted in Sec. 9.30) that null spaces and ranges of linear transformations are vector spaces.
5. Prove that to every $A \in L(R^n, R^1)$ corresponds a unique $\mathbf{y} \in R^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $\|A\| = |\mathbf{y}|$.
Hint: Under certain conditions, equality holds in the Schwarz inequality.
6. If $f(0, 0) = 0$ and

$$f(x, y) = \frac{xy}{x^2 + y^2} \quad \text{if } (x, y) \neq (0, 0),$$

prove that $(D_1f)(x, y)$ and $(D_2f)(x, y)$ exist at every point of R^2 , although f is not continuous at $(0, 0)$.

7. Suppose that f is a real-valued function defined in an open set $E \subset R^n$, and that the partial derivatives D_1f, \dots, D_nf are bounded in E . Prove that f is continuous in E .
Hint: Proceed as in the proof of Theorem 9.21.
8. Suppose that f is a differentiable real function in an open set $E \subset R^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = 0$.
9. If \mathbf{f} is a differentiable mapping of a *connected* open set $E \subset R^n$ into R^m , and if $\mathbf{f}'(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that \mathbf{f} is constant in E .
10. If f is a real function defined in a convex open set $E \subset R^n$, such that $(D_1f)(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $f(\mathbf{x})$ depends only on x_2, \dots, x_n .
Show that the convexity of E can be replaced by a weaker condition, but that some condition is required. For example, if $n = 2$ and E is shaped like a horseshoe, the statement may be false.
11. If f and g are differentiable real functions in R^n , prove that

$$\nabla(fg) = f \nabla g + g \nabla f$$

and that $\nabla(1/f) = -f^{-2}\nabla f$ wherever $f \neq 0$.

12. Fix two real numbers a and b , $0 < a < b$. Define a mapping $\mathbf{f} = (f_1, f_2, f_3)$ of R^2 into R^3 by

$$\begin{aligned} f_1(s, t) &= (b + a \cos s) \cos t \\ f_2(s, t) &= (b + a \cos s) \sin t \\ f_3(s, t) &= a \sin s. \end{aligned}$$

Describe the range K of f . (It is a certain compact subset of R^3 .)

(a) Show that there are exactly 4 points $p \in K$ such that

$$(\nabla f_1)(f^{-1}(p)) = 0.$$

Find these points.

(b) Determine the set of all $q \in K$ such that

$$(\nabla f_3)(f^{-1}(q)) = 0.$$

(c) Show that one of the points p found in part (a) corresponds to a local maximum of f_1 , one corresponds to a local minimum, and that the other two are neither (they are so-called "saddle points").

Which of the points q found in part (b) correspond to maxima or minima?

(d) Let λ be an irrational real number, and define $g(t) = f(t, \lambda t)$. Prove that g is a 1-1 mapping of R^1 onto a dense subset of K . Prove that

$$|g'(t)|^2 = a^2 + \lambda^2(b + a \cos t)^2.$$

13. Suppose f is a differentiable mapping of R^1 into R^3 such that $|f(t)| = 1$ for every t . Prove that $f'(t) \cdot f(t) = 0$.

Interpret this result geometrically.

14. Define $f(0, 0) = 0$ and

$$f(x, y) = \frac{x^3}{x^2 + y^2} \quad \text{if } (x, y) \neq (0, 0).$$

(a) Prove that $D_1 f$ and $D_2 f$ are bounded functions in R^2 . (Hence f is continuous.)

(b) Let u be any unit vector in R^2 . Show that the directional derivative $(D_u f)(0, 0)$ exists, and that its absolute value is at most 1.

(c) Let γ be a differentiable mapping of R^1 into R^2 (in other words, γ is a differentiable curve in R^2), with $\gamma(0) = (0, 0)$ and $|\gamma'(0)| > 0$. Put $g(t) = f(\gamma(t))$ and prove that g is differentiable for every $t \in R^1$.

If $\gamma \in \mathcal{C}'$, prove that $g \in \mathcal{C}'$.

(d) In spite of this, prove that f is not differentiable at $(0, 0)$.

Hint: Formula (40) fails.

15. Define $f(0, 0) = 0$, and put

$$f(x, y) = x^2 + y^2 - 2x^2y - \frac{4x^6y^2}{(x^4 + y^2)^2}$$

if $(x, y) \neq (0, 0)$.

(a) Prove, for all $(x, y) \in R^2$, that

$$4x^4y^2 \leq (x^4 + y^2)^2.$$

Conclude that f is continuous.

(b) For $0 \leq \theta \leq 2\pi$, $-\infty < t < \infty$, define

$$g_\theta(t) = f(t \cos \theta, t \sin \theta).$$

Show that $g_\theta(0) = 0$, $g'_\theta(0) = 0$, $g''_\theta(0) = 2$. Each g_θ has therefore a strict local minimum at $t = 0$.

In other words, the restriction of f to each line through $(0, 0)$ has a strict local minimum at $(0, 0)$.

(c) Show that $(0, 0)$ is nevertheless not a local minimum for f , since $f(x, x^2) = -x^4$.

16. Show that the continuity of \mathbf{f}' at the point \mathbf{a} is needed in the inverse function theorem, even in the case $n = 1$: If

$$f(t) = t + 2t^2 \sin\left(\frac{1}{t}\right)$$

for $t \neq 0$, and $f(0) = 0$, then $f'(0) = 1$, f' is bounded in $(-1, 1)$, but f is not one-to-one in any neighborhood of 0.

17. Let $\mathbf{f} = (f_1, f_2)$ be the mapping of R^2 into R^2 given by

$$f_1(x, y) = e^x \cos y, \quad f_2(x, y) = e^x \sin y.$$

(a) What is the range of f ?

(b) Show that the Jacobian of f is not zero at any point of R^2 . Thus every point of R^2 has a neighborhood in which f is one-to-one. Nevertheless, f is not one-to-one on R^2 .

(c) Put $\mathbf{a} = (0, \pi/3)$, $\mathbf{b} = f(\mathbf{a})$, let \mathbf{g} be the continuous inverse of \mathbf{f} , defined in a neighborhood of \mathbf{b} , such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$. Find an explicit formula for \mathbf{g} , compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, and verify the formula (52).

(d) What are the images under \mathbf{f} of lines parallel to the coordinate axes?

18. Answer analogous questions for the mapping defined by

$$u = x^2 - y^2, \quad v = 2xy.$$

19. Show that the system of equations

$$3x + y - z + u^2 = 0$$

$$x - y + 2z + u = 0$$

$$2x + 2y - 3z + 2u = 0$$

can be solved for x, y, u in terms of z ; for x, z, u in terms of y ; for y, z, u in terms of x ; but not for x, y, z in terms of u .

20. Take $n = m = 1$ in the implicit function theorem, and interpret the theorem (as well as its proof) graphically.
21. Define f in R^2 by

$$f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2.$$

(a) Find the four points in R^2 at which the gradient of f is zero. Show that f has exactly one local maximum and one local minimum in R^2 .

(b) Let S be the set of all $(x, y) \in \mathbb{R}^2$ at which $f(x, y) = 0$. Find those points of S that have no neighborhoods in which the equation $f(x, y) = 0$ can be solved for y in terms of x (or for x in terms of y). Describe S as precisely as you can.

22. Give a similar discussion for

$$f(x, y) = 2x^3 + 6xy^2 - 3x^2 + 3y^2.$$

23. Define f in \mathbb{R}^3 by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2.$$

Show that $f(0, 1, -1) = 0$, $(D_1 f)(0, 1, -1) \neq 0$, and that there exists therefore a differentiable function g in some neighborhood of $(1, -1)$ in \mathbb{R}^2 , such that $g(1, -1) = 0$ and

$$f(g(y_1, y_2), y_1, y_2) = 0.$$

Find $(D_1 g)(1, -1)$ and $(D_2 g)(1, -1)$.

24. For $(x, y) \neq (0, 0)$, define $\mathbf{f} = (f_1, f_2)$ by

$$f_1(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad f_2(x, y) = \frac{xy}{x^2 + y^2}.$$

Compute the rank of $\mathbf{f}'(x, y)$, and find the range of \mathbf{f} .

25. Suppose $A \in L(\mathbb{R}^n, \mathbb{R}^m)$, let r be the rank of A .

(a) Define S as in the proof of Theorem 9.32. Show that SA is a projection in \mathbb{R}^n whose null space is $\mathcal{N}(A)$ and whose range is $\mathcal{R}(S)$. *Hint:* By (68), $SASA = SA$.

(b) Use (a) to show that

$$\dim \mathcal{N}(A) + \dim \mathcal{R}(A) = n.$$

26. Show that the existence (and even the continuity) of $D_{12}f$ does not imply the existence of $D_{11}f$. For example, let $f(x, y) = g(x)$, where g is nowhere differentiable.

27. Put $f(0, 0) = 0$, and

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if $(x, y) \neq (0, 0)$. Prove that

(a) $f, D_1 f, D_2 f$ are continuous in \mathbb{R}^2 ;

(b) $D_{12}f$ and $D_{21}f$ exist at every point of \mathbb{R}^2 , and are continuous except at $(0, 0)$;

(c) $(D_{12}f)(0, 0) = 1$, and $(D_{21}f)(0, 0) = -1$.

28. For $t \geq 0$, put

$$\varphi(x, t) = \begin{cases} x & (0 \leq x \leq \sqrt{t}) \\ -x + 2\sqrt{t} & (\sqrt{t} \leq x \leq 2\sqrt{t}) \\ 0 & (\text{otherwise}), \end{cases}$$

and put $\varphi(x, t) = -\varphi(x, |t|)$ if $t < 0$.

Show that φ is continuous on R^2 , and

$$(D_2 \varphi)(x, 0) = 0$$

for all x . Define

$$f(t) = \int_{-1}^1 \varphi(x, t) dx.$$

Show that $f(t) = t$ if $|t| < \frac{1}{4}$. Hence

$$f'(0) \neq \int_{-1}^1 (D_2 \varphi)(x, 0) dx.$$

29. Let E be an open set in R^n . The classes $\mathcal{C}'(E)$ and $\mathcal{C}''(E)$ are defined in the text. By induction, $\mathcal{C}^{(k)}(E)$ can be defined as follows, for all positive integers k : To say that $f \in \mathcal{C}^{(k)}(E)$ means that the partial derivatives $D_1 f, \dots, D_n f$ belong to $\mathcal{C}^{(k-1)}(E)$.

Assume $f \in \mathcal{C}^{(k)}(E)$, and show (by repeated application of Theorem 9.41) that the k th-order derivative

$$D_{i_1 i_2 \dots i_k} f = D_{i_1} D_{i_2} \dots D_{i_k} f$$

is unchanged if the subscripts i_1, \dots, i_k are permuted.

For instance, if $n \geq 3$, then

$$D_{1213} f = D_{3112} f$$

for every $f \in \mathcal{C}^{(4)}$.

30. Let $f \in \mathcal{C}^{(m)}(E)$, where E is an open subset of R^n . Fix $\mathbf{a} \in E$, and suppose $\mathbf{x} \in R^n$ is so close to 0 that the points

$$\mathbf{p}(t) = \mathbf{a} + t\mathbf{x}$$

lie in E whenever $0 \leq t \leq 1$. Define

$$h(t) = f(\mathbf{p}(t))$$

for all $t \in R^1$ for which $\mathbf{p}(t) \in E$.

(a) For $1 \leq k \leq m$, show (by repeated application of the chain rule) that

$$h^{(k)}(t) = \sum (D_{i_1 \dots i_k} f)(\mathbf{p}(t)) x_{i_1} \dots x_{i_k}.$$

The sum extends over all ordered k -tuples (i_1, \dots, i_k) in which each i_j is one of the integers $1, \dots, n$.

(b) By Taylor's theorem (5.15),

$$h(1) = \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!} + \frac{h^{(m)}(t)}{m!}$$

for some $t \in (0, 1)$. Use this to prove Taylor's theorem in n variables by showing that the formula

$$f(\mathbf{a} + \mathbf{x}) = \sum_{k=0}^{m-1} \frac{1}{k!} \sum (D_{i_1 \dots i_k} f)(\mathbf{a}) x_{i_1} \dots x_{i_k} + r(\mathbf{x})$$

represents $f(\mathbf{a} + \mathbf{x})$ as the sum of its so-called “Taylor polynomial of degree $m - 1$,” plus a remainder that satisfies

$$\lim_{\mathbf{x} \rightarrow 0} \frac{r(\mathbf{x})}{|\mathbf{x}|^{m-1}} = 0.$$

Each of the inner sums extends over all ordered k -tuples (i_1, \dots, i_k) , as in part (a); as usual, the zero-order derivative of f is simply f , so that the constant term of the Taylor polynomial of f at \mathbf{a} is $f(\mathbf{a})$.

(c) Exercise 29 shows that repetition occurs in the Taylor polynomial as written in part (b). For instance, D_{113} occurs three times, as D_{113} , D_{131} , D_{311} . The sum of the corresponding three terms can be written in the form

$$3(D_1^2 D_3 f)(\mathbf{a}) x_1^2 x_3.$$

Prove (by calculating how often each derivative occurs) that the Taylor polynomial in (b) can be written in the form

$$\sum \frac{(D_1^{s_1} \dots D_n^{s_n} f)(\mathbf{a})}{s_1! \dots s_n!} x_1^{s_1} \dots x_n^{s_n}.$$

Here the summation extends over all ordered n -tuples (s_1, \dots, s_n) such that each s_i is a nonnegative integer, and $s_1 + \dots + s_n \leq m - 1$.

31. Suppose $f \in \mathcal{C}^{(3)}$ in some neighborhood of a point $\mathbf{a} \in R^2$, the gradient of f is 0 at \mathbf{a} , but not all second-order derivatives of f are 0 at \mathbf{a} . Show how one can then determine from the Taylor polynomial of f at \mathbf{a} (of degree 2) whether f has a local maximum, or a local minimum, or neither, at the point \mathbf{a} .

Extend this to R^n in place of R^2 .

INTEGRATION OF DIFFERENTIAL FORMS

Integration can be studied on many levels. In Chap. 6, the theory was developed for reasonably well-behaved functions on subintervals of the real line. In Chap. 11 we shall encounter a very highly developed theory of integration that can be applied to much larger classes of functions, whose domains are more or less arbitrary sets, not necessarily subsets of R^n . The present chapter is devoted to those aspects of integration theory that are closely related to the geometry of euclidean spaces, such as the change of variables formula, line integrals, and the machinery of differential forms that is used in the statement and proof of the n -dimensional analogue of the fundamental theorem of calculus, namely Stokes' theorem.

INTEGRATION

10.1 Definition Suppose I^k is a k -cell in R^k , consisting of all

$$\mathbf{x} = (x_1, \dots, x_k)$$

such that

$$(1) \quad a_i \leq x_i \leq b_i \quad (i = 1, \dots, k),$$

I^j is the j -cell in R^j defined by the first j inequalities (1), and f is a real continuous function on I^k .

Put $f = f_k$, and define f_{k-1} on I^{k-1} by

$$f_{k-1}(x_1, \dots, x_{k-1}) = \int_{a_k}^{b_k} f_k(x_1, \dots, x_{k-1}, x_k) dx_k.$$

The uniform continuity of f_k on I^k shows that f_{k-1} is continuous on I^{k-1} . Hence we can repeat this process and obtain functions f_j , continuous on I^j , such that f_{j-1} is the integral of f_j , with respect to x_j , over $[a_j, b_j]$. After k steps we arrive at a number f_0 , which we call the *integral of f over I^k* ; we write it in the form

$$(2) \quad \int_{I^k} f(\mathbf{x}) d\mathbf{x} \quad \text{or} \quad \int_{I^k} f.$$

A priori, this definition of the integral depends on the order in which the k integrations are carried out. However, this dependence is only apparent. To prove this, let us introduce the temporary notation $L(f)$ for the integral (2) and $L'(f)$ for the result obtained by carrying out the k integrations in some other order.

10.2 Theorem For every $f \in \mathcal{C}(I^k)$, $L(f) = L'(f)$.

Proof If $h(\mathbf{x}) = h_1(x_1) \cdots h_k(x_k)$, where $h_j \in \mathcal{C}([a_j, b_j])$, then

$$L(h) = \prod_{i=1}^k \int_{a_i}^{b_i} h_i(x_i) dx_i = L'(h).$$

If \mathcal{A} is the set of all finite sums of such functions h , it follows that $L(g) = L'(g)$ for all $g \in \mathcal{A}$. Also, \mathcal{A} is an algebra of functions on I^k to which the Stone-Weierstrass theorem applies.

Put $V = \prod_{i=1}^k (b_i - a_i)$. If $f \in \mathcal{C}(I^k)$ and $\varepsilon > 0$, there exists $g \in \mathcal{A}$ such that $\|f - g\| < \varepsilon/V$, where $\|f\|$ is defined as $\max |f(\mathbf{x})|$ ($\mathbf{x} \in I^k$). Then $|L(f - g)| < \varepsilon$, $|L'(f - g)| < \varepsilon$, and since

$$L(f) - L'(f) = L(f - g) + L'(g - f),$$

we conclude that $|L(f) - L'(f)| < 2\varepsilon$.

In this connection, Exercise 2 is relevant.

10.3 Definition The *support* of a (real or complex) function f on R^k is the closure of the set of all points $\mathbf{x} \in R^k$ at which $f(\mathbf{x}) \neq 0$. If f is a continuous

function with compact support, let I^k be any k -cell which contains the support of f , and define

$$(3) \quad \int_{R^k} f = \int_{I^k} f.$$

The integral so defined is evidently independent of the choice of I^k , provided only that I^k contains the support of f .

It is now tempting to extend the definition of the integral over R^k to functions which are limits (in some sense) of continuous functions with compact support. We do not want to discuss the conditions under which this can be done; the proper setting for this question is the Lebesgue integral. We shall merely describe one very simple example which will be used in the proof of Stokes' theorem.

10.4 Example Let Q^k be the k -simplex which consists of all points $\mathbf{x} = (x_1, \dots, x_k)$ in R^k for which $x_1 + \dots + x_k \leq 1$ and $x_i \geq 0$ for $i = 1, \dots, k$. If $k = 3$, for example, Q^k is a tetrahedron, with vertices at $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. If $f \in \mathcal{C}(Q^k)$, extend f to a function on I^k by setting $f(\mathbf{x}) = 0$ off Q^k , and define

$$(4) \quad \int_{Q^k} f = \int_{I^k} f.$$

Here I^k is the "unit cube" defined by

$$0 \leq x_i \leq 1 \quad (1 \leq i \leq k).$$

Since f may be discontinuous on I^k , the existence of the integral on the right of (4) needs proof. We also wish to show that this integral is independent of the order in which the k single integrations are carried out.

To do this, suppose $0 < \delta < 1$, put

$$(5) \quad \varphi(t) = \begin{cases} 1 & (t \leq 1 - \delta) \\ \frac{(1-t)}{\delta} & (1 - \delta < t \leq 1) \\ 0 & (1 < t), \end{cases}$$

and define

$$(6) \quad F(\mathbf{x}) = \varphi(x_1 + \dots + x_k) f(\mathbf{x}) \quad (\mathbf{x} \in I^k).$$

Then $F \in \mathcal{C}(I^k)$.

Put $\mathbf{y} = (x_1, \dots, x_{k-1})$, $\mathbf{x} = (\mathbf{y}, x_k)$. For each $\mathbf{y} \in I^{k-1}$, the set of all x_k such that $F(\mathbf{y}, x_k) \neq f(\mathbf{y}, x_k)$ is either empty or is a segment whose length does not exceed δ . Since $0 \leq \varphi \leq 1$, it follows that

$$(7) \quad |F_{k-1}(\mathbf{y}) - f_{k-1}(\mathbf{y})| \leq \delta \|f\| \quad (\mathbf{y} \in I^{k-1}),$$

where $\|f\|$ has the same meaning as in the proof of Theorem 10.2, and F_{k-1} , f_{k-1} are as in Definition 10.1.

As $\delta \rightarrow 0$, (7) exhibits f_{k-1} as a uniform limit of a sequence of continuous functions. Thus $f_{k-1} \in \mathcal{C}(I^{k-1})$, and the further integrations present no problem.

This proves the existence of the integral (4). Moreover, (7) shows that

$$(8) \quad \left| \int_{I^k} F(\mathbf{x}) d\mathbf{x} - \int_{I^k} f(\mathbf{x}) d\mathbf{x} \right| \leq \delta \|f\|.$$

Note that (8) is true, regardless of the order in which the k single integrations are carried out. Since $F \in \mathcal{C}(I^k)$, $\int F$ is unaffected by any change in this order. Hence (8) shows that the same is true of $\int f$.

This completes the proof.

Our next goal is the change of variables formula stated in Theorem 10.9. To facilitate its proof, we first discuss so-called primitive mappings, and partitions of unity. Primitive mappings will enable us to get a clearer picture of the local action of a \mathcal{C}' -mapping with invertible derivative, and partitions of unity are a very useful device that makes it possible to use local information in a global setting.

PRIMITIVE MAPPINGS

10.5 Definition If \mathbf{G} maps an open set $E \subset R^n$ into R^n , and if there is an integer m and a real function g with domain E such that

$$(9) \quad \mathbf{G}(\mathbf{x}) = \sum_{i \neq m} x_i \mathbf{e}_i + g(\mathbf{x}) \mathbf{e}_m \quad (\mathbf{x} \in E),$$

then we call \mathbf{G} *primitive*. A primitive mapping is thus one that changes at most one coordinate. Note that (9) can also be written in the form

$$(10) \quad \mathbf{G}(\mathbf{x}) = \mathbf{x} + [g(\mathbf{x}) - x_m] \mathbf{e}_m.$$

If g is differentiable at some point $\mathbf{a} \in E$, so is \mathbf{G} . The matrix $[\alpha_{ij}]$ of the operator $\mathbf{G}'(\mathbf{a})$ has

$$(11) \quad (D_1 g)(\mathbf{a}), \dots, (D_m g)(\mathbf{a}), \dots, (D_n g)(\mathbf{a})$$

as its m th row. For $j \neq m$, we have $\alpha_{jj} = 1$ and $\alpha_{ij} = 0$ if $i \neq j$. The Jacobian of \mathbf{G} at \mathbf{a} is thus given by

$$(12) \quad J_{\mathbf{G}}(\mathbf{a}) = \det[\mathbf{G}'(\mathbf{a})] = (D_m g)(\mathbf{a}),$$

and we see (by Theorem 9.36) that $\mathbf{G}'(\mathbf{a})$ is invertible if and only if $(D_m g)(\mathbf{a}) \neq 0$.

10.6 Definition A linear operator B on R^n that interchanges some pair of members of the standard basis and leaves the others fixed will be called a *flip*.

For example, the flip B on R^4 that interchanges \mathbf{e}_2 and \mathbf{e}_4 has the form

$$(13) \quad B(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3 + x_4 \mathbf{e}_4) = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_4 + x_3 \mathbf{e}_3 + x_4 \mathbf{e}_2$$

or, equivalently,

$$(14) \quad B(x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3 + x_4 \mathbf{e}_4) = x_1 \mathbf{e}_1 + x_4 \mathbf{e}_2 + x_3 \mathbf{e}_3 + x_2 \mathbf{e}_4.$$

Hence B can also be thought of as interchanging two of the coordinates, rather than two basis vectors.

In the proof that follows, we shall use the projections P_0, \dots, P_n in R^n , defined by $P_0 \mathbf{x} = \mathbf{0}$ and

$$(15) \quad P_m \mathbf{x} = x_1 \mathbf{e}_1 + \dots + x_m \mathbf{e}_m$$

for $1 \leq m \leq n$. Thus P_m is the projection whose range and null space are spanned by $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ and $\{\mathbf{e}_{m+1}, \dots, \mathbf{e}_n\}$, respectively.

10.7 Theorem Suppose F is a \mathcal{C}' -mapping of an open set $E \subset R^n$ into R^n , $\mathbf{0} \in E$, $F(\mathbf{0}) = \mathbf{0}$, and $F'(\mathbf{0})$ is invertible.

Then there is a neighborhood of $\mathbf{0}$ in R^n in which a representation

$$(16) \quad F(\mathbf{x}) = B_1 \cdots B_{n-1} G_n \circ \cdots \circ G_1(\mathbf{x})$$

is valid.

In (16), each G_i is a primitive \mathcal{C}' -mapping in some neighborhood of $\mathbf{0}$; $G_i(\mathbf{0}) = \mathbf{0}$, $G'_i(\mathbf{0})$ is invertible, and each B_i is either a flip or the identity operator.

Briefly, (16) represents F locally as a composition of primitive mappings and flips.

Proof Put $F = F_1$. Assume $1 \leq m \leq n-1$, and make the following induction hypothesis (which evidently holds for $m=1$):

V_m is a neighborhood of $\mathbf{0}$, $F_m \in \mathcal{C}'(V_m)$, $F_m(\mathbf{0}) = \mathbf{0}$, $F'_m(\mathbf{0})$ is invertible, and

$$(17) \quad P_{m-1} F_m(\mathbf{x}) = P_{m-1} \mathbf{x} \quad (\mathbf{x} \in V_m).$$

By (17), we have

$$(18) \quad F_m(\mathbf{x}) = P_{m-1} \mathbf{x} + \sum_{i=m}^n \alpha_i(\mathbf{x}) \mathbf{e}_i,$$

where $\alpha_m, \dots, \alpha_n$ are real \mathcal{C}' -functions in V_m . Hence

$$(19) \quad F'_m(\mathbf{0}) \mathbf{e}_m = \sum_{i=m}^n (D_m \alpha_i)(\mathbf{0}) \mathbf{e}_i.$$

Since $F'_m(\mathbf{0})$ is invertible, the left side of (19) is not $\mathbf{0}$, and therefore there is a k such that $m \leq k \leq n$ and $(D_m \alpha_k)(\mathbf{0}) \neq 0$.

Let B_m be the flip that interchanges m and this k (if $k = m$, B_m is the identity) and define

$$(20) \quad G_m(\mathbf{x}) = \mathbf{x} + [\alpha_k(\mathbf{x}) - x_m]\mathbf{e}_m \quad (\mathbf{x} \in V_m).$$

Then $G_m \in \mathcal{C}'(V_m)$, G_m is primitive, and $G'_m(\mathbf{0})$ is invertible, since $(D_m \alpha_k)(\mathbf{0}) \neq 0$.

The inverse function theorem shows therefore that there is an open set U_m , with $\mathbf{0} \in U_m \subset V_m$, such that G_m is a 1-1 mapping of U_m onto a neighborhood V_{m+1} of $\mathbf{0}$, in which G_m^{-1} is continuously differentiable. Define F_{m+1} by

$$(21) \quad F_{m+1}(\mathbf{y}) = B_m F_m \circ G_m^{-1}(\mathbf{y}) \quad (\mathbf{y} \in V_{m+1}).$$

Then $F_{m+1} \in \mathcal{C}'(V_{m+1})$, $F_{m+1}(\mathbf{0}) = \mathbf{0}$, and $F'_{m+1}(\mathbf{0})$ is invertible (by the chain rule). Also, for $\mathbf{x} \in U_m$,

$$(22) \quad \begin{aligned} P_m F_{m+1}(G_m(\mathbf{x})) &= P_m B_m F_m(\mathbf{x}) \\ &= P_m [P_{m-1} \mathbf{x} + \alpha_k(\mathbf{x})\mathbf{e}_m + \cdots] \\ &= P_{m-1} \mathbf{x} + \alpha_k(\mathbf{x})\mathbf{e}_m \\ &= P_m G_m(\mathbf{x}) \end{aligned}$$

so that

$$(23) \quad P_m F_{m+1}(\mathbf{y}) = P_m \mathbf{y} \quad (\mathbf{y} \in V_{m+1}).$$

Our induction hypothesis holds therefore with $m+1$ in place of m .

[In (22), we first used (21), then (18) and the definition of B_m , then the definition of P_m , and finally (20).]

Since $B_m B_m = I$, (21), with $\mathbf{y} = G_m(\mathbf{x})$, is equivalent to

$$(24) \quad F_m(\mathbf{x}) = B_m F_{m+1}(G_m(\mathbf{x})) \quad (\mathbf{x} \in U_m).$$

If we apply this with $m = 1, \dots, n-1$, we successively obtain

$$\begin{aligned} F &= F_1 = B_1 F_2 \circ G_1 \\ &= B_1 B_2 F_3 \circ G_2 \circ G_1 = \cdots \\ &= B_1 \cdots B_{n-1} F_n \circ G_{n-1} \circ \cdots \circ G_1 \end{aligned}$$

in some neighborhood of $\mathbf{0}$. By (17), F_n is primitive. This completes the proof.

PARTITIONS OF UNITY

10.8 Theorem Suppose K is a compact subset of R^n , and $\{V_\alpha\}$ is an open cover of K . Then there exist functions $\psi_1, \dots, \psi_s \in \mathcal{C}(R^n)$ such that

- (a) $0 \leq \psi_i \leq 1$ for $1 \leq i \leq s$;
- (b) each ψ_i has its support in some V_α , and
- (c) $\psi_1(\mathbf{x}) + \dots + \psi_s(\mathbf{x}) = 1$ for every $\mathbf{x} \in K$.

Because of (c), $\{\psi_i\}$ is called a *partition of unity*, and (b) is sometimes expressed by saying that $\{\psi_i\}$ is *subordinate* to the cover $\{V_\alpha\}$.

Corollary If $f \in \mathcal{C}(R^n)$ and the support of f lies in K , then

$$(25) \quad f = \sum_{i=1}^s \psi_i f.$$

Each $\psi_i f$ has its support in some V_α .

The point of (25) is that it furnishes a representation of f as a sum of continuous functions $\psi_i f$ with “small” supports.

Proof Associate with each $\mathbf{x} \in K$ an index $\alpha(\mathbf{x})$ so that $\mathbf{x} \in V_{\alpha(\mathbf{x})}$. Then there are open balls $B(\mathbf{x})$ and $W(\mathbf{x})$, centered at \mathbf{x} , with

$$(26) \quad \overline{B(\mathbf{x})} \subset W(\mathbf{x}) \subset \overline{W(\mathbf{x})} \subset V_{\alpha(\mathbf{x})}.$$

Since K is compact, there are points $\mathbf{x}_1, \dots, \mathbf{x}_s$ in K such that

$$(27) \quad K \subset B(\mathbf{x}_1) \cup \dots \cup B(\mathbf{x}_s).$$

By (26), there are functions $\varphi_1, \dots, \varphi_s \in \mathcal{C}(R^n)$, such that $\varphi_i(\mathbf{x}) = 1$ on $B(\mathbf{x}_i)$, $\varphi_i(\mathbf{x}) = 0$ outside $W(\mathbf{x}_i)$, and $0 \leq \varphi_i(\mathbf{x}) \leq 1$ on R^n . Define $\psi_1 = \varphi_1$ and

$$(28) \quad \psi_{i+1} = (1 - \varphi_1) \cdots (1 - \varphi_i) \varphi_{i+1}$$

for $i = 1, \dots, s-1$.

Properties (a) and (b) are clear. The relation

$$(29) \quad \psi_1 + \dots + \psi_i = 1 - (1 - \varphi_1) \cdots (1 - \varphi_i)$$

is trivial for $i = 1$. If (29) holds for some $i < s$, addition of (28) and (29) yields (29) with $i+1$ in place of i . It follows that

$$(30) \quad \sum_{i=1}^s \psi_i(\mathbf{x}) = 1 - \prod_{i=1}^s [1 - \varphi_i(\mathbf{x})] \quad (\mathbf{x} \in R^n).$$

If $\mathbf{x} \in K$, then $\mathbf{x} \in B(\mathbf{x}_i)$ for some i , hence $\varphi_i(\mathbf{x}) = 1$, and the product in (30) is 0. This proves (c).

CHANGE OF VARIABLES

We can now describe the effect of a change of variables on a multiple integral. For simplicity, we confine ourselves here to continuous functions with compact support, although this is too restrictive for many applications. This is illustrated by Exercises 9 to 13.

10.9 Theorem Suppose T is a 1-1 \mathcal{C}' -mapping of an open set $E \subset R^k$ into R^k such that $J_T(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in E$. If f is a continuous function on R^k whose support is compact and lies in $T(E)$, then

$$(31) \quad \int_{R^k} f(\mathbf{y}) \, d\mathbf{y} = \int_{R^k} f(T(\mathbf{x})) |J_T(\mathbf{x})| \, d\mathbf{x}.$$

We recall that J_T is the Jacobian of T . The assumption $J_T(\mathbf{x}) \neq 0$ implies, by the inverse function theorem, that T^{-1} is continuous on $T(E)$, and this ensures that the integrand on the right of (31) has compact support in E (Theorem 4.14).

The appearance of the *absolute value* of $J_T(\mathbf{x})$ in (31) may call for a comment. Take the case $k = 1$, and suppose T is a 1-1 \mathcal{C}' -mapping of R^1 onto R^1 . Then $J_T(x) = T'(x)$; and if T is *increasing*, we have

$$(32) \quad \int_{R^1} f(y) \, dy = \int_{R^1} f(T(x)) T'(x) \, dx,$$

by Theorems 6.19 and 6.17, for all continuous f with compact support. But if T decreases, then $T'(x) < 0$; and if f is positive in the interior of its support, the left side of (32) is positive and the right side is negative. A correct equation is obtained if T' is replaced by $|T'|$ in (32).

The point is that the integrals we are now considering are integrals of functions over subsets of R^k , and we associate no direction or orientation with these subsets. We shall adopt a different point of view when we come to integration of differential forms over surfaces.

Proof It follows from the remarks just made that (31) is true if T is a primitive \mathcal{C}' -mapping (see Definition 10.5), and Theorem 10.2 shows that (31) is true if T is a linear mapping which merely interchanges two coordinates.

If the theorem is true for transformations P , Q , and if $S(\mathbf{x}) = P(Q(\mathbf{x}))$, then

$$\begin{aligned} \int f(\mathbf{z}) \, d\mathbf{z} &= \int f(P(\mathbf{y})) |J_P(\mathbf{y})| \, d\mathbf{y} \\ &= \int f(P(Q(\mathbf{x}))) |J_P(Q(\mathbf{x}))| |J_Q(\mathbf{x})| \, d\mathbf{x} \\ &= \int f(S(\mathbf{x})) |J_S(\mathbf{x})| \, d\mathbf{x}, \end{aligned}$$

since

$$\begin{aligned} J_P(Q(\mathbf{x}))J_Q(\mathbf{x}) &= \det P'(Q(\mathbf{x})) \det Q'(\mathbf{x}) \\ &= \det P'(Q(\mathbf{x}))Q'(\mathbf{x}) = \det S'(\mathbf{x}) = J_S(\mathbf{x}), \end{aligned}$$

by the multiplication theorem for determinants and the chain rule. Thus the theorem is also true for S .

Each point $\mathbf{a} \in E$ has a neighborhood $U \subset E$ in which

$$(33) \quad T(\mathbf{x}) = T(\mathbf{a}) + B_1 \cdots B_{k-1} \mathbf{G}_k \circ \mathbf{G}_{k-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x} - \mathbf{a}),$$

where \mathbf{G}_i and B_i are as in Theorem 10.7. Setting $V = T(U)$, it follows that (31) holds if the support of f lies in V . Thus:

Each point $\mathbf{y} \in T(E)$ lies in an open set $V_{\mathbf{y}} \subset T(E)$ such that (31) holds for all continuous functions whose support lies in $V_{\mathbf{y}}$.

Now let f be a continuous function with compact support $K \subset T(E)$. Since $\{V_{\mathbf{y}}\}$ covers K , the Corollary to Theorem 10.8 shows that $f = \sum \psi_i f$, where each ψ_i is continuous, and each ψ_i has its support in some $V_{\mathbf{y}}$. Thus (31) holds for each $\psi_i f$, and hence also for their sum f .

DIFFERENTIAL FORMS

We shall now develop some of the machinery that is needed for the n -dimensional version of the fundamental theorem of calculus which is usually called *Stokes' theorem*. The original form of Stokes' theorem arose in applications of vector analysis to electromagnetism and was stated in terms of the curl of a vector field. Green's theorem and the divergence theorem are other special cases. These topics are briefly discussed at the end of the chapter.

It is a curious feature of Stokes' theorem that the only thing that is difficult about it is the elaborate structure of definitions that are needed for its statement. These definitions concern differential forms, their derivatives, boundaries, and orientation. Once these concepts are understood, the statement of the theorem is very brief and succinct, and its proof presents little difficulty.

Up to now we have considered derivatives of functions of several variables only for functions defined in *open* sets. This was done to avoid difficulties that can occur at boundary points. It will now be convenient, however, to discuss differentiable functions on *compact* sets. We therefore adopt the following convention:

To say that \mathbf{f} is a \mathcal{C}' -mapping (or a \mathcal{C}'' -mapping) of a compact set $D \subset R^k$ into R^n means that there is a \mathcal{C}' -mapping (or a \mathcal{C}'' -mapping) \mathbf{g} of an open set $W \subset R^k$ into R^n such that $D \subset W$ and such that $\mathbf{g}(\mathbf{x}) = \mathbf{f}(\mathbf{x})$ for all $\mathbf{x} \in D$.

10.10 Definition Suppose E is an open set in R^n . A k -surface in E is a \mathcal{C}' -mapping Φ from a compact set $D \subset R^k$ into E .

D is called the *parameter domain* of Φ . Points of D will be denoted by $\mathbf{u} = (u_1, \dots, u_k)$.

We shall confine ourselves to the simple situation in which D is either a k -cell or the k -simplex Q^k described in Example 10.4. The reason for this is that we shall have to integrate over D , and we have not yet discussed integration over more complicated subsets of R^k . It will be seen that this restriction on D (which will be tacitly made from now on) entails no significant loss of generality in the resulting theory of differential forms.

We stress that k -surfaces in E are defined to be *mappings* into E , not subsets of E . This agrees with our earlier definition of curves (Definition 6.26). In fact, 1-surfaces are precisely the same as continuously differentiable curves.

10.11 Definition Suppose E is an open set in R^n . A *differential form of order $k \geq 1$ in E* (briefly, a *k -form in E*) is a function ω , symbolically represented by the sum

$$(34) \quad \omega = \sum a_{i_1 \dots i_k}(\mathbf{x}) dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

(the indices i_1, \dots, i_k range independently from 1 to n), which assigns to each k -surface Φ in E a number $\omega(\Phi) = \int_{\Phi} \omega$, according to the rule

$$(35) \quad \int_{\Phi} \omega = \int_D \sum a_{i_1 \dots i_k}(\Phi(\mathbf{u})) \frac{\partial(x_{i_1}, \dots, x_{i_k})}{\partial(u_1, \dots, u_k)} d\mathbf{u},$$

where D is the parameter domain of Φ .

The functions $a_{i_1 \dots i_k}$ are assumed to be real and continuous in E . If ϕ_1, \dots, ϕ_n are the components of Φ , the Jacobian in (35) is the one determined by the mapping

$$(u_1, \dots, u_k) \rightarrow (\phi_{i_1}(\mathbf{u}), \dots, \phi_{i_k}(\mathbf{u})).$$

Note that the right side of (35) is an integral over D , as defined in Definition 10.1 (or Example 10.4) and that (35) is the *definition* of the symbol $\int_{\Phi} \omega$.

A k -form ω is said to be of class \mathcal{C}' or \mathcal{C}'' if the functions $a_{i_1 \dots i_k}$ in (34) are all of class \mathcal{C}' or \mathcal{C}'' .

A 0-form in E is defined to be a continuous function in E .

10.12 Examples

(a) Let γ be a 1-surface (a curve of class \mathcal{C}') in R^3 , with parameter domain $[0, 1]$.

Write (x, y, z) in place of (x_1, x_2, x_3) , and put

$$\omega = x dy + y dx.$$

Then

$$\int_{\gamma} \omega = \int_0^1 [\gamma_1(t)\gamma_2'(t) + \gamma_2(t)\gamma_1'(t)] dt = \gamma_1(1)\gamma_2(1) - \gamma_1(0)\gamma_2(0).$$

Note that in this example $\int_{\gamma} \omega$ depends only on the initial point $\gamma(0)$ and on the end point $\gamma(1)$ of γ . In particular, $\int_{\gamma} \omega = 0$ for every closed curve γ . (As we shall see later, this is true for every 1-form ω which is *exact*.)

Integrals of 1-forms are often called *line integrals*.

(b) Fix $a > 0$, $b > 0$, and define

$$\gamma(t) = (a \cos t, b \sin t) \quad (0 \leq t \leq 2\pi),$$

so that γ is a closed curve in R^2 . (Its range is an ellipse.) Then

$$\int_{\gamma} x dy = \int_0^{2\pi} ab \cos^2 t dt = \pi ab,$$

whereas

$$\int_{\gamma} y dx = - \int_0^{2\pi} ab \sin^2 t dt = -\pi ab.$$

Note that $\int_{\gamma} x dy$ is the area of the region bounded by γ . This is a special case of Green's theorem.

(c) Let D be the 3-cell defined by

$$0 \leq r \leq 1, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi.$$

Define $\Phi(r, \theta, \varphi) = (x, y, z)$, where

$$\begin{aligned} x &= r \sin \theta \cos \varphi \\ y &= r \sin \theta \sin \varphi \\ z &= r \cos \theta. \end{aligned}$$

Then

$$J_{\Phi}(r, \theta, \varphi) = \frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} = r^2 \sin \theta.$$

Hence

$$(36) \quad \int_{\Phi} dx \wedge dy \wedge dz = \int_D J_{\Phi} = \frac{4\pi}{3}.$$

Note that Φ maps D onto the closed unit ball of R^3 , that the mapping is 1-1 in the interior of D (but certain boundary points are identified by Φ), and that the integral (36) is equal to the volume of $\Phi(D)$.

10.13 Elementary properties Let $\omega, \omega_1, \omega_2$ be k -forms in E . We write $\omega_1 = \omega_2$ if and only if $\omega_1(\Phi) = \omega_2(\Phi)$ for every k -surface Φ in E . In particular, $\omega = 0$ means that $\omega(\Phi) = 0$ for every k -surface Φ in E . If c is a real number, then $c\omega$ is the k -form defined by

$$(37) \quad \int_{\Phi} c\omega = c \int_{\Phi} \omega,$$

and $\omega = \omega_1 + \omega_2$ means that

$$(38) \quad \int_{\Phi} \omega = \int_{\Phi} \omega_1 + \int_{\Phi} \omega_2$$

for every k -surface Φ in E . As a special case of (37), note that $-\omega$ is defined so that

$$(39) \quad \int_{\Phi} (-\omega) = - \int_{\Phi} \omega.$$

Consider a k -form

$$(40) \quad \omega = a(\mathbf{x}) dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

and let $\bar{\omega}$ be the k -form obtained by interchanging some pair of subscripts in (40). If (35) and (39) are combined with the fact that a determinant changes sign if two of its rows are interchanged, we see that

$$(41) \quad \bar{\omega} = -\omega.$$

As a special case of this, note that the *anticommutative relation*

$$(42) \quad dx_i \wedge dx_j = -dx_j \wedge dx_i$$

holds for all i and j . In particular,

$$(43) \quad dx_i \wedge dx_i = 0 \quad (i = 1, \dots, n).$$

More generally, let us return to (40), and assume that $i_r = i_s$ for some $r \neq s$. If these two subscripts are interchanged, then $\bar{\omega} = \omega$, hence $\omega = 0$, by (41).

In other words, if ω is given by (40), then $\omega = 0$ unless the subscripts i_1, \dots, i_k are all distinct.

If ω is as in (34), the summands with repeated subscripts can therefore be omitted without changing ω .

It follows that 0 is the only k -form in any open subset of R^n , if $k > n$.

The anticommutativity expressed by (42) is the reason for the inordinate amount of attention that has to be paid to minus signs when studying differential forms.

10.14 Basic k -forms If i_1, \dots, i_k are integers such that $1 \leq i_1 < i_2 < \dots < i_k \leq n$, and if I is the ordered k -tuple $\{i_1, \dots, i_k\}$, then we call I an *increasing k -index*, and we use the brief notation

$$(44) \quad dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k}.$$

These forms dx_I are the so-called *basic k -forms in R^n* .

It is not hard to verify that there are precisely $n!/k!(n-k)!$ basic k -forms in R^n ; we shall make no use of this, however.

Much more important is the fact that every k -form can be represented in terms of basic k -forms. To see this, note that every k -tuple $\{j_1, \dots, j_k\}$ of distinct integers can be converted to an increasing k -index I by a finite number of interchanges of pairs; each of these amounts to a multiplication by -1 , as we saw in Sec. 10.13; hence

$$(45) \quad dx_{j_1} \wedge \dots \wedge dx_{j_k} = \varepsilon(j_1, \dots, j_k) dx_I$$

where $\varepsilon(j_1, \dots, j_k)$ is 1 or -1 , depending on the number of interchanges that are needed. In fact, it is easy to see that

$$(46) \quad \varepsilon(j_1, \dots, j_k) = s(j_1, \dots, j_k)$$

where s is as in Definition 9.33.

For example,

$$dx_1 \wedge dx_5 \wedge dx_3 \wedge dx_2 = -dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_5$$

and

$$dx_4 \wedge dx_2 \wedge dx_3 = dx_2 \wedge dx_3 \wedge dx_4.$$

If every k -tuple in (34) is converted to an increasing k -index, then we obtain the so-called *standard presentation* of ω :

$$(47) \quad \omega = \sum_I b_I(\mathbf{x}) dx_I.$$

The summation in (47) extends over all increasing k -indices I . [Of course, every increasing k -index arises from many (from $k!$, to be precise) k -tuples. Each b_I in (47) may thus be a sum of several of the coefficients that occur in (34).]

For example,

$$x_1 dx_2 \wedge dx_1 - x_2 dx_3 \wedge dx_2 + x_3 dx_2 \wedge dx_3 + dx_1 \wedge dx_2$$

is a 2-form in R^3 whose standard presentation is

$$(1 - x_1) dx_1 \wedge dx_2 + (x_2 + x_3) dx_2 \wedge dx_3.$$

The following uniqueness theorem is one of the main reasons for the introduction of the standard presentation of a k -form.

10.15 Theorem *Suppose*

$$(48) \quad \omega = \sum_I b_I(\mathbf{x}) dx_I$$

is the standard presentation of a k -form ω in an open set $E \subset R^n$. If $\omega = 0$ in E , then $b_I(\mathbf{x}) = 0$ for every increasing k -index I and for every $\mathbf{x} \in E$.

Note that the analogous statement would be false for sums such as (34), since, for example,

$$dx_1 \wedge dx_2 + dx_2 \wedge dx_1 = 0.$$

Proof Assume, to reach a contradiction, that $b_J(\mathbf{v}) > 0$ for some $\mathbf{v} \in E$ and for some increasing k -index $J = \{j_1, \dots, j_k\}$. Since b_J is continuous, there exists $h > 0$ such that $b_J(\mathbf{x}) > 0$ for all $\mathbf{x} \in R^n$ whose coordinates satisfy $|x_i - v_i| \leq h$. Let D be the k -cell in R^k such that $\mathbf{u} \in D$ if and only if $|u_r| \leq h$ for $r = 1, \dots, k$. Define

$$(49) \quad \Phi(\mathbf{u}) = \mathbf{v} + \sum_{r=1}^k u_r \mathbf{e}_{j_r} \quad (\mathbf{u} \in D).$$

Then Φ is a k -surface in E , with parameter domain D , and $b_J(\Phi(\mathbf{u})) > 0$ for every $\mathbf{u} \in D$.

We claim that

$$(50) \quad \int_{\Phi} \omega = \int_D b_J(\Phi(\mathbf{u})) d\mathbf{u}.$$

Since the right side of (50) is positive, it follows that $\omega(\Phi) \neq 0$. Hence (50) gives our contradiction.

To prove (50), apply (35) to the presentation (48). More specifically, compute the Jacobians that occur in (35). By (49),

$$\frac{\partial(x_{j_1}, \dots, x_{j_k})}{\partial(u_1, \dots, u_k)} = 1.$$

For any other increasing k -index $I \neq J$, the Jacobian is 0, since it is the determinant of a matrix with at least one row of zeros.

10.16 Products of basic k -forms *Suppose*

$$(51) \quad I = \{i_1, \dots, i_p\}, \quad J = \{j_1, \dots, j_q\}$$

where $1 \leq i_1 < \dots < i_p \leq n$ and $1 \leq j_1 < \dots < j_q \leq n$. The *product* of the corresponding basic forms dx_I and dx_J in R^n is a $(p+q)$ -form in R^n , denoted by the symbol $dx_I \wedge dx_J$, and defined by

$$(52) \quad dx_I \wedge dx_J = dx_{i_1} \wedge \dots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_q}.$$

If I and J have an element in common, then the discussion in Sec. 10.13 shows that $dx_I \wedge dx_J = 0$.

If I and J have no element in common, let us write $[I, J]$ for the increasing $(p + q)$ -index which is obtained by arranging the members of $I \cup J$ in increasing order. Then $dx_{[I, J]}$ is a basic $(p + q)$ -form. We claim that

$$(53) \quad dx_I \wedge dx_J = (-1)^\alpha dx_{[I, J]}$$

where α is the number of differences $j_t - i_s$ that are *negative*. (The number of positive differences is thus $pq - \alpha$.)

To prove (53), perform the following operations on the numbers

$$(54) \quad i_1, \dots, i_p; j_1, \dots, j_q.$$

Move i_p to the right, step by step, until its right neighbor is larger than i_p . The number of steps is the number of subscripts t such that $i_p < j_t$. (Note that 0 steps are a distinct possibility.) Then do the same for i_{p-1}, \dots, i_1 . The total number of steps taken is α . The final arrangement reached is $[I, J]$. Each step, when applied to the right side of (52), multiplies $dx_I \wedge dx_J$ by -1 . Hence (53) holds.

Note that the right side of (53) is the standard presentation of $dx_I \wedge dx_J$.

Next, let $K = (k_1, \dots, k_r)$ be an increasing r -index in $\{1, \dots, n\}$. We shall use (53) to prove that

$$(55) \quad (dx_I \wedge dx_J) \wedge dx_K = dx_I \wedge (dx_J \wedge dx_K).$$

If any two of the sets I, J, K have an element in common, then each side of (55) is 0, hence they are equal.

So let us assume that I, J, K are pairwise disjoint. Let $[I, J, K]$ denote the increasing $(p + q + r)$ -index obtained from their union. Associate β with the ordered pair (J, K) and γ with the ordered pair (I, K) in the way that α was associated with (I, J) in (53). The left side of (55) is then

$$(-1)^\alpha dx_{[I, J]} \wedge dx_K = (-1)^\alpha (-1)^{\beta+\gamma} dx_{[I, J, K]}$$

by two applications of (53), and the right side of (55) is

$$(-1)^\beta dx_I \wedge dx_{[J, K]} = (-1)^\beta (-1)^{\alpha+\gamma} dx_{[I, J, K]}.$$

Hence (55) is correct.

10.17 Multiplication Suppose ω and λ are p - and q -forms, respectively, in some open set $E \subset R^n$, with standard presentations

$$(56) \quad \omega = \sum_I b_I(\mathbf{x}) dx_I, \quad \lambda = \sum_J c_J(\mathbf{x}) dx_J$$

where I and J range over all increasing p -indices and over all increasing q -indices taken from the set $\{1, \dots, n\}$.

Their product, denoted by the symbol $\omega \wedge \lambda$, is defined to be

$$(57) \quad \omega \wedge \lambda = \sum_{I, J} b_I(\mathbf{x}) c_J(\mathbf{x}) dx_I \wedge dx_J.$$

In this sum, I and J range independently over their possible values, and $dx_I \wedge dx_J$ is as in Sec. 10.16. Thus $\omega \wedge \lambda$ is a $(p+q)$ -form in E .

It is quite easy to see (we leave the details as an exercise) that the distributive laws

$$(\omega_1 + \omega_2) \wedge \lambda = (\omega_1 \wedge \lambda) + (\omega_2 \wedge \lambda)$$

and

$$\omega \wedge (\lambda_1 + \lambda_2) = (\omega \wedge \lambda_1) + (\omega \wedge \lambda_2)$$

hold, with respect to the addition defined in Sec. 10.13. If these distributive laws are combined with (55), we obtain the associative law

$$(58) \quad (\omega \wedge \lambda) \wedge \sigma = \omega \wedge (\lambda \wedge \sigma)$$

for arbitrary forms ω, λ, σ in E .

In this discussion it was tacitly assumed that $p \geq 1$ and $q \geq 1$. The product of a 0-form f with the p -form ω given by (56) is simply defined to be the p -form

$$f\omega = \omega f = \sum_I f(\mathbf{x}) b_I(\mathbf{x}) dx_I.$$

It is customary to write $f\omega$, rather than $f \wedge \omega$, when f is a 0-form.

10.18 Differentiation We shall now define a differentiation operator d which associates a $(k+1)$ -form $d\omega$ to each k -form ω of class \mathcal{C}' in some open set $E \subset R^n$.

A 0-form of class \mathcal{C}' in E is just a real function $f \in \mathcal{C}'(E)$, and we define

$$(59) \quad df = \sum_{i=1}^n (D_i f)(\mathbf{x}) dx_i.$$

If $\omega = \sum b_I(\mathbf{x}) dx_I$ is the standard presentation of a k -form ω , and $b_I \in \mathcal{C}'(E)$ for each increasing k -index I , then we define

$$(60) \quad d\omega = \sum_I (db_I) \wedge dx_I.$$

10.19 Example Suppose E is open in R^n , $f \in \mathcal{C}'(E)$, and γ is a continuously differentiable curve in E , with domain $[0, 1]$. By (59) and (35),

$$(61) \quad \int_{\gamma} df = \int_0^1 \sum_{i=1}^n (D_i f)(\gamma(t)) \gamma'_i(t) dt.$$

By the chain rule, the last integrand is $(f \circ \gamma)'(t)$. Hence

$$(62) \quad \int_{\gamma} df = f(\gamma(1)) - f(\gamma(0)),$$

and we see that $\int_{\gamma} df$ is the same for all γ with the same initial point and the same end point, as in (a) of Example 10.12.

Comparison with Example 10.12(b) shows therefore that the 1-form $x dy$ is not the derivative of any 0-form f . This could also be deduced from part (b) of the following theorem, since

$$d(x dy) = dx \wedge dy \neq 0.$$

10.20 Theorem

(a) If ω and λ are k - and m -forms, respectively, of class \mathcal{C}' in E , then

$$(63) \quad d(\omega \wedge \lambda) = (d\omega) \wedge \lambda + (-1)^k \omega \wedge d\lambda.$$

(b) If ω is of class \mathcal{C}'' in E , then $d^2\omega = 0$.

Here $d^2\omega$ means, of course, $d(d\omega)$.

Proof Because of (57) and (60), (a) follows if (63) is proved for the special case

$$(64) \quad \omega = f dx_I, \quad \lambda = g dx_J$$

where $f, g \in \mathcal{C}'(E)$, dx_I is a basic k -form, and dx_J is a basic m -form. [If k or m or both are 0, simply omit dx_I or dx_J in (64); the proof that follows is unaffected by this.] Then

$$\omega \wedge \lambda = fg dx_I \wedge dx_J.$$

Let us assume that I and J have no element in common. [In the other case each of the three terms in (63) is 0.] Then, using (53),

$$d(\omega \wedge \lambda) = d(fg dx_I \wedge dx_J) = (-1)^a d(fg dx_{[I, J]}).$$

By (59), $d(fg) = f dg + g df$. Hence (60) gives

$$\begin{aligned} d(\omega \wedge \lambda) &= (-1)^a (f dg + g df) \wedge dx_{[I, J]} \\ &= (g df + f dg) \wedge dx_I \wedge dx_J. \end{aligned}$$

Since dg is a 1-form and dx_I is a k -form, we have

$$dg \wedge dx_I = (-1)^k dx_I \wedge dg,$$

by (42). Hence

$$\begin{aligned} d(\omega \wedge \lambda) &= (df \wedge dx_I) \wedge (g dx_J) + (-1)^k (f dx_I) \wedge (dg \wedge dx_J) \\ &= (d\omega) \wedge \lambda + (-1)^k \omega \wedge d\lambda, \end{aligned}$$

which proves (a).

Note that the associative law (58) was used freely.

Let us prove (b) first for a 0-form $f \in \mathcal{C}''$:

$$\begin{aligned} d^2 f &= d \left(\sum_{j=1}^n (D_j f)(\mathbf{x}) dx_j \right) \\ &= \sum_{j=1}^n d(D_j f) \wedge dx_j \\ &= \sum_{i,j=1}^n (D_{ij} f)(\mathbf{x}) dx_i \wedge dx_j. \end{aligned}$$

Since $D_{ij} f = D_{ji} f$ (Theorem 9.41) and $dx_i \wedge dx_j = -dx_j \wedge dx_i$, we see that $d^2 f = 0$.

If $\omega = f dx_I$, as in (64), then $d\omega = (df) \wedge dx_I$. By (60), $d(dx_I) = 0$. Hence (63) shows that

$$d^2 \omega = (d^2 f) \wedge dx_I = 0.$$

10.21 Change of variables Suppose E is an open set in R^n , T is a \mathcal{C}' -mapping of E into an open set $V \subset R^m$, and ω is a k -form in V , whose standard presentation is

$$(65) \quad \omega = \sum_I b_I(\mathbf{y}) dy_I.$$

(We use \mathbf{y} for points of V , \mathbf{x} for points of E .)

Let t_1, \dots, t_m be the components of T : If

$$\mathbf{y} = (y_1, \dots, y_m) = T(\mathbf{x})$$

then $y_i = t_i(\mathbf{x})$. As in (59),

$$(66) \quad dt_i = \sum_{j=1}^n (D_j t_i)(\mathbf{x}) dx_j \quad (1 \leq i \leq m).$$

Thus each dt_i is a 1-form in E .

The mapping T transforms ω into a k -form ω_T in E , whose definition is

$$(67) \quad \omega_T = \sum_I b_I(T(\mathbf{x})) dt_{i_1} \wedge \cdots \wedge dt_{i_k}.$$

In each summand of (67), $I = \{i_1, \dots, i_k\}$ is an increasing k -index.

Our next theorem shows that addition, multiplication, and differentiation of forms are defined in such a way that they commute with changes of variables.

10.22 Theorem *With E and T as in Sec. 10.21, let ω and λ be k - and m -forms in V , respectively. Then*

- (a) $(\omega + \lambda)_T = \omega_T + \lambda_T$ if $k = m$;
- (b) $(\omega \wedge \lambda)_T = \omega_T \wedge \lambda_T$;
- (c) $d(\omega_T) = (d\omega)_T$ if ω is of class \mathcal{C}' and T is of class \mathcal{C}'' .

Proof Part (a) follows immediately from the definitions. Part (b) is almost as obvious, once we realize that

$$(68) \quad (dy_{i_1} \wedge \cdots \wedge dy_{i_r})_T = dt_{i_1} \wedge \cdots \wedge dt_{i_r}$$

regardless of whether $\{i_1, \dots, i_r\}$ is increasing or not; (68) holds because the same number of minus signs are needed on each side of (68) to produce increasing rearrangements.

We turn to the proof of (c). If f is a 0-form of class \mathcal{C}' in V , then

$$f_T(\mathbf{x}) = f(T(\mathbf{x})), \quad df = \sum_i (D_i f)(\mathbf{y}) dy_i.$$

By the chain rule, it follows that

$$(69) \quad \begin{aligned} d(f_T) &= \sum_j (D_j f_T)(\mathbf{x}) dx_j \\ &= \sum_j \sum_i (D_i f)(T(\mathbf{x}))(D_j t_i)(\mathbf{x}) dx_j \\ &= \sum_i (D_i f)(T(\mathbf{x})) dt_i \\ &= (df)_T. \end{aligned}$$

If $dy_I = dy_{i_1} \wedge \cdots \wedge dy_{i_k}$, then $(dy_I)_T = dt_{i_1} \wedge \cdots \wedge dt_{i_k}$, and Theorem 10.20 shows that

$$(70) \quad d((dy_I)_T) = 0.$$

(This is where the assumption $T \in \mathcal{C}''$ is used.)

Assume now that $\omega = f dy_I$. Then

$$\omega_T = f_T(\mathbf{x}) (dy_I)_T$$

and the preceding calculations lead to

$$\begin{aligned} d(\omega_T) &= d(f_T) \wedge (dy_I)_T = (df)_T \wedge (dy_I)_T \\ &= ((df) \wedge dy_I)_T = (d\omega)_T. \end{aligned}$$

The first equality holds by (63) and (70), the second by (69), the third by part (b), and the last by the definition of $d\omega$.

The general case of (c) follows from the special case just proved, if we apply (a). This completes the proof.

Our next objective is Theorem 10.25. This will follow directly from two other important transformation properties of differential forms, which we state first.

10.23 Theorem *Suppose T is a \mathcal{C}' -mapping of an open set $E \subset R^n$ into an open set $V \subset R^m$, S is a \mathcal{C}' -mapping of V into an open set $W \subset R^p$, and ω is a k -form in W , so that ω_S is a k -form in V and both $(\omega_S)_T$ and ω_{ST} are k -forms in E , where ST is defined by $(ST)(\mathbf{x}) = S(T(\mathbf{x}))$. Then*

$$(71) \quad (\omega_S)_T = \omega_{ST}.$$

Proof If ω and λ are forms in W , Theorem 10.22 shows that

$$((\omega \wedge \lambda)_S)_T = (\omega_S \wedge \lambda_S)_T = (\omega_S)_T \wedge (\lambda_S)_T$$

and

$$(\omega \wedge \lambda)_{ST} = \omega_{ST} \wedge \lambda_{ST}.$$

Thus if (71) holds for ω and for λ , it follows that (71) also holds for $\omega \wedge \lambda$. Since every form can be built up from 0-forms and 1-forms by addition and multiplication, and since (71) is trivial for 0-forms, it is enough to prove (71) in the case $\omega = dz_q$, $q = 1, \dots, p$. (We denote the points of E, V, W by $\mathbf{x}, \mathbf{y}, \mathbf{z}$, respectively.)

Let t_1, \dots, t_m be the components of T , let s_1, \dots, s_p be the components of S , and let r_1, \dots, r_p be the components of ST . If $\omega = dz_q$, then

$$\omega_S = ds_q = \sum_j (D_j s_q)(\mathbf{y}) dy_j,$$

so that the chain rule implies

$$\begin{aligned} (\omega_S)_T &= \sum_j (D_j s_q)(T(\mathbf{x})) dt_j \\ &= \sum_j (D_j s_q)(T(\mathbf{x})) \sum_i (D_i t_j)(\mathbf{x}) dx_i \\ &= \sum_i (D_i r_q)(\mathbf{x}) dx_i = dr_q = \omega_{ST}. \end{aligned}$$

10.24 Theorem *Suppose ω is a k -form in an open set $E \subset R^n$, Φ is a k -surface in E , with parameter domain $D \subset R^k$, and Δ is the k -surface in R^k , with parameter domain D , defined by $\Delta(\mathbf{u}) = \mathbf{u}(\mathbf{u} \in D)$. Then*

$$\int_{\Phi} \omega = \int_{\Delta} \omega_{\Phi}.$$

Proof We need only consider the case

$$\omega = a(\mathbf{x}) dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

If ϕ_1, \dots, ϕ_n are the components of Φ , then

$$\omega_\Phi = a(\Phi(\mathbf{u})) d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k}.$$

The theorem will follow if we can show that

$$(72) \quad d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = J(\mathbf{u}) du_1 \wedge \cdots \wedge du_k,$$

where

$$J(\mathbf{u}) = \frac{\partial(x_{i_1}, \dots, x_{i_k})}{\partial(u_1, \dots, u_k)},$$

since (72) implies

$$\begin{aligned} \int_\Phi \omega &= \int_D a(\Phi(\mathbf{u})) J(\mathbf{u}) d\mathbf{u} \\ &= \int_\Delta a(\Phi(\mathbf{u})) J(\mathbf{u}) du_1 \wedge \cdots \wedge du_k = \int_\Delta \omega_\Phi. \end{aligned}$$

Let $[A]$ be the k by k matrix with entries

$$\alpha(p, q) = (D_q \phi_{i_p})(\mathbf{u}) \quad (p, q = 1, \dots, k).$$

Then

$$d\phi_{i_p} = \sum_q \alpha(p, q) du_q$$

so that

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = \sum \alpha(1, q_1) \cdots \alpha(k, q_k) du_{q_1} \wedge \cdots \wedge du_{q_k}.$$

In this last sum, q_1, \dots, q_k range independently over $1, \dots, k$. The anti-commutative relation (42) implies that

$$du_{q_1} \wedge \cdots \wedge du_{q_k} = s(q_1, \dots, q_k) du_1 \wedge \cdots \wedge du_k,$$

where s is as in Definition 9.33; applying this definition, we see that

$$d\phi_{i_1} \wedge \cdots \wedge d\phi_{i_k} = \det [A] du_1 \wedge \cdots \wedge du_k;$$

and since $J(\mathbf{u}) = \det [A]$, (72) is proved.

The final result of this section combines the two preceding theorems.

10.25 Theorem Suppose T is a \mathcal{C}' -mapping of an open set $E \subset R^n$ into an open set $V \subset R^m$, Φ is a k -surface in E , and ω is a k -form in V .

Then

$$\int_{T\Phi} \omega = \int_\Phi \omega_T.$$

Proof Let D be the parameter domain of Φ (hence also of $T\Phi$) and define Δ as in Theorem 10.24.

Then

$$\int_{T\Phi} \omega = \int_{\Delta} \omega_{T\Phi} = \int_{\Delta} (\omega_T)_{\Phi} = \int_{\Phi} \omega_T.$$

The first of these equalities is Theorem 10.24, applied to $T\Phi$ in place of Φ . The second follows from Theorem 10.23. The third is Theorem 10.24, with ω_T in place of ω .

SIMPLEXES AND CHAINS

10.26 Affine simplexes A mapping \mathbf{f} that carries a vector space X into a vector space Y is said to be *affine* if $\mathbf{f} - \mathbf{f}(0)$ is linear. In other words, the requirement is that

$$(73) \quad \mathbf{f}(\mathbf{x}) = \mathbf{f}(0) + A\mathbf{x}$$

for some $A \in L(X, Y)$.

An affine mapping of R^k into R^n is thus determined if we know $\mathbf{f}(0)$ and $\mathbf{f}(\mathbf{e}_i)$ for $1 \leq i \leq k$; as usual, $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$ is the standard basis of R^k .

We define the *standard simplex* Q^k to be the set of all $\mathbf{u} \in R^k$ of the form

$$(74) \quad \mathbf{u} = \sum_{i=1}^k \alpha_i \mathbf{e}_i$$

such that $\alpha_i \geq 0$ for $i = 1, \dots, k$ and $\sum \alpha_i \leq 1$.

Assume now that $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k$ are points of R^n . The *oriented affine k -simplex*

$$(75) \quad \sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$$

is defined to be the k -surface in R^n with parameter domain Q^k which is given by the affine mapping

$$(76) \quad \sigma(\alpha_1 \mathbf{e}_1 + \dots + \alpha_k \mathbf{e}_k) = \mathbf{p}_0 + \sum_{i=1}^k \alpha_i (\mathbf{p}_i - \mathbf{p}_0).$$

Note that σ is characterized by

$$(77) \quad \sigma(0) = \mathbf{p}_0, \quad \sigma(\mathbf{e}_i) = \mathbf{p}_i \quad (\text{for } 1 \leq i \leq k),$$

and that

$$(78) \quad \sigma(\mathbf{u}) = \mathbf{p}_0 + A\mathbf{u} \quad (\mathbf{u} \in Q^k)$$

where $A \in L(R^k, R^n)$ and $A\mathbf{e}_i = \mathbf{p}_i - \mathbf{p}_0$ for $1 \leq i \leq k$.

We call σ *oriented* to emphasize that the ordering of the vertices $\mathbf{p}_0, \dots, \mathbf{p}_k$ is taken into account. If

$$(79) \quad \bar{\sigma} = [p_{i_0}, p_{i_1}, \dots, p_{i_k}],$$

where $\{i_0, i_1, \dots, i_k\}$ is a permutation of the ordered set $\{0, 1, \dots, k\}$, we adopt the notation

$$(80) \quad \bar{\sigma} = s(i_0, i_1, \dots, i_k)\sigma,$$

where s is the function defined in Definition 9.33. Thus $\bar{\sigma} = \pm\sigma$, depending on whether $s = 1$ or $s = -1$. Strictly speaking, having adopted (75) and (76) as the definition of σ , we should not write $\bar{\sigma} = \sigma$ unless $i_0 = 0, \dots, i_k = k$, even if $s(i_0, \dots, i_k) = 1$; what we have here is an equivalence relation, not an equality. However, for our purposes the notation is justified by Theorem 10.27.

If $\bar{\sigma} = \varepsilon\sigma$ (using the above convention) and if $\varepsilon = 1$, we say that $\bar{\sigma}$ and σ have the *same orientation*; if $\varepsilon = -1$, $\bar{\sigma}$ and σ are said to have *opposite orientations*. Note that we have not defined what we mean by the “orientation of a simplex.” What we have defined is a relation between pairs of simplexes having the same set of vertices, the relation being that of “having the same orientation.”

There is, however, one situation where the orientation of a simplex can be defined in a natural way. This happens when $n = k$ and when the vectors $\mathbf{p}_i - \mathbf{p}_0$ ($1 \leq i \leq k$) are *independent*. In that case, the linear transformation A that appears in (78) is invertible, and its determinant (which is the same as the Jacobian of σ) is not 0. Then σ is said to be *positively* (or *negatively*) oriented if $\det A$ is positive (or negative). In particular, the simplex $[\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k]$ in R^k , given by the identity mapping, has positive orientation.

So far we have assumed that $k \geq 1$. An *oriented 0-simplex* is defined to be a point with a sign attached. We write $\sigma = +\mathbf{p}_0$ or $\sigma = -\mathbf{p}_0$. If $\sigma = \varepsilon\mathbf{p}_0$ ($\varepsilon = \pm 1$) and if f is a 0-form (i.e., a real function), we define

$$\int_{\sigma} f = \varepsilon f(\mathbf{p}_0).$$

10.27 Theorem *If σ is an oriented rectilinear k -simplex in an open set $E \subset R^n$ and if $\bar{\sigma} = \varepsilon\sigma$ then*

$$(81) \quad \int_{\bar{\sigma}} \omega = \varepsilon \int_{\sigma} \omega$$

for every k -form ω in E .

Proof For $k = 0$, (81) follows from the preceding definition. So we assume $k \geq 1$ and assume that σ is given by (75).

Suppose $1 \leq j \leq k$, and suppose $\bar{\sigma}$ is obtained from σ by interchanging \mathbf{p}_0 and \mathbf{p}_j . Then $\varepsilon = -1$, and

$$\bar{\sigma}(\mathbf{u}) = \mathbf{p}_j + B\mathbf{u} \quad (\mathbf{u} \in Q^k),$$

where B is the linear mapping of R^k into R^n defined by $Be_j = \mathbf{p}_0 - \mathbf{p}_j$, $Be_i = \mathbf{p}_i - \mathbf{p}_j$ if $i \neq j$. If we write $Ae_i = \mathbf{x}_i$ ($1 \leq i \leq k$), where A is given by (78), the column vectors of B (that is, the vectors Be_i) are

$$\mathbf{x}_1 - \mathbf{x}_j, \dots, \mathbf{x}_{j-1} - \mathbf{x}_j, -\mathbf{x}_j, \mathbf{x}_{j+1} - \mathbf{x}_j, \dots, \mathbf{x}_k - \mathbf{x}_j.$$

If we subtract the j th column from each of the others, none of the determinants in (35) are affected, and we obtain columns $\mathbf{x}_1, \dots, \mathbf{x}_{j-1}, -\mathbf{x}_j, \mathbf{x}_{j+1}, \dots, \mathbf{x}_k$. These differ from those of A only in the sign of the j th column. Hence (81) holds for this case.

Suppose next that $0 < i < j \leq k$ and that $\bar{\sigma}$ is obtained from σ by interchanging \mathbf{p}_i and \mathbf{p}_j . Then $\bar{\sigma}(\mathbf{u}) = \mathbf{p}_0 + C\mathbf{u}$, where C has the same columns as A , except that the i th and j th columns have been interchanged. This again implies that (81) holds, since $\varepsilon = -1$.

The general case follows, since every permutation of $\{0, 1, \dots, k\}$ is a composition of the special cases we have just dealt with.

10.28 Affine chains An *affine k -chain* Γ in an open set $E \subset R^n$ is a collection of finitely many oriented affine k -simplexes $\sigma_1, \dots, \sigma_r$ in E . These need not be distinct; a simplex may thus occur in Γ with a certain multiplicity.

If Γ is as above, and if ω is a k -form in E , we define

$$(82) \quad \int_{\Gamma} \omega = \sum_{i=1}^r \int_{\sigma_i} \omega.$$

We may view a k -surface Φ in E as a function whose domain is the collection of all k -forms in E and which assigns the number $\int_{\Phi} \omega$ to ω . Since real-valued functions can be added (as in Definition 4.3), this suggests the use of the notation

$$(83) \quad \Gamma = \sigma_1 + \dots + \sigma_r$$

or, more compactly,

$$(84) \quad \Gamma = \sum_{i=1}^r \sigma_i$$

to state the fact that (82) holds for every k -form ω in E .

To avoid misunderstanding, we point out explicitly that the notations introduced by (83) and (80) have to be handled with care. The point is that every oriented affine k -simplex σ in R^n is a function in two ways, with different domains and different ranges, and that therefore two entirely different operations

of addition are possible. Originally, σ was defined as an R^n -valued function with domain Q^k ; accordingly, $\sigma_1 + \sigma_2$ *could* be interpreted to be the function σ that assigns the vector $\sigma_1(\mathbf{u}) + \sigma_2(\mathbf{u})$ to every $\mathbf{u} \in Q^k$; note that σ is then again an oriented affine k -simplex in R^n ! This is *not* what is meant by (83).

For example, if $\sigma_2 = -\sigma_1$ as in (80) (that is to say, if σ_1 and σ_2 have the same set of vertices but are oppositely oriented) and if $\Gamma = \sigma_1 + \sigma_2$, then $\int_\Gamma \omega = 0$ for all ω , and we may express this by writing $\Gamma = 0$ or $\sigma_1 + \sigma_2 = 0$. This does not mean that $\sigma_1(\mathbf{u}) + \sigma_2(\mathbf{u})$ is the null vector of R^n .

10.29 Boundaries For $k \geq 1$, the *boundary* of the oriented affine k -simplex

$$\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$$

is defined to be the affine $(k-1)$ -chain

$$(85) \quad \partial\sigma = \sum_{j=0}^k (-1)^j [\mathbf{p}_0, \dots, \mathbf{p}_{j-1}, \mathbf{p}_{j+1}, \dots, \mathbf{p}_k].$$

For example, if $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2]$, then

$$\partial\sigma = [\mathbf{p}_1, \mathbf{p}_2] - [\mathbf{p}_0, \mathbf{p}_2] + [\mathbf{p}_0, \mathbf{p}_1] = [\mathbf{p}_0, \mathbf{p}_1] + [\mathbf{p}_1, \mathbf{p}_2] + [\mathbf{p}_2, \mathbf{p}_0],$$

which coincides with the usual notion of the oriented boundary of a triangle.

For $1 \leq j \leq k$, observe that the simplex $\sigma_j = [\mathbf{p}_0, \dots, \mathbf{p}_{j-1}, \mathbf{p}_{j+1}, \dots, \mathbf{p}_k]$ which occurs in (85) has Q^{k-1} as its parameter domain and that it is defined by

$$(86) \quad \sigma_j(\mathbf{u}) = \mathbf{p}_0 + B\mathbf{u} \quad (\mathbf{u} \in Q^{k-1}),$$

where B is the linear mapping from R^{k-1} to R^n determined by

$$\begin{aligned} B\mathbf{e}_i &= \mathbf{p}_i - \mathbf{p}_0 & (\text{if } 1 \leq i \leq j-1), \\ B\mathbf{e}_i &= \mathbf{p}_{i+1} - \mathbf{p}_0 & (\text{if } j \leq i \leq k-1). \end{aligned}$$

The simplex

$$\sigma_0 = [\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_k],$$

which also occurs in (85), is given by the mapping

$$\sigma_0(\mathbf{u}) = \mathbf{p}_1 + B\mathbf{u},$$

where $B\mathbf{e}_i = \mathbf{p}_{i+1} - \mathbf{p}_1$ for $1 \leq i \leq k-1$.

10.30 Differentiable simplexes and chains Let T be a \mathcal{C}'' -mapping of an open set $E \subset R^n$ into an open set $V \subset R^m$; T need not be one-to-one. If σ is an oriented affine k -simplex in E , then the composite mapping $\Phi = T \circ \sigma$ (which we shall sometimes write in the simpler form $T\sigma$) is a k -surface in V , with parameter domain Q^k . We call Φ an *oriented k -simplex of class \mathcal{C}''* .

A finite collection Ψ of oriented k -simplexes Φ_1, \dots, Φ_r of class \mathcal{C}'' in V is called a k -chain of class \mathcal{C}'' in V . If ω is a k -form in V , we define

$$(87) \quad \int_{\Psi} \omega = \sum_{i=1}^r \int_{\Phi_i} \omega$$

and use the corresponding notation $\Psi = \Sigma \Phi_i$.

If $\Gamma = \Sigma \sigma_i$ is an affine chain and if $\Phi_i = T \circ \sigma_i$, we also write $\Psi = T \circ \Gamma$, or

$$(88) \quad T(\Sigma \sigma_i) = \Sigma T\sigma_i.$$

The boundary $\partial\Phi$ of the oriented k -simplex $\Phi = T \circ \sigma$ is defined to be the $(k-1)$ chain

$$(89) \quad \partial\Phi = T(\partial\sigma).$$

In justification of (89), observe that if T is affine, then $\Phi = T \circ \sigma$ is an oriented affine k -simplex, in which case (89) is not a matter of definition, but is seen to be a *consequence* of (85). Thus (89) generalizes this special case.

It is immediate that $\partial\Phi$ is of class \mathcal{C}'' if this is true of Φ .

Finally, we define the boundary $\partial\Psi$ of the k -chain $\Psi = \Sigma \Phi_i$ to be the $(k-1)$ chain

$$(90) \quad \partial\Psi = \Sigma \partial\Phi_i.$$

10.31 Positively oriented boundaries So far we have associated boundaries to chains, not to subsets of R^n . This notion of boundary is exactly the one that is most suitable for the statement and proof of Stokes' theorem. However, in applications, especially in R^2 or R^3 , it is customary and convenient to talk about "oriented boundaries" of certain sets as well. We shall now describe this briefly.

Let Q^n be the standard simplex in R^n , let σ_0 be the identity mapping with domain Q^n . As we saw in Sec. 10.26, σ_0 may be regarded as a positively oriented n -simplex in R^n . Its boundary $\partial\sigma_0$ is an affine $(n-1)$ -chain. This chain is called the *positively oriented boundary of the set Q^n* .

For example, the positively oriented boundary of Q^3 is

$$[e_1, e_2, e_3] - [0, e_2, e_3] + [0, e_1, e_3] - [0, e_1, e_2].$$

Now let T be a 1-1 mapping of Q^n into R^n , of class \mathcal{C}'' , whose Jacobian is positive (at least in the interior of Q^n). Let $E = T(Q^n)$. By the inverse function theorem, E is the closure of an open subset of R^n . We define the positively oriented boundary of the set E to be the $(n-1)$ -chain

$$\partial T = T(\partial\sigma_0),$$

and we may denote this $(n-1)$ -chain by ∂E .

An obvious question occurs here: If $E = T_1(Q^n) = T_2(Q^n)$, and if both T_1 and T_2 have positive Jacobians, is it true that $\partial T_1 = \partial T_2$? That is to say, does the equality

$$\int_{\partial T_1} \omega = \int_{\partial T_2} \omega$$

hold for every $(n-1)$ -form ω ? The answer is yes, but we shall omit the proof. (To see an example, compare the end of this section with Exercise 17.)

One can go further. Let

$$\Omega = E_1 \cup \cdots \cup E_r,$$

where $E_i = T_i(Q^n)$, each T_i has the properties that T had above, and the interiors of the sets E_i are pairwise disjoint. Then the $(n-1)$ -chain

$$\partial T_1 + \cdots + \partial T_r = \partial \Omega$$

is called the positively oriented boundary of Ω .

For example, the unit square I^2 in R^2 is the union of $\sigma_1(Q^2)$ and $\sigma_2(Q^2)$, where

$$\sigma_1(\mathbf{u}) = \mathbf{u}, \quad \sigma_2(\mathbf{u}) = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{u}.$$

Both σ_1 and σ_2 have Jacobian $1 > 0$. Since

$$\sigma_1 = [0, \mathbf{e}_1, \mathbf{e}_2], \quad \sigma_2 = [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1]$$

we have

$$\begin{aligned} \partial \sigma_1 &= [\mathbf{e}_1, \mathbf{e}_2] - [0, \mathbf{e}_2] + [0, \mathbf{e}_1], \\ \partial \sigma_2 &= [\mathbf{e}_2, \mathbf{e}_1] - [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2]; \end{aligned}$$

The sum of these two boundaries is

$$\partial I^2 = [0, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, 0],$$

the positively oriented boundary of I^2 . Note that $[\mathbf{e}_1, \mathbf{e}_2]$ canceled $[\mathbf{e}_2, \mathbf{e}_1]$.

If Φ is a 2-surface in R^m , with parameter domain I^2 , then Φ (regarded as a function on 2-forms) is the same as the 2-chain

$$\Phi \circ \sigma_1 + \Phi \circ \sigma_2.$$

Thus

$$\begin{aligned} \partial \Phi &= \partial(\Phi \circ \sigma_1) + \partial(\Phi \circ \sigma_2) \\ &= \Phi(\partial \sigma_1) + \Phi(\partial \sigma_2) = \Phi(\partial I^2). \end{aligned}$$

In other words, if the parameter domain of Φ is the square I^2 , we need not refer back to the simplex Q^2 , but can obtain $\partial \Phi$ directly from ∂I^2 .

Other examples may be found in Exercises 17 to 19.

10.32 Example For $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$, define

$$\Sigma(u, v) = (\sin u \cos v, \sin u \sin v, \cos u).$$

Then Σ is a 2-surface in R^3 , whose parameter domain is a rectangle $D \subset R^2$, and whose range is the unit sphere in R^3 . Its boundary is

$$\partial\Sigma = \Sigma(\partial D) = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$$

where

$$\gamma_1(u) = \Sigma(u, 0) = (\sin u, 0, \cos u),$$

$$\gamma_2(v) = \Sigma(\pi, v) = (0, 0, -1),$$

$$\gamma_3(u) = \Sigma(\pi - u, 2\pi) = (\sin u, 0, -\cos u),$$

$$\gamma_4(v) = \Sigma(0, 2\pi - v) = (0, 0, 1),$$

with $[0, \pi]$ and $[0, 2\pi]$ as parameter intervals for u and v , respectively.

Since γ_2 and γ_4 are constant, their derivatives are 0, hence the integral of any 1-form over γ_2 or γ_4 is 0. [See Example 1.12(a).]

Since $\gamma_3(u) = \gamma_1(\pi - u)$, direct application of (35) shows that

$$\int_{\gamma_3} \omega = - \int_{\gamma_1} \omega$$

for every 1-form ω . Thus $\int_{\partial\Sigma} \omega = 0$, and we conclude that $\partial\Sigma = 0$.

(In geographic terminology, $\partial\Sigma$ starts at the north pole N , runs to the south pole S along a meridian, pauses at S , returns to N along the same meridian, and finally pauses at N . The two passages along the meridian are in opposite directions. The corresponding two line integrals therefore cancel each other. In Exercise 32 there is also one curve which occurs twice in the boundary, but without cancellation.)

STOKES' THEOREM

10.33 Theorem If Ψ is a k -chain of class \mathcal{C}'' in an open set $V \subset R^m$ and if ω is a $(k-1)$ -form of class \mathcal{C}' in V , then

$$(91) \quad \int_{\Psi} d\omega = \int_{\partial\Psi} \omega.$$

The case $k = m = 1$ is nothing but the fundamental theorem of calculus (with an additional differentiability assumption). The case $k = m = 2$ is Green's theorem, and $k = m = 3$ gives the so-called "divergence theorem" of Gauss. The case $k = 2$, $m = 3$ is the one originally discovered by Stokes. (Spivak's

book describes some of the historical background.) These special cases will be discussed further at the end of the present chapter.

Proof It is enough to prove that

$$(92) \quad \int_{\Phi} d\omega = \int_{\partial\Phi} \omega$$

for every oriented k -simplex Φ of class \mathcal{C}'' in V . For if (92) is proved and if $\Psi = \Sigma \Phi_i$, then (87) and (89) imply (91).

Fix such a Φ and put

$$(93) \quad \sigma = [0, \mathbf{e}_1, \dots, \mathbf{e}_k].$$

Thus σ is the oriented affine k -simplex with parameter domain Q^k which is defined by the identity mapping. Since Φ is also defined on Q^k (see Definition 10.30) and $\Phi \in \mathcal{C}''$, there is an open set $E \subset R^k$ which contains Q^k , and there is a \mathcal{C}'' -mapping T of E into V such that $\Phi = T \circ \sigma$. By Theorems 10.25 and 10.22(c), the left side of (92) is equal to

$$\int_{T\sigma} d\omega = \int_{\sigma} (d\omega)_T = \int_{\sigma} d(\omega_T).$$

Another application of Theorem 10.25 shows, by (89), that the right side of (92) is

$$\int_{\partial(T\sigma)} \omega = \int_{T(\partial\sigma)} \omega = \int_{\partial\sigma} \omega_T.$$

Since ω_T is a $(k-1)$ -form in E , we see that *in order to prove (92) we merely have to show that*

$$(94) \quad \int_{\sigma} d\lambda = \int_{\partial\sigma} \lambda$$

for the special simplex (93) and for every $(k-1)$ -form λ of class \mathcal{C}' in E .

If $k=1$, the definition of an oriented 0-simplex shows that (94) merely asserts that

$$(95) \quad \int_0^1 f'(u) du = f(1) - f(0)$$

for every continuously differentiable function f on $[0, 1]$, which is true by the fundamental theorem of calculus.

From now on we assume that $k > 1$, fix an integer r ($1 \leq r \leq k$), and choose $f \in \mathcal{C}'(E)$. It is then enough to prove (94) for the case

$$(96) \quad \lambda = f(\mathbf{x}) dx_1 \wedge \cdots \wedge dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_k$$

since every $(k-1)$ -form is a sum of these special ones, for $r = 1, \dots, k$.

By (85), the boundary of the simplex (93) is

$$\partial\sigma = [\mathbf{e}_1, \dots, \mathbf{e}_k] + \sum_{i=1}^k (-1)^i \tau_i$$

where

$$\tau_i = [0, \mathbf{e}_1, \dots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \dots, \mathbf{e}_k]$$

for $i = 1, \dots, k$. Put

$$\tau_0 = [\mathbf{e}_r, \mathbf{e}_1, \dots, \mathbf{e}_{r-1}, \mathbf{e}_{r+1}, \dots, \mathbf{e}_k].$$

Note that τ_0 is obtained from $[\mathbf{e}_1, \dots, \mathbf{e}_k]$ by $r - 1$ successive interchanges of \mathbf{e}_r and its left neighbors. Thus

$$(97) \quad \partial\sigma = (-1)^{r-1} \tau_0 + \sum_{i=1}^k (-1)^i \tau_i.$$

Each τ_i has Q^{k-1} as parameter domain.

If $\mathbf{x} = \tau_0(\mathbf{u})$ and $\mathbf{u} \in Q^{k-1}$, then

$$(98) \quad x_j = \begin{cases} u_j & (1 \leq j < r), \\ 1 - (u_1 + \dots + u_{k-1}) & (j = r), \\ u_{j-1} & (r < j \leq k). \end{cases}$$

If $1 \leq i \leq k$, $\mathbf{u} \in Q^{k-1}$, and $\mathbf{x} = \tau_i(\mathbf{u})$, then

$$(99) \quad x_j = \begin{cases} u_j & (1 \leq j < i), \\ 0 & (j = i), \\ u_{j-1} & (i < j \leq k). \end{cases}$$

For $0 \leq i \leq k$, let J_i be the Jacobian of the mapping

$$(100) \quad (u_1, \dots, u_{k-1}) \rightarrow (x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_k)$$

induced by τ_i . When $i = 0$ and when $i = r$, (98) and (99) show that (100) is the identity mapping. Thus $J_0 = 1$, $J_r = 1$. For other i , the fact that $x_i = 0$ in (99) shows that J_i has a row of zeros, hence $J_i = 0$. Thus

$$(101) \quad \int_{\tau_i} \lambda = 0 \quad (i \neq 0, i \neq r),$$

by (35) and (96). Consequently, (97) gives

$$(102) \quad \begin{aligned} \int_{\partial\sigma} \lambda &= (-1)^{r-1} \int_{\tau_0} \lambda + (-1)^r \int_{\tau_r} \lambda \\ &= (-1)^{r-1} \int [f(\tau_0(\mathbf{u})) - f(\tau_r(\mathbf{u}))] d\mathbf{u}. \end{aligned}$$

On the other hand,

$$\begin{aligned} d\lambda &= (D_r f)(\mathbf{x}) dx_r \wedge dx_1 \wedge \cdots \wedge dx_{r-1} \wedge dx_{r+1} \wedge \cdots \wedge dx_k \\ &= (-1)^{r-1} (D_r f)(\mathbf{x}) dx_1 \wedge \cdots \wedge dx_k \end{aligned}$$

so that

$$(103) \quad \int_{\sigma} d\lambda = (-1)^{r-1} \int_{Q^k} (D_r f)(\mathbf{x}) d\mathbf{x}.$$

We evaluate (103) by first integrating with respect to x_r , over the interval

$$[0, 1 - (x_1 + \cdots + x_{r-1} + x_{r+1} + \cdots + x_k)],$$

put $(x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_k) = (u_1, \dots, u_{k-1})$, and see with the aid of (98) that the integral over Q^k in (103) is equal to the integral over Q^{k-1} in (102). Thus (94) holds, and the proof is complete.

CLOSED FORMS AND EXACT FORMS

10.34 Definition Let ω be a k -form in an open set $E \subset R^n$. If there is a $(k-1)$ -form λ in E such that $\omega = d\lambda$, then ω is said to be *exact in E* .

If ω is of class \mathcal{C}' and $d\omega = 0$, then ω is said to be *closed*.

Theorem 10.20(b) shows that every exact form of class \mathcal{C}' is closed.

In certain sets E , for example in convex ones, the converse is true; this is the content of Theorem 10.39 (usually known as *Poincaré's lemma*) and Theorem 10.40. However, Examples 10.36 and 10.37 will exhibit closed forms that are not exact.

10.35 Remarks

(a) Whether a given k -form ω is or is not closed can be verified by simply differentiating the coefficients in the standard presentation of ω . For example, a 1-form

$$(104) \quad \omega = \sum_{i=1}^n f_i(\mathbf{x}) dx_i,$$

with $f_i \in \mathcal{C}'(E)$ for some open set $E \subset R^n$, is closed if and only if the equations

$$(105) \quad (D_j f_i)(\mathbf{x}) = (D_i f_j)(\mathbf{x})$$

hold for all i, j in $\{1, \dots, n\}$ and for all $\mathbf{x} \in E$.

Note that (105) is a “pointwise” condition; it does not involve any global properties that depend on the shape of E .

On the other hand, to show that ω is exact in E , one has to prove the existence of a form λ , defined in E , such that $d\lambda = \omega$. This amounts to solving a system of partial differential equations, not just locally, but in all of E . For example, to show that (104) is exact in a set E , one has to find a function (or 0-form) $g \in \mathcal{C}'(E)$ such that

$$(106) \quad (D_i g)(\mathbf{x}) = f_i(\mathbf{x}) \quad (\mathbf{x} \in E, 1 \leq i \leq n).$$

Of course, (105) is a necessary condition for the solvability of (106).

(b) Let ω be an *exact* k -form in E . Then there is a $(k-1)$ -form λ in E with $d\lambda = \omega$, and Stokes' theorem asserts that

$$(107) \quad \int_{\Psi} \omega = \int_{\Psi} d\lambda = \int_{\partial\Psi} \lambda$$

for every k -chain Ψ of class \mathcal{C}'' in E .

If Ψ_1 and Ψ_2 are such chains, and if they have the same boundaries, it follows that

$$\int_{\Psi_1} \omega = \int_{\Psi_2} \omega.$$

In particular, *the integral of an exact k -form in E is 0 over every k -chain in E whose boundary is 0.*

As an important special case of this, note that integrals of exact 1-forms in E are 0 over closed (differentiable) curves in E .

(c) Let ω be a *closed* k -form in E . Then $d\omega = 0$, and Stokes' theorem asserts that

$$(108) \quad \int_{\partial\Psi} \omega = \int_{\Psi} d\omega = 0$$

for every $(k+1)$ -chain Ψ of class \mathcal{C}'' in E .

In other words, *integrals of closed k -forms in E are 0 over k -chains that are boundaries of $(k+1)$ -chains in E .*

(d) Let Ψ be a $(k+1)$ -chain in E and let λ be a $(k-1)$ -form in E , both of class \mathcal{C}'' . Since $d^2\lambda = 0$, two applications of Stokes' theorem show that

$$(109) \quad \int_{\partial\partial\Psi} \lambda = \int_{\partial\Psi} d\lambda = \int_{\Psi} d^2\lambda = 0.$$

We conclude that $\partial^2\Psi = 0$. In other words, *the boundary of a boundary is 0.*

See Exercise 16 for a more direct proof of this.

10.36 Example Let $E = R^2 - \{0\}$, the plane with the origin removed. The 1-form

$$(110) \quad \eta = \frac{x \, dy - y \, dx}{x^2 + y^2}$$

is *closed* in $R^2 - \{0\}$. This is easily verified by differentiation. Fix $r > 0$, and define

$$(111) \quad \gamma(t) = (r \cos t, r \sin t) \quad (0 \leq t \leq 2\pi).$$

Then γ is a curve (an “oriented 1-simplex”) in $R^2 - \{0\}$. Since $\gamma(0) = \gamma(2\pi)$, we have

$$(112) \quad \partial\gamma = 0.$$

Direct computation shows that

$$(113) \quad \int_{\gamma} \eta = 2\pi \neq 0.$$

The discussion in Remarks 10.35(b) and (c) shows that we can draw two conclusions from (113):

First, η is *not exact* in $R^2 - \{0\}$, for otherwise (112) would force the integral (113) to be 0.

Secondly, γ is *not the boundary of any 2-chain in $R^2 - \{0\}$* (of class \mathcal{C}''), for otherwise the fact that η is closed would force the integral (113) to be 0.

10.37 Example Let $E = R^3 - \{0\}$, 3-space with the origin removed. Define

$$(114) \quad \zeta = \frac{x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy}{(x^2 + y^2 + z^2)^{3/2}}$$

where we have written (x, y, z) in place of (x_1, x_2, x_3) . Differentiation shows that $d\zeta = 0$, so that ζ is a closed 2-form in $R^3 - \{0\}$.

Let Σ be the 2-chain in $R^3 - \{0\}$ that was constructed in Example 10.32; recall that Σ is a parametrization of the unit sphere in R^3 . Using the rectangle D of Example 10.32 as parameter domain, it is easy to compute that

$$(115) \quad \int_{\Sigma} \zeta = \int_D \sin u \, du \, dv = 4\pi \neq 0.$$

As in the preceding example, we can now conclude that ζ is *not exact* in $R^3 - \{0\}$ (since $\partial\Sigma = 0$, as was shown in Example 10.32) and that the sphere Σ is *not the boundary of any 3-chain in $R^3 - \{0\}$* (of class \mathcal{C}''), although $\partial\Sigma = 0$.

The following result will be used in the proof of Theorem 10.39.

10.38 Theorem Suppose E is a convex open set in R^n , $f \in \mathcal{C}'(E)$, p is an integer, $1 \leq p \leq n$, and

$$(116) \quad (D_j f)(\mathbf{x}) = 0 \quad (p < j \leq n, \mathbf{x} \in E).$$

Then there exists an $F \in \mathcal{C}'(E)$ such that

$$(117) \quad (D_p F)(\mathbf{x}) = f(\mathbf{x}), \quad (D_j F)(\mathbf{x}) = 0 \quad (p < j \leq n, \mathbf{x} \in E).$$

Proof Write $\mathbf{x} = (\mathbf{x}', x_p, \mathbf{x}'')$, where

$$\mathbf{x}' = (x_1, \dots, x_{p-1}), \quad \mathbf{x}'' = (x_{p+1}, \dots, x_n).$$

(When $p = 1$, \mathbf{x}' is absent; when $p = n$, \mathbf{x}'' is absent.) Let V be the set of all $(\mathbf{x}', x_p) \in R^p$ such that $(\mathbf{x}', x_p, \mathbf{x}'') \in E$ for some \mathbf{x}'' . Being a projection of E , V is a convex open set in R^p . Since E is convex and (116) holds, $f(\mathbf{x})$ does not depend on \mathbf{x}'' . Hence there is a function φ , with domain V , such that

$$f(\mathbf{x}) = \varphi(\mathbf{x}', x_p)$$

for all $\mathbf{x} \in E$.

If $p = 1$, V is a segment in R^1 (possibly unbounded). Pick $c \in V$ and define

$$F(\mathbf{x}) = \int_c^{x_1} \varphi(t) dt \quad (\mathbf{x} \in E).$$

If $p > 1$, let U be the set of all $\mathbf{x}' \in R^{p-1}$ such that $(\mathbf{x}', x_p) \in V$ for some x_p . Then U is a convex open set in R^{p-1} , and there is a function $\alpha \in \mathcal{C}'(U)$ such that $(\mathbf{x}', \alpha(\mathbf{x}')) \in V$ for every $\mathbf{x}' \in U$; in other words, the graph of α lies in V (Exercise 29). Define

$$F(\mathbf{x}) = \int_{\alpha(\mathbf{x}')}^{x_p} \varphi(\mathbf{x}', t) dt \quad (\mathbf{x} \in E).$$

In either case, F satisfies (117).

(Note: Recall the usual convention that \int_a^b means $-\int_b^a$ if $b < a$.)

10.39 Theorem If $E \subset R^n$ is convex and open, if $k \geq 1$, if ω is a k -form of class \mathcal{C}' in E , and if $d\omega = 0$, then there is a $(k-1)$ -form λ in E such that $\omega = d\lambda$.

Briefly, closed forms are exact in convex sets.

Proof For $p = 1, \dots, n$, let Y_p denote the set of all k -forms ω , of class \mathcal{C}' in E , whose standard presentation

$$(118) \quad \omega = \sum_I f_I(\mathbf{x}) dx_I$$

does not involve dx_{p+1}, \dots, dx_n . In other words, $I \subset \{1, \dots, p\}$ if $f_I(\mathbf{x}) \neq 0$ for some $\mathbf{x} \in E$.

We shall proceed by induction on p .

Assume first that $\omega \in Y_1$. Then $\omega = f(\mathbf{x}) dx_1$. Since $d\omega = 0$, $(D_j f)(\mathbf{x}) = 0$ for $1 < j \leq n$, $\mathbf{x} \in E$. By Theorem 10.38 there is an $F \in \mathcal{C}'(E)$ such that $D_1 F = f$ and $D_j F = 0$ for $1 < j \leq n$. Thus

$$dF = (D_1 F)(\mathbf{x}) dx_1 = f(\mathbf{x}) dx_1 = \omega.$$

Now we take $p > 1$ and make the following induction hypothesis:
Every closed k -form that belongs to Y_{p-1} is exact in E .

Choose $\omega \in Y_p$ so that $d\omega = 0$. By (118),

$$(119) \quad \sum_I \sum_{j=1}^n (D_j f_I)(\mathbf{x}) dx_j \wedge dx_I = d\omega = 0.$$

Consider a fixed j , with $p < j \leq n$. Each I that occurs in (118) lies in $\{1, \dots, p\}$. If I_1, I_2 are two of these k -indices, and if $I_1 \neq I_2$, then the $(k+1)$ -indices $(I_1, j), (I_2, j)$ are distinct. Thus there is no cancellation, and we conclude from (119) that every coefficient in (118) satisfies

$$(120) \quad (D_j f_I)(\mathbf{x}) = 0 \quad (\mathbf{x} \in E, p < j \leq n).$$

We now gather those terms in (118) that contain dx_p and rewrite ω in the form

$$(121) \quad \omega = \alpha + \sum_{I_0} f_{I_0}(\mathbf{x}) dx_{I_0} \wedge dx_p,$$

where $\alpha \in Y_{p-1}$, each I_0 is an increasing $(k-1)$ -index in $\{1, \dots, p-1\}$, and $I = (I_0, p)$. By (120), Theorem 10.38 furnishes functions $F_I \in \mathcal{C}'(E)$ such that

$$(122) \quad D_p F_I = f_I, \quad D_j F_I = 0 \quad (p < j \leq n).$$

Put

$$(123) \quad \beta = \sum_{I_0} F_{I_0}(\mathbf{x}) dx_{I_0}$$

and define $\gamma = \omega - (-1)^{k-1} d\beta$. Since β is a $(k-1)$ -form, it follows that

$$\begin{aligned} \gamma &= \omega - \sum_{I_0} \sum_{j=1}^p (D_j F_{I_0})(\mathbf{x}) dx_{I_0} \wedge dx_j \\ &= \alpha - \sum_{I_0} \sum_{j=1}^{p-1} (D_j F_{I_0})(\mathbf{x}) dx_{I_0} \wedge dx_j, \end{aligned}$$

which is clearly in Y_{p-1} . Since $d\omega = 0$ and $d^2\beta = 0$, we have $d\gamma = 0$. Our induction hypothesis shows therefore that $\gamma = d\mu$ for some $(k-1)$ -form μ in E . If $\lambda = \mu + (-1)^{k-1}\beta$, we conclude that $\omega = d\lambda$.

By induction, this completes the proof.

10.40 Theorem Fix k , $1 \leq k \leq n$. Let $E \subset R^n$ be an open set in which every closed k -form is exact. Let T be a 1-1 \mathcal{C}^n -mapping of E onto an open set $U \subset R^n$ whose inverse S is also of class \mathcal{C}^n .

Then every closed k -form in U is exact in U .

Note that every convex open set E satisfies the present hypothesis, by Theorem 10.39. The relation between E and U may be expressed by saying that they are \mathcal{C}^n -equivalent.

Thus every closed form is exact in any set which is \mathcal{C}^n -equivalent to a convex open set.

Proof Let ω be a k -form in U , with $d\omega = 0$. By Theorem 10.22(c), ω_T is a k -form in E for which $d(\omega_T) = 0$. Hence $\omega_T = d\lambda$ for some $(k-1)$ -form λ in E . By Theorem 10.23, and another application of Theorem 10.22(c),

$$\omega = (\omega_T)_S = (d\lambda)_S = d(\lambda_S).$$

Since λ_S is a $(k-1)$ -form in U , ω is exact in U .

10.41 Remark In applications, cells (see Definition 2.17) are often more convenient parameter domains than simplexes. If our whole development had been based on cells rather than simplexes, the computation that occurs in the proof of Stokes' theorem would be even simpler. (It is done that way in Spivak's book.) The reason for preferring simplexes is that the definition of the boundary of an oriented simplex seems easier and more natural than is the case for a cell. (See Exercise 19.) Also, the partitioning of sets into simplexes (called "triangulation") plays an important role in topology, and there are strong connections between certain aspects of topology, on the one hand, and differential forms, on the other. These are hinted at in Sec. 10.35. The book by Singer and Thorpe contains a good introduction to this topic.

Since every cell can be triangulated, we may regard it as a chain. For dimension 2, this was done in Example 10.32; for dimension 3, see Exercise 18.

Poincaré's lemma (Theorem 10.39) can be proved in several ways. See, for example, page 94 in Spivak's book, or page 280 in Fleming's. Two simple proofs for certain special cases are indicated in Exercises 24 and 27.

VECTOR ANALYSIS

We conclude this chapter with a few applications of the preceding material to theorems concerning vector analysis in R^3 . These are special cases of theorems about differential forms, but are usually stated in different terminology. We are thus faced with the job of translating from one language to another.

10.42 Vector fields Let $\mathbf{F} = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2 + F_3 \mathbf{e}_3$ be a continuous mapping of an open set $E \subset R^3$ into R^3 . Since \mathbf{F} associates a vector to each point of E , \mathbf{F} is sometimes called a vector field, especially in physics. With every such \mathbf{F} is associated a 1-form

$$(124) \quad \lambda_{\mathbf{F}} = F_1 dx + F_2 dy + F_3 dz$$

and a 2-form

$$(125) \quad \omega_{\mathbf{F}} = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy.$$

Here, and in the rest of this chapter, we use the customary notation (x, y, z) in place of (x_1, x_2, x_3) .

It is clear, conversely, that every 1-form λ in E is $\lambda_{\mathbf{F}}$ for some vector field \mathbf{F} in E , and that every 2-form ω is $\omega_{\mathbf{F}}$ for some \mathbf{F} . In R^3 , the study of 1-forms and 2-forms is thus coextensive with the study of vector fields.

If $u \in \mathcal{C}'(E)$ is a real function, then its *gradient*

$$\nabla u = (D_1 u) \mathbf{e}_1 + (D_2 u) \mathbf{e}_2 + (D_3 u) \mathbf{e}_3$$

is an example of a vector field in E .

Suppose now that \mathbf{F} is a vector field in E , of class \mathcal{C}' . Its *curl* $\nabla \times \mathbf{F}$ is the vector field defined in E by

$$\nabla \times \mathbf{F} = (D_2 F_3 - D_3 F_2) \mathbf{e}_1 + (D_3 F_1 - D_1 F_3) \mathbf{e}_2 + (D_1 F_2 - D_2 F_1) \mathbf{e}_3$$

and its *divergence* is the real function $\nabla \cdot \mathbf{F}$ defined in E by

$$\nabla \cdot \mathbf{F} = D_1 F_1 + D_2 F_2 + D_3 F_3.$$

These quantities have various physical interpretations. We refer to the book by O. D. Kellogg for more details.

Here are some relations between gradients, curls, and divergences.

10.43 Theorem Suppose E is an open set in R^3 , $u \in \mathcal{C}''(E)$, and \mathbf{G} is a vector field in E , of class \mathcal{C}'' .

- (a) If $\mathbf{F} = \nabla u$, then $\nabla \times \mathbf{F} = \mathbf{0}$.
- (b) If $\mathbf{F} = \nabla \times \mathbf{G}$, then $\nabla \cdot \mathbf{F} = 0$.

Furthermore, if E is \mathcal{C}'' -equivalent to a convex set, then (a) and (b) have converses, in which we assume that \mathbf{F} is a vector field in E , of class \mathcal{C}' :

- (a') If $\nabla \times \mathbf{F} = \mathbf{0}$, then $\mathbf{F} = \nabla u$ for some $u \in \mathcal{C}''(E)$.
- (b') If $\nabla \cdot \mathbf{F} = 0$, then $\mathbf{F} = \nabla \times \mathbf{G}$ for some vector field \mathbf{G} in E , of class \mathcal{C}'' .

Proof If we compare the definitions of ∇u , $\nabla \times \mathbf{F}$, and $\nabla \cdot \mathbf{F}$ with the differential forms $\lambda_{\mathbf{F}}$ and $\omega_{\mathbf{F}}$ given by (124) and (125), we obtain the following four statements:

$$\begin{aligned}
\mathbf{F} = \nabla u & \quad \text{if and only if} \quad \lambda_{\mathbf{F}} = du. \\
\nabla \times \mathbf{F} = \mathbf{0} & \quad \text{if and only if} \quad d\lambda_{\mathbf{F}} = 0. \\
\mathbf{F} = \nabla \times \mathbf{G} & \quad \text{if and only if} \quad \omega_{\mathbf{F}} = d\lambda_{\mathbf{G}}. \\
\nabla \cdot \mathbf{F} = 0 & \quad \text{if and only if} \quad d\omega_{\mathbf{F}} = 0.
\end{aligned}$$

Now if $\mathbf{F} = \nabla u$, then $\lambda_{\mathbf{F}} = du$, hence $d\lambda_{\mathbf{F}} = d^2u = 0$ (Theorem 10.20), which means that $\nabla \times \mathbf{F} = \mathbf{0}$. Thus (a) is proved.

As regards (a'), the hypothesis amounts to saying that $d\lambda_{\mathbf{F}} = 0$ in E . By Theorem 10.40, $\lambda_{\mathbf{F}} = du$ for some 0-form u . Hence $\mathbf{F} = \nabla u$.

The proofs of (b) and (b') follow exactly the same pattern.

10.44 Volume elements The k -form

$$dx_1 \wedge \cdots \wedge dx_k$$

is called the volume element in R^k . It is often denoted by dV (or by dV_k if it seems desirable to indicate the dimension explicitly), and the notation

$$(126) \quad \int_{\Phi} f(\mathbf{x}) dx_1 \wedge \cdots \wedge dx_k = \int_{\Phi} f dV$$

is used when Φ is a positively oriented k -surface in R^k and f is a continuous function on the range of Φ .

The reason for using this terminology is very simple: If D is a parameter domain in R^k , and if Φ is a 1-1 \mathcal{C}' -mapping of D into R^k , with positive Jacobian J_{Φ} , then the left side of (126) is

$$\int_D f(\Phi(\mathbf{u})) J_{\Phi}(\mathbf{u}) d\mathbf{u} = \int_{\Phi(D)} f(\mathbf{x}) d\mathbf{x},$$

by (35) and Theorem 10.9.

In particular, when $f = 1$, (126) defines the *volume* of Φ . We already saw a special case of this in (36).

The usual notation for dV_2 is dA .

10.45 Green's theorem Suppose E is an open set in R^2 , $\alpha \in \mathcal{C}'(E)$, $\beta \in \mathcal{C}'(E)$, and Ω is a closed subset of E , with positively oriented boundary $\partial\Omega$, as described in Sec. 10.31. Then

$$(127) \quad \int_{\partial\Omega} (\alpha dx + \beta dy) = \int_{\Omega} \left(\frac{\partial\beta}{\partial x} - \frac{\partial\alpha}{\partial y} \right) dA.$$

Proof Put $\lambda = \alpha dx + \beta dy$. Then

$$\begin{aligned} d\lambda &= (D_2\alpha) dy \wedge dx + (D_1\beta) dx \wedge dy \\ &= (D_1\beta - D_2\alpha) dA, \end{aligned}$$

and (127) is the same as

$$\int_{\partial\Omega} \lambda = \int_{\Omega} d\lambda,$$

which is true by Theorem 10.33.

With $\alpha(x, y) = -y$ and $\beta(x, y) = x$, (127) becomes

$$(128) \quad \frac{1}{2} \int_{\partial\Omega} (x dy - y dx) = A(\Omega),$$

the area of Ω .

With $\alpha = 0$, $\beta = x$, a similar formula is obtained. Example 10.12(b) contains a special case of this.

10.46 Area elements in R^3 Let Φ be a 2-surface in R^3 , of class \mathcal{C}' , with parameter domain $D \subset R^2$. Associate with each point $(u, v) \in D$ the vector

$$(129) \quad \mathbf{N}(u, v) = \frac{\partial(y, z)}{\partial(u, v)} \mathbf{e}_1 + \frac{\partial(z, x)}{\partial(u, v)} \mathbf{e}_2 + \frac{\partial(x, y)}{\partial(u, v)} \mathbf{e}_3.$$

The Jacobians in (129) correspond to the equation

$$(130) \quad (x, y, z) = \Phi(u, v).$$

If f is a continuous function on $\Phi(D)$, the *area integral* of f over Φ is defined to be

$$(131) \quad \int_{\Phi} f dA = \int_D f(\Phi(u, v)) |\mathbf{N}(u, v)| du dv.$$

In particular, when $f = 1$ we obtain the *area* of Φ , namely,

$$(132) \quad A(\Phi) = \int_D |\mathbf{N}(u, v)| du dv.$$

The following discussion will show that (131) and its special case (132) are reasonable definitions. It will also describe the geometric features of the vector \mathbf{N} .

Write $\Phi = \varphi_1 \mathbf{e}_1 + \varphi_2 \mathbf{e}_2 + \varphi_3 \mathbf{e}_3$, fix a point $\mathbf{p}_0 = (u_0, v_0) \in D$, put $\mathbf{N} = \mathbf{N}(\mathbf{p}_0)$, put

$$(133) \quad \alpha_i = (D_1 \varphi_i)(\mathbf{p}_0), \quad \beta_i = (D_2 \varphi_i)(\mathbf{p}_0) \quad (i = 1, 2, 3)$$

and let $T \in L(R^2, R^3)$ be the linear transformation given by

$$(134) \quad T(u, v) = \sum_{i=1}^3 (\alpha_i u + \beta_i v) \mathbf{e}_i.$$

Note that $T = \Phi'(\mathbf{p}_0)$, in accordance with Definition 9.11.

Let us now assume that the rank of T is 2. (If it is 1 or 0, then $\mathbf{N} = \mathbf{0}$, and the tangent plane mentioned below degenerates to a line or to a point.) The range of the affine mapping

$$(u, v) \rightarrow \Phi(\mathbf{p}_0) + T(u, v)$$

is then a plane Π , called the *tangent plane* to Φ at \mathbf{p}_0 . [One would like to call Π the tangent plane at $\Phi(\mathbf{p}_0)$, rather than at \mathbf{p}_0 ; if Φ is not one-to-one, this runs into difficulties.]

If we use (133) in (129), we obtain

$$(135) \quad \mathbf{N} = (\alpha_2 \beta_3 - \alpha_3 \beta_2) \mathbf{e}_1 + (\alpha_3 \beta_1 - \alpha_1 \beta_3) \mathbf{e}_2 + (\alpha_1 \beta_2 - \alpha_2 \beta_1) \mathbf{e}_3,$$

and (134) shows that

$$(136) \quad T\mathbf{e}_1 = \sum_{i=1}^3 \alpha_i \mathbf{e}_i, \quad T\mathbf{e}_2 = \sum_{i=1}^3 \beta_i \mathbf{e}_i.$$

A straightforward computation now leads to

$$(137) \quad \mathbf{N} \cdot (T\mathbf{e}_1) = 0 = \mathbf{N} \cdot (T\mathbf{e}_2).$$

Hence \mathbf{N} is perpendicular to Π . It is therefore called the *normal to Φ at \mathbf{p}_0* .

A second property of \mathbf{N} , also verified by a direct computation based on (135) and (136), is that the determinant of the linear transformation of R^3 that takes $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{T\mathbf{e}_1, T\mathbf{e}_2, \mathbf{N}\}$ is $|\mathbf{N}|^2 > 0$ (Exercise 30). The 3-simplex

$$(138) \quad [0, T\mathbf{e}_1, T\mathbf{e}_2, \mathbf{N}]$$

is thus *positively oriented*.

The third property of \mathbf{N} that we shall use is a consequence of the first two: The above-mentioned determinant, whose value is $|\mathbf{N}|^2$, is the volume of the parallelepiped with edges $[0, T\mathbf{e}_1]$, $[0, T\mathbf{e}_2]$, $[0, \mathbf{N}]$. By (137), $[0, \mathbf{N}]$ is perpendicular to the other two edges. *The area of the parallelogram with vertices*

$$(139) \quad 0, T\mathbf{e}_1, T\mathbf{e}_2, T(\mathbf{e}_1 + \mathbf{e}_2)$$

is therefore $|\mathbf{N}|$.

This parallelogram is the image under T of the unit square in R^2 . If E is any rectangle in R^2 , it follows (by the linearity of T) that the area of the parallelogram $T(E)$ is

$$(140) \quad A(T(E)) = |\mathbf{N}| A(E) = \int_E |\mathbf{N}(u_0, v_0)| \, du \, dv.$$

We conclude that (132) is correct when Φ is affine. To justify the definition (132) in the general case, divide D into small rectangles, pick a point (u_0, v_0) in each, and replace Φ in each rectangle by the corresponding tangent plane. The sum of the areas of the resulting parallelograms, obtained via (140), is then an approximation to $A(\Phi)$. Finally, one can justify (131) from (132) by approximating f by step functions.

10.47 Example Let $0 < a < b$ be fixed. Let K be the 3-cell determined by

$$0 \leq t \leq a, \quad 0 \leq u \leq 2\pi, \quad 0 \leq v \leq 2\pi.$$

The equations

$$\begin{aligned} x &= t \cos u \\ (141) \quad y &= (b + t \sin u) \cos v \\ z &= (b + t \sin u) \sin v \end{aligned}$$

describe a mapping Ψ of R^3 into R^3 which is 1-1 in the interior of K , such that $\Psi(K)$ is a solid torus. Its Jacobian is

$$J_\Psi = \frac{\partial(x, y, z)}{\partial(t, u, v)} = t(b + t \sin u)$$

which is positive on K , except on the face $t = 0$. If we integrate J_Ψ over K , we obtain

$$\text{vol}(\Psi(K)) = 2\pi^2 a^2 b$$

as the volume of our solid torus.

Now consider the 2-chain $\Phi = \partial\Psi$. (See Exercise 19.) Ψ maps the faces $u = 0$ and $u = 2\pi$ of K onto the same cylindrical strip, but with opposite orientations. Ψ maps the faces $v = 0$ and $v = 2\pi$ onto the same circular disc, but with opposite orientations. Ψ maps the face $t = 0$ onto a circle, which contributes 0 to the 2-chain $\partial\Psi$. (The relevant Jacobians are 0.) Thus Φ is simply the 2-surface obtained by setting $t = a$ in (141), with parameter domain D the square defined by $0 \leq u \leq 2\pi, 0 \leq v \leq 2\pi$.

According to (129) and (141), the normal to Φ at $(u, v) \in D$ is thus the vector

$$\mathbf{N}(u, v) = a(b + a \sin u) \mathbf{n}(u, v)$$

where

$$\mathbf{n}(u, v) = (\cos u) \mathbf{e}_1 + (\sin u \cos v) \mathbf{e}_2 + (\sin u \sin v) \mathbf{e}_3.$$

Since $|\mathbf{n}(u, v)| = 1$, we have $|\mathbf{N}(u, v)| = a(b + a \sin u)$, and if we integrate this over D , (131) gives

$$A(\Phi) = 4\pi^2 ab$$

as the surface area of our torus.

If we think of $\mathbf{N} = \mathbf{N}(u, v)$ as a directed line segment, pointing from $\Phi(u, v)$ to $\Phi(u, v) + \mathbf{N}(u, v)$, then \mathbf{N} points *outward*, that is to say, away from $\Psi(K)$. This is so because $\mathbf{J}_\Psi > 0$ when $t = a$.

For example, take $u = v = \pi/2$, $t = a$. This gives the largest value of z on $\Psi(K)$, and $\mathbf{N} = a(b + a)\mathbf{e}_3$ points “upward” for this choice of (u, v) .

10.48 Integrals of 1-forms in R^3 Let γ be a \mathcal{C}' -curve in an open set $E \subset R^3$, with parameter interval $[0, 1]$, let \mathbf{F} be a vector field in E , as in Sec. 10.42, and define $\lambda_{\mathbf{F}}$ by (124). The integral of $\lambda_{\mathbf{F}}$ over γ can be rewritten in a certain way which we now describe.

For any $u \in [0, 1]$,

$$\gamma'(u) = \gamma'_1(u)\mathbf{e}_1 + \gamma'_2(u)\mathbf{e}_2 + \gamma'_3(u)\mathbf{e}_3$$

is called the *tangent vector* to γ at u . We define $\mathbf{t} = \mathbf{t}(u)$ to be the unit vector in the direction of $\gamma'(u)$. Thus

$$\gamma'(u) = |\gamma'(u)| \mathbf{t}(u).$$

[If $\gamma'(u) = \mathbf{0}$ for some u , put $\mathbf{t}(u) = \mathbf{e}_1$; any other choice would do just as well.] By (35),

$$\begin{aligned} \int_{\gamma} \lambda_{\mathbf{F}} &= \sum_{i=1}^3 \int_0^1 F_i(\gamma(u)) \gamma'_i(u) du \\ (142) \quad &= \int_0^1 \mathbf{F}(\gamma(u)) \cdot \gamma'(u) du \\ &= \int_0^1 \mathbf{F}(\gamma(u)) \cdot \mathbf{t}(u) |\gamma'(u)| du. \end{aligned}$$

Theorem 6.27 makes it reasonable to call $|\gamma'(u)| du$ the *element of arc length along γ* . A customary notation for it is ds , and (142) is rewritten in the form

$$(143) \quad \int_{\gamma} \lambda_{\mathbf{F}} = \int_{\gamma} (\mathbf{F} \cdot \mathbf{t}) ds.$$

Since \mathbf{t} is a unit tangent vector to γ , $\mathbf{F} \cdot \mathbf{t}$ is called the *tangential component* of \mathbf{F} along γ .

The right side of (143) should be regarded as just an abbreviation for the last integral in (142). The point is that \mathbf{F} is defined on the range of γ , but \mathbf{t} is defined on $[0, 1]$; thus $\mathbf{F} \cdot \mathbf{t}$ has to be properly interpreted. Of course, when γ is one-to-one, then $\mathbf{t}(u)$ can be replaced by $\mathbf{t}(\gamma(u))$, and this difficulty disappears.

10.49 Integrals of 2-forms in R^3 Let Φ be a 2-surface in an open set $E \subset R^3$, of class \mathcal{C}' , with parameter domain $D \subset R^2$. Let \mathbf{F} be a vector field in E , and define $\omega_{\mathbf{F}}$ by (125). As in the preceding section, we shall obtain a different representation of the integral of $\omega_{\mathbf{F}}$ over Φ .

By (35) and (129),

$$\begin{aligned} \int_{\Phi} \omega_{\mathbf{F}} &= \int_{\Phi} (F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy) \\ &= \int_D \left\{ (F_1 \circ \Phi) \frac{\partial(y, z)}{\partial(u, v)} + (F_2 \circ \Phi) \frac{\partial(z, x)}{\partial(u, v)} + (F_3 \circ \Phi) \frac{\partial(x, y)}{\partial(u, v)} \right\} du dv \\ &= \int_D \mathbf{F}(\Phi(u, v)) \cdot \mathbf{N}(u, v) du dv. \end{aligned}$$

Now let $\mathbf{n} = \mathbf{n}(u, v)$ be the unit vector in the direction of $\mathbf{N}(u, v)$. [If $\mathbf{N}(u, v) = \mathbf{0}$ for some $(u, v) \in D$, take $\mathbf{n}(u, v) = \mathbf{e}_1$.] Then $\mathbf{N} = |\mathbf{N}|\mathbf{n}$, and therefore the last integral becomes

$$\int_D \mathbf{F}(\Phi(u, v)) \cdot \mathbf{n}(u, v) |\mathbf{N}(u, v)| du dv.$$

By (131), we can finally write this in the form

$$(144) \quad \int_{\Phi} \omega_{\mathbf{F}} = \int_{\Phi} (\mathbf{F} \cdot \mathbf{n}) dA.$$

With regard to the meaning of $\mathbf{F} \cdot \mathbf{n}$, the remark made at the end of Sec. 10.48 applies here as well.

We can now state the original form of Stokes' theorem.

10.50 Stokes' formula If \mathbf{F} is a vector field of class \mathcal{C}' in an open set $E \subset R^3$, and if Φ is a 2-surface of class \mathcal{C}'' in E , then

$$(145) \quad \int_{\Phi} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dA = \int_{\partial\Phi} (\mathbf{F} \cdot \mathbf{t}) ds.$$

Proof Put $\mathbf{H} = \nabla \times \mathbf{F}$. Then, as in the proof of Theorem 10.43, we have

$$(146) \quad \omega_{\mathbf{H}} = d\lambda_{\mathbf{F}}.$$

Hence

$$\begin{aligned}\int_{\Phi} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dA &= \int_{\Phi} (\mathbf{H} \cdot \mathbf{n}) \, dA = \int_{\Phi} \omega_{\mathbf{H}} \\ &= \int_{\Phi} d\lambda_{\mathbf{F}} = \int_{\partial\Phi} \lambda_{\mathbf{F}} = \int_{\partial\Phi} (\mathbf{F} \cdot \mathbf{t}) \, ds.\end{aligned}$$

Here we used the definition of \mathbf{H} , then (144) with \mathbf{H} in place of \mathbf{F} , then (146), then—the main step—Theorem 10.33, and finally (143), extended in the obvious way from curves to 1-chains.

10.51 The divergence theorem *If \mathbf{F} is a vector field of class \mathcal{C}' in an open set $E \subset R^3$, and if Ω is a closed subset of E with positively oriented boundary $\partial\Omega$ (as described in Sec. 10.31) then*

$$(147) \quad \int_{\Omega} (\nabla \cdot \mathbf{F}) \, dV = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) \, dA.$$

Proof By (125),

$$d\omega_{\mathbf{F}} = (\nabla \cdot \mathbf{F}) \, dx \wedge dy \wedge dz = (\nabla \cdot \mathbf{F}) \, dV.$$

Hence

$$\int_{\Omega} (\nabla \cdot \mathbf{F}) \, dV = \int_{\Omega} d\omega_{\mathbf{F}} = \int_{\partial\Omega} \omega_{\mathbf{F}} = \int_{\partial\Omega} (\mathbf{F} \cdot \mathbf{n}) \, dA,$$

by Theorem 10.33, applied to the 2-form $\omega_{\mathbf{F}}$, and (144).

EXERCISES

1. Let H be a compact convex set in R^k , with nonempty interior. Let $f \in \mathcal{C}(H)$, put $f(\mathbf{x}) = 0$ in the complement of H , and define $\int_H f$ as in Definition 10.3.

Prove that $\int_H f$ is independent of the order in which the k integrations are carried out.

Hint: Approximate f by functions that are continuous on R^k and whose supports are in H , as was done in Example 10.4.

2. For $i = 1, 2, 3, \dots$, let $\varphi_i \in \mathcal{C}(R^1)$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y)$$

Then f has compact support in R^2 , f is continuous except at $(0, 0)$, and

$$\int dy \int f(x, y) \, dx = 0 \quad \text{but} \quad \int dx \int f(x, y) \, dy = 1.$$

Observe that f is unbounded in every neighborhood of $(0, 0)$.

3. (a) If F is as in Theorem 10.7, put $A = F'(0)$, $F_1(\mathbf{x}) = A^{-1}F(\mathbf{x})$. Then $F'_1(0) = I$. Show that

$$F_1(\mathbf{x}) = G_n \circ G_{n-1} \circ \cdots \circ G_1(\mathbf{x})$$

in some neighborhood of 0 , for certain primitive mappings G_1, \dots, G_n . This gives another version of Theorem 10.7:

$$F(\mathbf{x}) = F'(0)G_n \circ G_{n-1} \circ \cdots \circ G_1(\mathbf{x}).$$

(b) Prove that the mapping $(x, y) \rightarrow (y, x)$ of R^2 onto R^2 is not the composition of any two primitive mappings, in any neighborhood of the origin. (This shows that the flips B_i cannot be omitted from the statement of Theorem 10.7.)

4. For $(x, y) \in R^2$, define

$$F(x, y) = (e^x \cos y - 1, e^x \sin y).$$

Prove that $F = G_2 \circ G_1$, where

$$G_1(x, y) = (e^x \cos y - 1, y)$$

$$G_2(u, v) = (u, (1 + u) \tan v)$$

are primitive in some neighborhood of $(0, 0)$.

Compute the Jacobians of G_1, G_2, F at $(0, 0)$. Define

$$H_2(x, y) = (x, e^x \sin y)$$

and find

$$H_1(u, v) = (h(u, v), v)$$

so that $F = H_1 \circ H_2$ is some neighborhood of $(0, 0)$.

5. Formulate and prove an analogue of Theorem 10.8, in which K is a compact subset of an arbitrary metric space. (Replace the functions φ_i that occur in the proof of Theorem 10.8 by functions of the type constructed in Exercise 22 of Chap. 4.)
6. Strengthen the conclusion of Theorem 10.8 by showing that the functions ψ_i can be made differentiable, and even infinitely differentiable. (Use Exercise 1 of Chap. 8 in the construction of the auxiliary functions φ_i .)
7. (a) Show that the simplex Q^k is the smallest convex subset of R^k that contains $0, e_1, \dots, e_k$.
(b) Show that affine mappings take convex sets to convex sets.
8. Let H be the parallelogram in R^2 whose vertices are $(1, 1), (3, 2), (4, 5), (2, 4)$. Find the affine map T which sends $(0, 0)$ to $(1, 1)$, $(1, 0)$ to $(3, 2)$, $(0, 1)$ to $(2, 4)$. Show that $J_T = 5$. Use T to convert the integral

$$\alpha = \int_H e^{x-y} dx dy$$

to an integral over I^2 and thus compute α .

9. Define $(x, y) = T(r, \theta)$ on the rectangle

$$0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi$$

by the equations

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Show that T maps this rectangle onto the closed disc D with center at $(0, 0)$ and radius a , that T is one-to-one in the interior of the rectangle, and that $J_T(r, \theta) = r$. If $f \in \mathcal{C}(D)$, prove the formula for integration in polar coordinates:

$$\int_D f(x, y) dx dy = \int_0^a \int_0^{2\pi} f(T(r, \theta)) r dr d\theta.$$

Hint: Let D_0 be the interior of D , minus the interval from $(0, 0)$ to $(0, a)$. As it stands, Theorem 10.9 applies to continuous functions f whose support lies in D_0 . To remove this restriction, proceed as in Example 10.4.

10. Let $a \rightarrow \infty$ in Exercise 9 and prove that

$$\int_{\mathbb{R}^2} f(x, y) dx dy = \int_0^\infty \int_0^{2\pi} f(T(r, \theta)) r dr d\theta,$$

for continuous functions f that decrease sufficiently rapidly as $|x| + |y| \rightarrow \infty$. (Find a more precise formulation.) Apply this to

$$f(x, y) = \exp(-x^2 - y^2)$$

to derive formula (101) of Chap. 8.

11. Define $(u, v) = T(s, t)$ on the strip

$$0 < s < \infty, \quad 0 < t < 1$$

by setting $u = s - st, v = st$. Show that T is a 1-1 mapping of the strip onto the positive quadrant Q in \mathbb{R}^2 . Show that $J_T(s, t) = s$.

For $x > 0, y > 0$, integrate

$$u^{x-1} e^{-u} v^{y-1} e^{-v}$$

over Q , use Theorem 10.9 to convert the integral to one over the strip, and derive formula (96) of Chap. 8 in this way.

(For this application, Theorem 10.9 has to be extended so as to cover certain improper integrals. Provide this extension.)

12. Let I^k be the set of all $\mathbf{u} = (u_1, \dots, u_k) \in \mathbb{R}^k$ with $0 \leq u_i \leq 1$ for all i ; let Q^k be the set of all $\mathbf{x} = (x_1, \dots, x_k) \in \mathbb{R}^k$ with $x_i \geq 0, \sum x_i \leq 1$. (I^k is the unit cube; Q^k is the standard simplex in \mathbb{R}^k .) Define $\mathbf{x} = T(\mathbf{u})$ by

$$\begin{aligned} x_1 &= u_1 \\ x_2 &= (1 - u_1)u_2 \\ &\dots\dots\dots \\ x_k &= (1 - u_1) \cdots (1 - u_{k-1})u_k. \end{aligned}$$

Show that

$$\sum_{i=1}^k x_i = 1 - \prod_{i=1}^k (1 - u_i).$$

Show that T maps I^k onto Q^k , that T is 1-1 in the interior of I^k , and that its inverse S is defined in the interior of Q^k by $u_1 = x_1$ and

$$u_i = \frac{x_i}{1 - x_1 - \cdots - x_{i-1}}$$

for $i = 2, \dots, k$. Show that

$$J_T(\mathbf{u}) = (1 - u_1)^{k-1} (1 - u_2)^{k-2} \cdots (1 - u_{k-1}),$$

and

$$J_S(\mathbf{x}) = [(1 - x_1)(1 - x_1 - x_2) \cdots (1 - x_1 - \cdots - x_{k-1})]^{-1}.$$

13. Let r_1, \dots, r_k be nonnegative integers, and prove that

$$\int_{Q^k} x_1^{r_1} \cdots x_k^{r_k} dx = \frac{r_1! \cdots r_k!}{(k + r_1 + \cdots + r_k)!}$$

Hint: Use Exercise 12, Theorems 10.9 and 8.20.

Note that the special case $r_1 = \cdots = r_k = 0$ shows that the volume of Q^k is $1/k!$.

14. Prove formula (46).

15. If ω and λ are k - and m -forms, respectively, prove that

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

16. If $k \geq 2$ and $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$ is an oriented affine k -simplex, prove that $\partial^2 \sigma = 0$, directly from the definition of the boundary operator ∂ . Deduce from this that $\partial^2 \Psi = 0$ for every chain Ψ .

Hint: For orientation, do it first for $k = 2, k = 3$. In general, if $i < j$, let σ_{ij} be the $(k - 2)$ -simplex obtained by deleting \mathbf{p}_i and \mathbf{p}_j from σ . Show that each σ_{ij} occurs twice in $\partial^2 \sigma$, with opposite sign.

17. Put $J^2 = \tau_1 + \tau_2$, where

$$\tau_1 = [0, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2], \quad \tau_2 = -[0, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1].$$

Explain why it is reasonable to call J^2 the positively oriented unit square in R^2 . Show that ∂J^2 is the sum of 4 oriented affine 1-simplexes. Find these. What is $\partial(\tau_1 - \tau_2)$?

18. Consider the oriented affine 3-simplex

$$\sigma_1 = [0, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]$$

in R^3 . Show that σ_1 (regarded as a linear transformation) has determinant 1. Thus σ_1 is positively oriented.

Let $\sigma_2, \dots, \sigma_6$ be five other oriented 3-simplexes, obtained as follows: There are five permutations (i_1, i_2, i_3) of $(1, 2, 3)$, distinct from $(1, 2, 3)$. Associate with each (i_1, i_2, i_3) the simplex

$$s(i_1, i_2, i_3)[0, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$$

where s is the sign that occurs in the definition of the determinant. (This is how τ_2 was obtained from τ_1 in Exercise 17.)

Show that $\sigma_2, \dots, \sigma_6$ are positively oriented.

Put $J^3 = \sigma_1 + \dots + \sigma_6$. Then J^3 may be called the positively oriented unit cube in R^3 .

Show that ∂J^3 is the sum of 12 oriented affine 2-simplexes. (These 12 triangles cover the surface of the unit cube I^3 .)

Show that $\mathbf{x} = (x_1, x_2, x_3)$ is in the range of σ_1 if and only if $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$.

Show that the ranges of $\sigma_1, \dots, \sigma_6$ have disjoint interiors, and that their union covers I^3 . (Compare with Exercise 13; note that $3! = 6$.)

19. Let J^2 and J^3 be as in Exercise 17 and 18. Define

$$\begin{aligned} B_{01}(u, v) &= (0, u, v), & B_{11}(u, v) &= (1, u, v), \\ B_{02}(u, v) &= (u, 0, v), & B_{12}(u, v) &= (u, 1, v), \\ B_{03}(u, v) &= (u, v, 0), & B_{13}(u, v) &= (u, v, 1). \end{aligned}$$

These are affine, and map R^2 into R^3 .

Put $\beta_{ri} = B_{ri}(J^2)$, for $r = 0, 1, i = 1, 2, 3$. Each β_{ri} is an affine-oriented 2-chain. (See Sec. 10.30.) Verify that

$$\partial J^3 = \sum_{i=1}^3 (-1)^i (\beta_{0i} - \beta_{1i}),$$

in agreement with Exercise 18.

20. State conditions under which the formula

$$\int_{\bullet} f d\omega = \int_{\partial \bullet} f\omega - \int_{\bullet} (df) \wedge \omega$$

is valid, and show that it generalizes the formula for integration by parts.

Hint: $d(f\omega) = (df) \wedge \omega + f d\omega$.

21. As in Example 10.36, consider the 1-form

$$\eta = \frac{x dy - y dx}{x^2 + y^2}$$

in $R^2 - \{0\}$.

(a) Carry out the computation that leads to formula (113), and prove that $d\eta = 0$.

(b) Let $\gamma(t) = (r \cos t, r \sin t)$, for some $r > 0$, and let Γ be a \mathcal{C}^∞ -curve in $R^2 - \{0\}$,

with parameter interval $[0, 2\pi]$, with $\Gamma(0) = \Gamma(2\pi)$, such that the intervals $[\gamma(t), \Gamma(t)]$ do not contain 0 for any $t \in [0, 2\pi]$. Prove that

$$\int_{\Gamma} \eta = 2\pi.$$

Hint: For $0 \leq t \leq 2\pi$, $0 \leq u \leq 1$, define

$$\Phi(t, u) = (1 - u) \Gamma(t) + u\gamma(t).$$

Then Φ is a 2-surface in $R^2 - \{0\}$ whose parameter domain is the indicated rectangle. Because of cancellations (as in Example 10.32),

$$\partial\Phi = \Gamma - \gamma.$$

Use Stokes' theorem to deduce that

$$\int_{\Gamma} \eta = \int_{\gamma} \eta$$

because $d\eta = 0$.

(c) Take $\Gamma(t) = (a \cos t, b \sin t)$ where $a > 0$, $b > 0$ are fixed. Use part (b) to show that

$$\int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t} dt = 2\pi.$$

(d) Show that

$$\eta = d\left(\arctan \frac{y}{x}\right)$$

in any convex open set in which $x \neq 0$, and that

$$\eta = d\left(-\arctan \frac{x}{y}\right)$$

in any convex open set in which $y \neq 0$.

Explain why this justifies the notation $\eta = d\theta$, in spite of the fact that η is not exact in $R^2 - \{0\}$.

(e) Show that (b) can be derived from (d).

(f) If Γ is any closed \mathcal{C}' -curve in $R^2 - \{0\}$, prove that

$$\frac{1}{2\pi} \int_{\Gamma} \eta = \text{Ind}(\Gamma).$$

(See Exercise 23 of Chap. 8 for the definition of the index of a curve.)

22. As in Example 10.37, define ζ in $R^3 - \{0\}$ by

$$\zeta = \frac{x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy}{r^3}$$

where $r = (x^2 + y^2 + z^2)^{1/2}$, let D be the rectangle given by $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$, and let Σ be the 2-surface in R^3 , with parameter domain D , given by

$$x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = \cos u.$$

(a) Prove that $d\zeta = 0$ in $R^3 - \{0\}$.

(b) Let S denote the restriction of Σ to a parameter domain $E \subset D$. Prove that

$$\int_S \zeta = \int_E \sin u \, du \, dv = A(S),$$

where A denotes area, as in Sec. 10.43. Note that this contains (115) as a special case.

(c) Suppose g, h_1, h_2, h_3 , are \mathcal{C}'' -functions on $[0, 1]$, $g > 0$. Let $(x, y, z) = \Phi(s, t)$ define a 2-surface Φ , with parameter domain I^2 , by

$$x = g(t)h_1(s), \quad y = g(t)h_2(s), \quad z = g(t)h_3(s).$$

Prove that

$$\int_{\Phi} \zeta = 0,$$

directly from (35).

Note the shape of the range of Φ : For fixed s , $\Phi(s, t)$ runs over an interval on a line through 0 . The range of Φ thus lies in a "cone" with vertex at the origin.

(d) Let E be a closed rectangle in D , with edges parallel to those of D . Suppose $f \in \mathcal{C}''(D)$, $f > 0$. Let Ω be the 2-surface with parameter domain E , defined by

$$\Omega(u, v) = f(u, v) \Sigma(u, v).$$

Define S as in (b) and prove that

$$\int_{\Omega} \zeta = \int_S \zeta = A(S).$$

(Since S is the "radial projection" of Ω into the unit sphere, this result makes it reasonable to call $\int_{\Omega} \zeta$ the "solid angle" subtended by the range of Ω at the origin.)

Hint: Consider the 3-surface Ψ given by

$$\Psi(t, u, v) = [1 - t + tf(u, v)] \Sigma(u, v),$$

where $(u, v) \in E$, $0 \leq t \leq 1$. For fixed v , the mapping $(t, u) \rightarrow \Psi(t, u, v)$ is a 2-sur-

face Φ to which (c) can be applied to show that $\int_{\Phi} \zeta = 0$. The same thing holds when u is fixed. By (a) and Stokes' theorem,

$$\int_{\partial\Psi} \zeta = \int_{\Psi} d\zeta = 0.$$

(e) Put $\lambda = -(z/r)\eta$, where

$$\eta = \frac{x dy - y dx}{x^2 + y^2},$$

as in Exercise 21. Then λ is a 1-form in the open set $V \subset \mathbb{R}^3$ in which $x^2 + y^2 > 0$. Show that ζ is exact in V by showing that

$$\zeta = d\lambda.$$

(f) Derive (d) from (e), without using (c).

Hint: To begin with, assume $0 < u < \pi$ on E . By (e),

$$\int_{\Omega} \zeta = \int_{\partial\Omega} \lambda \quad \text{and} \quad \int_S \zeta = \int_{\partial S} \lambda.$$

Show that the two integrals of λ are equal, by using part (d) of Exercise 21, and by noting that z/r is the same at $\Sigma(u, v)$ as at $\Omega(u, v)$.

(g) Is ζ exact in the complement of every line through the origin?

23. Fix n . Define $r_k = (x_1^2 + \cdots + x_k^2)^{1/2}$ for $1 \leq k \leq n$, let E_k be the set of all $\mathbf{x} \in \mathbb{R}^n$ at which $r_k > 0$, and let ω_k be the $(k-1)$ -form defined in E_k by

$$\omega_k = (r_k)^{-k} \sum_{i=1}^k (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k.$$

Note that $\omega_2 = \eta$, $\omega_3 = \zeta$, in the terminology of Exercises 21 and 22. Note also that

$$E_1 \subset E_2 \subset \cdots \subset E_n = \mathbb{R}^n - \{0\}.$$

(a) Prove that $d\omega_k = 0$ in E_k .

(b) For $k = 2, \dots, n$, prove that ω_k is exact in E_{k-1} , by showing that

$$\omega_k = d(f_k \omega_{k-1}) = (df_k) \wedge \omega_{k-1},$$

where $f_k(\mathbf{x}) = (-1)^k g_k(x_k/r_k)$ and

$$g_k(t) = \int_{-1}^t (1-s^2)^{(k-3)/2} ds \quad (-1 < t < 1).$$

Hint: f_k satisfies the differential equations

$$\mathbf{x} \cdot (\nabla f_k)(\mathbf{x}) = 0$$

and

$$(D_k f_k)(\mathbf{x}) = \frac{(-1)^k (r_{k-1})^{k-1}}{(r_k)^k}.$$

(c) Is ω_n exact in E_n ?

(d) Note that (b) is a generalization of part (e) of Exercise 22. Try to extend some of the other assertions of Exercises 21 and 22 to ω_n , for arbitrary n .

24. Let $\omega = \sum a_i(\mathbf{x}) dx_i$ be a 1-form of class \mathcal{C}'' in a convex open set $E \subset R^n$. Assume $d\omega = 0$ and prove that ω is exact in E , by completing the following outline:

Fix $\mathbf{p} \in E$. Define

$$f(\mathbf{x}) = \int_{[\mathbf{p}, \mathbf{x}]} \omega \quad (\mathbf{x} \in E).$$

Apply Stokes' theorem to affine-oriented 2-simplexes $[\mathbf{p}, \mathbf{x}, \mathbf{y}]$ in E . Deduce that

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y}) dt$$

for $\mathbf{x} \in E, \mathbf{y} \in E$. Hence $(D_i f)(\mathbf{x}) = a_i(\mathbf{x})$.

25. Assume that ω is a 1-form in an open set $E \subset R^n$ such that

$$\int_{\gamma} \omega = 0$$

for every closed curve γ in E , of class \mathcal{C}' . Prove that ω is exact in E , by imitating part of the argument sketched in Exercise 24.

26. Assume ω is a 1-form in $R^3 - \{0\}$, of class \mathcal{C}' and $d\omega = 0$. Prove that ω is exact in $R^3 - \{0\}$.

Hint: Every closed continuously differentiable curve in $R^3 - \{0\}$ is the boundary of a 2-surface in $R^3 - \{0\}$. Apply Stokes' theorem and Exercise 25.

27. Let E be an open 3-cell in R^3 , with edges parallel to the coordinate axes. Suppose $(a, b, c) \in E, f_i \in \mathcal{C}''(E)$ for $i=1, 2, 3$,

$$\omega = f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy,$$

and assume that $d\omega = 0$ in E . Define

$$\lambda = g_1 dx + g_2 dy$$

where

$$g_1(x, y, z) = \int_c^z f_2(x, y, s) ds - \int_b^y f_3(x, t, c) dt$$

$$g_2(x, y, z) = - \int_c^z f_1(x, y, s) ds,$$

for $(x, y, z) \in E$. Prove that $d\lambda = \omega$ in E .

Evaluate these integrals when $\omega = \zeta$ and thus find the form λ that occurs in part (e) of Exercise 22.

28. Fix $b > a > 0$, define

$$\Phi(r, \theta) = (r \cos \theta, r \sin \theta)$$

for $a \leq r \leq b$, $0 \leq \theta \leq 2\pi$. (The range of Φ is an annulus in R^2 .) Put $\omega = x^3 dy$, and compute both

$$\int_{\circ} d\omega \quad \text{and} \quad \int_{\circ\circ} \omega$$

to verify that they are equal.

29. Prove the existence of a function α with the properties needed in the proof of Theorem 10.38, and prove that the resulting function F is of class \mathcal{C}' . (Both assertions become trivial if E is an open cell or an open ball, since α can then be taken to be a constant. Refer to Theorem 9.42.)
30. If N is the vector given by (135), prove that

$$\det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix} = |N|^2.$$

Also, verify Eq. (137).

31. Let $E \subset R^3$ be open, suppose $g \in \mathcal{C}''(E)$, $h \in \mathcal{C}''(E)$, and consider the vector field

$$F = g \nabla h.$$

(a) Prove that

$$\nabla \cdot F = g \nabla^2 h + (\nabla g) \cdot (\nabla h)$$

where $\nabla^2 h = \nabla \cdot (\nabla h) = \sum \partial^2 h / \partial x_i^2$ is the so-called “Laplacian” of h .

(b) If Ω is a closed subset of E with positively oriented boundary $\partial\Omega$ (as in Theorem 10.51), prove that

$$\int_{\Omega} [g \nabla^2 h + (\nabla g) \cdot (\nabla h)] dV = \int_{\partial\Omega} g \frac{\partial h}{\partial n} dA$$

where (as is customary) we have written $\partial h / \partial n$ in place of $(\nabla h) \cdot \mathbf{n}$. (Thus $\partial h / \partial n$ is the directional derivative of h in the direction of the outward normal to $\partial\Omega$, the so-called *normal derivative* of h .) Interchange g and h , subtract the resulting formula from the first one, to obtain

$$\int_{\Omega} (g \nabla^2 h - h \nabla^2 g) dV = \int_{\partial\Omega} \left(g \frac{\partial h}{\partial n} - h \frac{\partial g}{\partial n} \right) dA.$$

These two formulas are usually called *Green's identities*.

(c) Assume that h is *harmonic* in E ; this means that $\nabla^2 h = 0$. Take $g = 1$ and conclude that

$$\int_{\partial\Omega} \frac{\partial h}{\partial n} dA = 0.$$

Take $g = h$, and conclude that $h = 0$ in Ω if $h = 0$ on $\partial\Omega$.

(d) Show that Green's identities are also valid in R^2 .

32. Fix δ , $0 < \delta < 1$. Let D be the set of all $(\theta, t) \in R^2$ such that $0 \leq \theta \leq \pi$, $-\delta \leq t \leq \delta$. Let Φ be the 2-surface in R^3 , with parameter domain D , given by

$$\begin{aligned}x &= (1 - t \sin \theta) \cos 2\theta \\y &= (1 - t \sin \theta) \sin 2\theta \\z &= t \cos \theta\end{aligned}$$

where $(x, y, z) = \Phi(\theta, t)$. Note that $\Phi(\pi, t) = \Phi(0, -t)$, and that Φ is one-to-one on the rest of D .

The range $M = \Phi(D)$ of Φ is known as a *Möbius band*. It is the simplest example of a nonorientable surface.

Prove the various assertions made in the following description: Put $\mathbf{p}_1 = (0, -\delta)$, $\mathbf{p}_2 = (\pi, -\delta)$, $\mathbf{p}_3 = (\pi, \delta)$, $\mathbf{p}_4 = (0, \delta)$, $\mathbf{p}_5 = \mathbf{p}_1$. Put $\gamma_i = [\mathbf{p}_i, \mathbf{p}_{i+1}]$, $i = 1, \dots, 4$, and put $\Gamma_i = \Phi \circ \gamma_i$. Then

$$\partial\Phi = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4.$$

Put $\mathbf{a} = (1, 0, -\delta)$, $\mathbf{b} = (1, 0, \delta)$. Then

$$\Phi(\mathbf{p}_1) = \Phi(\mathbf{p}_3) = \mathbf{a}, \quad \Phi(\mathbf{p}_2) = \Phi(\mathbf{p}_4) = \mathbf{b},$$

and $\partial\Phi$ can be described as follows.

Γ_1 spirals up from \mathbf{a} to \mathbf{b} ; its projection into the (x, y) -plane has winding number $+1$ around the origin. (See Exercise 23, Chap. 8.)

$$\Gamma_2 = [\mathbf{b}, \mathbf{a}].$$

Γ_3 spirals up from \mathbf{a} to \mathbf{b} ; its projection into the (x, y) plane has winding number -1 around the origin.

$$\Gamma_4 = [\mathbf{b}, \mathbf{a}].$$

$$\text{Thus } \partial\Phi = \Gamma_1 + \Gamma_3 + 2\Gamma_2.$$

If we go from \mathbf{a} to \mathbf{b} along Γ_1 and continue along the "edge" of M until we return to \mathbf{a} , the curve traced out is

$$\Gamma = \Gamma_1 - \Gamma_3,$$

which may also be represented on the parameter interval $[0, 2\pi]$ by the equations

$$\begin{aligned}x &= (1 + \delta \sin \theta) \cos 2\theta \\y &= (1 + \delta \sin \theta) \sin 2\theta \\z &= -\delta \cos \theta.\end{aligned}$$

It should be emphasized that $\Gamma \neq \partial\Phi$: Let η be the 1-form discussed in Exercises 21 and 22. Since $d\eta = 0$, Stokes' theorem shows that

$$\int_{\partial\Phi} \eta = 0.$$

But although Γ is the “geometric” boundary of M , we have

$$\int_{\Gamma} \eta = 4\pi.$$

In order to avoid this possible source of confusion, Stokes’ formula (Theorem 10.50) is frequently stated only for orientable surfaces Φ .

11

THE LEBESGUE THEORY

It is the purpose of this chapter to present the fundamental concepts of the Lebesgue theory of measure and integration and to prove some of the crucial theorems in a rather general setting, without obscuring the main lines of the development by a mass of comparatively trivial detail. Therefore proofs are only sketched in some cases, and some of the easier propositions are stated without proof. However, the reader who has become familiar with the techniques used in the preceding chapters will certainly find no difficulty in supplying the missing steps.

The theory of the Lebesgue integral can be developed in several distinct ways. Only one of these methods will be discussed here. For alternative procedures we refer to the more specialized treatises on integration listed in the Bibliography.

SET FUNCTIONS

If A and B are any two sets, we write $A - B$ for the set of all elements x such that $x \in A$, $x \notin B$. The notation $A - B$ does not imply that $B \subset A$. We denote the empty set by 0 , and say that A and B are disjoint if $A \cap B = 0$.

11.1 Definition A family \mathcal{R} of sets is called a *ring* if $A \in \mathcal{R}$ and $B \in \mathcal{R}$ implies

$$(1) \quad A \cup B \in \mathcal{R}, \quad A - B \in \mathcal{R}.$$

Since $A \cap B = A - (A - B)$, we also have $A \cap B \in \mathcal{R}$ if \mathcal{R} is a ring.

A ring \mathcal{R} is called a *σ -ring* if

$$(2) \quad \bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$$

whenever $A_n \in \mathcal{R}$ ($n = 1, 2, 3, \dots$). Since

$$\bigcap_{n=1}^{\infty} A_n = A_1 - \bigcup_{n=1}^{\infty} (A_1 - A_n),$$

we also have

$$\bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$$

if \mathcal{R} is a σ -ring.

11.2 Definition We say that ϕ is a set function defined on \mathcal{R} if ϕ assigns to every $A \in \mathcal{R}$ a number $\phi(A)$ of the extended real number system. ϕ is *additive* if $A \cap B = 0$ implies

$$(3) \quad \phi(A \cup B) = \phi(A) + \phi(B),$$

and ϕ is *countably additive* if $A_i \cap A_j = 0$ ($i \neq j$) implies

$$(4) \quad \phi\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \phi(A_n).$$

We shall always assume that the range of ϕ does not contain both $+\infty$ and $-\infty$; for if it did, the right side of (3) could become meaningless. Also, we exclude set functions whose only value is $+\infty$ or $-\infty$.

It is interesting to note that the left side of (4) is independent of the order in which the A_n 's are arranged. Hence the rearrangement theorem shows that the right side of (4) converges absolutely if it converges at all; if it does not converge, the partial sums tend to $+\infty$, or to $-\infty$.

If ϕ is additive, the following properties are easily verified:

$$(5) \quad \phi(0) = 0.$$

$$(6) \quad \phi(A_1 \cup \dots \cup A_n) = \phi(A_1) + \dots + \phi(A_n)$$

if $A_i \cap A_j = 0$ whenever $i \neq j$.

$$(7) \quad \phi(A_1 \cup A_2) + \phi(A_1 \cap A_2) = \phi(A_1) + \phi(A_2).$$

If $\phi(A) \geq 0$ for all A , and $A_1 \subset A_2$, then

$$(8) \quad \phi(A_1) \leq \phi(A_2).$$

Because of (8), nonnegative additive set functions are often called monotonic.

$$(9) \quad \phi(A - B) = \phi(A) - \phi(B)$$

if $B \subset A$, and $|\phi(B)| < +\infty$.

11.3 Theorem Suppose ϕ is countably additive on a ring \mathcal{R} . Suppose $A_n \in \mathcal{R}$ ($n = 1, 2, 3, \dots$), $A_1 \subset A_2 \subset A_3 \subset \dots$, $A \in \mathcal{R}$, and

$$A = \bigcup_{n=1}^{\infty} A_n.$$

Then, as $n \rightarrow \infty$,

$$\phi(A_n) \rightarrow \phi(A).$$

Proof Put $B_1 = A_1$, and

$$B_n = A_n - A_{n-1} \quad (n = 2, 3, \dots).$$

Then $B_i \cap B_j = \emptyset$ for $i \neq j$, $A_n = B_1 \cup \dots \cup B_n$, and $A = \bigcup B_n$. Hence

$$\phi(A_n) = \sum_{i=1}^n \phi(B_i)$$

and

$$\phi(A) = \sum_{i=1}^{\infty} \phi(B_i).$$

CONSTRUCTION OF THE LEBESGUE MEASURE

11.4 Definition Let R^p denote p -dimensional euclidean space. By an *interval* in R^p we mean the set of points $\mathbf{x} = (x_1, \dots, x_p)$ such that

$$(10) \quad a_i \leq x_i \leq b_i \quad (i = 1, \dots, p),$$

or the set of points which is characterized by (10) with any or all of the \leq signs replaced by $<$. The possibility that $a_i = b_i$ for any value of i is not ruled out; in particular, the empty set is included among the intervals.

If A is the union of a finite number of intervals, A is said to be an *elementary set*.

If I is an interval, we define

$$m(I) = \prod_{i=1}^p (b_i - a_i),$$

no matter whether equality is included or excluded in any of the inequalities (10).

If $A = I_1 \cup \cdots \cup I_n$, and if these intervals are pairwise disjoint, we set

$$(11) \quad m(A) = m(I_1) + \cdots + m(I_n).$$

We let \mathcal{E} denote the family of all elementary subsets of R^p .

At this point, the following properties should be verified:

- (12) \mathcal{E} is a ring, but not a σ -ring.
- (13) If $A \in \mathcal{E}$, then A is the union of a finite number of *disjoint* intervals.
- (14) If $A \in \mathcal{E}$, $m(A)$ is well defined by (11); that is, if two different decompositions of A into disjoint intervals are used, each gives rise to the same value of $m(A)$.
- (15) m is additive on \mathcal{E} .

Note that if $p = 1, 2, 3$, then m is length, area, and volume, respectively.

11.5 Definition A nonnegative additive set function ϕ defined on \mathcal{E} is said to be *regular* if the following is true: To every $A \in \mathcal{E}$ and to every $\varepsilon > 0$ there exist sets $F \in \mathcal{E}$, $G \in \mathcal{E}$ such that F is closed, G is open, $F \subset A \subset G$, and

$$(16) \quad \phi(G) - \varepsilon \leq \phi(A) \leq \phi(F) + \varepsilon.$$

11.6 Examples

(a) *The set function m is regular.*

If A is an interval, it is trivial that the requirements of Definition 11.5 are satisfied. The general case follows from (13).

(b) Take $R^p = R^1$, and let α be a monotonically increasing function, defined for all real x . Put

$$\begin{aligned} \mu([a, b)) &= \alpha(b-) - \alpha(a-), \\ \mu([a, b]) &= \alpha(b+) - \alpha(a-), \\ \mu((a, b]) &= \alpha(b+) - \alpha(a+), \\ \mu((a, b)) &= \alpha(b-) - \alpha(a+). \end{aligned}$$

Here $[a, b)$ is the set $a \leq x < b$, etc. Because of the possible discontinuities of α , these cases have to be distinguished. If μ is defined for

elementary sets as in (11), μ is regular on \mathcal{E} . The proof is just like that of (a).

Our next objective is to show that every regular set function on \mathcal{E} can be extended to a countably additive set function on a σ -ring which contains \mathcal{E} .

11.7 Definition Let μ be additive, regular, nonnegative, and finite on \mathcal{E} . Consider countable coverings of any set $E \subset R^p$ by open elementary sets A_n :

$$E \subset \bigcup_{n=1}^{\infty} A_n.$$

Define

$$(17) \quad \mu^*(E) = \inf \sum_{n=1}^{\infty} \mu(A_n),$$

the inf being taken over all countable coverings of E by open elementary sets. $\mu^*(E)$ is called the *outer measure* of E , corresponding to μ .

It is clear that $\mu^*(E) \geq 0$ for all E and that

$$(18) \quad \mu^*(E_1) \leq \mu^*(E_2)$$

if $E_1 \subset E_2$.

11.8 Theorem

(a) For every $A \in \mathcal{E}$, $\mu^*(A) = \mu(A)$.

(b) If $E = \bigcup_{n=1}^{\infty} E_n$, then

$$(19) \quad \mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(E_n).$$

Note that (a) asserts that μ^* is an extension of μ from \mathcal{E} to the family of all subsets of R^p . The property (19) is called *subadditivity*.

Proof Choose $A \in \mathcal{E}$ and $\varepsilon > 0$.

The regularity of μ shows that A is contained in an open elementary set G such that $\mu(G) \leq \mu(A) + \varepsilon$. Since $\mu^*(A) \leq \mu(G)$ and since ε was arbitrary, we have

$$(20) \quad \mu^*(A) \leq \mu(A).$$

The definition of μ^* shows that there is a sequence $\{A_n\}$ of open elementary sets whose union contains A , such that

$$\sum_{n=1}^{\infty} \mu(A_n) \leq \mu^*(A) + \varepsilon.$$

The regularity of μ shows that A contains a closed elementary set F such that $\mu(F) \geq \mu(A) - \varepsilon$; and since F is compact, we have

$$F \subset A_1 \cup \cdots \cup A_N$$

for some N . Hence

$$\mu(A) \leq \mu(F) + \varepsilon \leq \mu(A_1 \cup \cdots \cup A_N) + \varepsilon \leq \sum_1^N \mu(A_n) + \varepsilon \leq \mu^*(A) + 2\varepsilon.$$

In conjunction with (20), this proves (a).

Next, suppose $E = \bigcup E_n$, and assume that $\mu^*(E_n) < +\infty$ for all n . Given $\varepsilon > 0$, there are coverings $\{A_{nk}\}$, $k = 1, 2, 3, \dots$, of E_n by open elementary sets such that

$$(21) \quad \sum_{k=1}^{\infty} \mu(A_{nk}) \leq \mu^*(E_n) + 2^{-n}\varepsilon.$$

Then

$$\mu^*(E) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(A_{nk}) \leq \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon,$$

and (19) follows. In the excluded case, i.e., if $\mu^*(E_n) = +\infty$ for some n , (19) is of course trivial.

11.9 Definition For any $A \subset R^p$, $B \subset R^p$, we define

$$(22) \quad S(A, B) = (A - B) \cup (B - A),$$

$$(23) \quad d(A, B) = \mu^*(S(A, B)).$$

We write $A_n \rightarrow A$ if

$$\lim_{n \rightarrow \infty} d(A, A_n) = 0.$$

If there is a sequence $\{A_n\}$ of elementary sets such that $A_n \rightarrow A$, we say that A is *finitely μ -measurable* and write $A \in \mathfrak{M}_f(\mu)$.

If A is the union of a countable collection of finitely μ -measurable sets, we say that A is *μ -measurable* and write $A \in \mathfrak{M}(\mu)$.

$S(A, B)$ is the so-called “symmetric difference” of A and B . We shall see that $d(A, B)$ is essentially a distance function.

The following theorem will enable us to obtain the desired extension of μ .

11.10 Theorem $\mathfrak{M}(\mu)$ is a σ -ring, and μ^* is countably additive on $\mathfrak{M}(\mu)$.

Before we turn to the proof of this theorem, we develop some of the properties of $S(A, B)$ and $d(A, B)$. We have

$$(24) \quad S(A, B) = S(B, A), \quad S(A, A) = 0.$$

$$(25) \quad S(A, B) \subset S(A, C) \cup S(C, B).$$

$$(26) \quad \left. \begin{array}{l} S(A_1 \cup A_2, B_1 \cup B_2) \\ S(A_1 \cap A_2, B_1 \cap B_2) \\ S(A_1 - A_2, B_1 - B_2) \end{array} \right\} \subset S(A_1, B_1) \cup S(A_2, B_2).$$

(24) is clear, and (25) follows from

$$(A - B) \subset (A - C) \cup (C - B), \quad (B - A) \subset (C - A) \cup (B - C).$$

The first formula of (26) is obtained from

$$(A_1 \cup A_2) - (B_1 \cup B_2) \subset (A_1 - B_1) \cup (A_2 - B_2).$$

Next, writing E^c for the complement of E , we have

$$\begin{aligned} S(A_1 \cap A_2, B_1 \cap B_2) &= S(A_1^c \cup A_2^c, B_1^c \cup B_2^c) \\ &\subset S(A_1^c, B_1^c) \cup S(A_2^c, B_2^c) = S(A_1, B_1) \cup S(A_2, B_2); \end{aligned}$$

and the last formula of (26) is obtained if we note that

$$A_1 - A_2 = A_1 \cap A_2^c.$$

By (23), (19), and (18), these properties of $S(A, B)$ imply

$$(27) \quad d(A, B) = d(B, A), \quad d(A, A) = 0,$$

$$(28) \quad d(A, B) \leq d(A, C) + d(C, B),$$

$$(29) \quad \left. \begin{array}{l} d(A_1 \cup A_2, B_1 \cup B_2) \\ d(A_1 \cap A_2, B_1 \cap B_2) \\ d(A_1 - A_2, B_1 - B_2) \end{array} \right\} \leq d(A_1, B_1) + d(A_2, B_2).$$

The relations (27) and (28) show that $d(A, B)$ satisfies the requirements of Definition 2.15, except that $d(A, B) = 0$ does not imply $A = B$. For instance, if $\mu = m$, A is countable, and B is empty, we have

$$d(A, B) = m^*(A) = 0;$$

to see this, cover the n th point of A by an interval I_n such that

$$m(I_n) < 2^{-n}\epsilon.$$

But if we define two sets A and B to be equivalent, provided

$$d(A, B) = 0,$$

we divide the subsets of R^p into equivalence classes, and $d(A, B)$ makes the set of these equivalence classes into a metric space. $\mathfrak{M}_F(\mu)$ is then obtained as the closure of \mathcal{E} . This interpretation is not essential for the proof, but it explains the underlying idea.

We need one more property of $d(A, B)$, namely,

$$(30) \quad |\mu^*(A) - \mu^*(B)| \leq d(A, B),$$

if at least one of $\mu^*(A)$, $\mu^*(B)$ is finite. For suppose $0 \leq \mu^*(B) \leq \mu^*(A)$. Then (28) shows that

$$d(A, 0) \leq d(A, B) + d(B, 0),$$

that is,

$$\mu^*(A) \leq d(A, B) + \mu^*(B).$$

Since $\mu^*(B)$ is finite, it follows that

$$\mu^*(A) - \mu^*(B) \leq d(A, B).$$

Proof of Theorem 11.10 Suppose $A \in \mathfrak{M}_F(\mu)$, $B \in \mathfrak{M}_F(\mu)$. Choose $\{A_n\}$, $\{B_n\}$ such that $A_n \in \mathcal{E}$, $B_n \in \mathcal{E}$, $A_n \rightarrow A$, $B_n \rightarrow B$. Then (29) and (30) show that

$$(31) \quad A_n \cup B_n \rightarrow A \cup B,$$

$$(32) \quad A_n \cap B_n \rightarrow A \cap B,$$

$$(33) \quad A_n - B_n \rightarrow A - B,$$

$$(34) \quad \mu^*(A_n) \rightarrow \mu^*(A),$$

and $\mu^*(A) < +\infty$ since $d(A_n, A) \rightarrow 0$. By (31) and (33), $\mathfrak{M}_F(\mu)$ is a ring. By (7),

$$\mu(A_n) + \mu(B_n) = \mu(A_n \cup B_n) + \mu(A_n \cap B_n).$$

Letting $n \rightarrow \infty$, we obtain, by (34) and Theorem 11.8(a),

$$\mu^*(A) + \mu^*(B) = \mu^*(A \cup B) + \mu^*(A \cap B).$$

If $A \cap B = 0$, then $\mu^*(A \cap B) = 0$.

It follows that μ^* is additive on $\mathfrak{M}_F(\mu)$.

Now let $A \in \mathfrak{M}(\mu)$. Then A can be represented as the union of a countable collection of *disjoint* sets of $\mathfrak{M}_F(\mu)$. For if $A = \bigcup A'_n$ with $A'_n \in \mathfrak{M}_F(\mu)$, write $A_1 = A'_1$, and

$$A_n = (A'_1 \cup \cdots \cup A'_n) - (A'_1 \cup \cdots \cup A'_{n-1}) \quad (n = 2, 3, 4, \dots).$$

Then

$$(35) \quad A = \bigcup_{n=1}^{\infty} A_n$$

is the required representation. By (19)

$$(36) \quad \mu^*(A) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

On the other hand, $A \supset A_1 \cup \cdots \cup A_n$; and by the additivity of μ^* on $\mathfrak{M}_F(\mu)$ we obtain

$$(37) \quad \mu^*(A) \geq \mu^*(A_1 \cup \cdots \cup A_n) = \mu^*(A_1) + \cdots + \mu^*(A_n).$$

Equations (36) and (37) imply

$$(38) \quad \mu^*(A) = \sum_{n=1}^{\infty} \mu^*(A_n).$$

Suppose $\mu^*(A)$ is finite. Put $B_n = A_1 \cup \cdots \cup A_n$. Then (38) shows that

$$d(A, B_n) = \mu^*\left(\bigcup_{i=n+1}^{\infty} A_i\right) = \sum_{i=n+1}^{\infty} \mu^*(A_i) \rightarrow 0$$

as $n \rightarrow \infty$. Hence $B_n \rightarrow A$; and since $B_n \in \mathfrak{M}_F(\mu)$, it is easily seen that $A \in \mathfrak{M}_F(\mu)$.

We have thus shown that $A \in \mathfrak{M}_F(\mu)$ if $A \in \mathfrak{M}(\mu)$ and $\mu^*(A) < +\infty$. It is now clear that μ^* is countably additive on $\mathfrak{M}(\mu)$. For if

$$A = \bigcup A_n,$$

where $\{A_n\}$ is a sequence of disjoint sets of $\mathfrak{M}(\mu)$, we have shown that (38) holds if $\mu^*(A_n) < +\infty$ for every n , and in the other case (38) is trivial.

Finally, we have to show that $\mathfrak{M}(\mu)$ is a σ -ring. If $A_n \in \mathfrak{M}(\mu)$, $n = 1, 2, 3, \dots$, it is clear that $\bigcup A_n \in \mathfrak{M}(\mu)$ (Theorem 2.12). Suppose $A \in \mathfrak{M}(\mu)$, $B \in \mathfrak{M}(\mu)$, and

$$A = \bigcup_{n=1}^{\infty} A_n, \quad B = \bigcup_{n=1}^{\infty} B_n,$$

where $A_n, B_n \in \mathfrak{M}_F(\mu)$. Then the identity

$$A_n \cap B = \bigcup_{i=1}^{\infty} (A_n \cap B_i)$$

shows that $A_n \cap B \in \mathfrak{M}(\mu)$; and since

$$\mu^*(A_n \cap B) \leq \mu^*(A_n) < +\infty,$$

$A_n \cap B \in \mathfrak{M}_F(\mu)$. Hence $A_n - B \in \mathfrak{M}_F(\mu)$, and $A - B \in \mathfrak{M}(\mu)$ since $A - B = \bigcup_{n=1}^{\infty} (A_n - B)$.

We now replace $\mu^*(A)$ by $\mu(A)$ if $A \in \mathfrak{M}(\mu)$. Thus μ , originally only defined on \mathcal{E} , is extended to a countably additive set function on the σ -ring $\mathfrak{M}(\mu)$. This extended set function is called a *measure*. The special case $\mu = m$ is called the *Lebesgue measure* on R^p .

11.11 Remarks

(a) If A is open, then $A \in \mathfrak{M}(\mu)$. For every open set in R^p is the union of a countable collection of open intervals. To see this, it is sufficient to construct a countable base whose members are open intervals.

By taking complements, it follows that every closed set is in $\mathfrak{M}(\mu)$.

(b) If $A \in \mathfrak{M}(\mu)$ and $\varepsilon > 0$, there exist sets F and G such that

$$F \subset A \subset G,$$

F is closed, G is open, and

$$(39) \quad \mu(G - A) < \varepsilon, \quad \mu(A - F) < \varepsilon.$$

The first inequality holds since μ^* was defined by means of coverings by *open* elementary sets. The second inequality then follows by taking complements.

(c) We say that E is a *Borel set* if E can be obtained by a countable number of operations, starting from open sets, each operation consisting in taking unions, intersections, or complements. The collection \mathcal{B} of all Borel sets in R^p is a σ -ring; in fact, it is the smallest σ -ring which contains all open sets. By Remark (a), $E \in \mathfrak{M}(\mu)$ if $E \in \mathcal{B}$.

(d) If $A \in \mathfrak{M}(\mu)$, there exist Borel sets F and G such that $F \subset A \subset G$, and

$$(40) \quad \mu(G - A) = \mu(A - F) = 0.$$

This follows from (b) if we take $\varepsilon = 1/n$ and let $n \rightarrow \infty$.

Since $A = F \cup (A - F)$, we see that every $A \in \mathfrak{M}(\mu)$ is the union of a Borel set and a set of measure zero.

The Borel sets are μ -measurable for every μ . But the sets of measure zero [that is, the sets E for which $\mu^*(E) = 0$] may be different for different μ 's.

(e) For every μ , the sets of measure zero form a σ -ring.

(f) In case of the Lebesgue measure, every countable set has measure zero. But there are uncountable (in fact, perfect) sets of measure zero. The Cantor set may be taken as an example: Using the notation of Sec. 2.44, it is easily seen that

$$m(E_n) = \left(\frac{2}{3}\right)^n \quad (n = 1, 2, 3, \dots);$$

and since $P = \bigcap E_n$, $P \subset E_n$ for every n , so that $m(P) = 0$.

MEASURE SPACES

11.12 Definition Suppose X is a set, not necessarily a subset of a euclidean space, or indeed of any metric space. X is said to be a *measure space* if there exists a σ -ring \mathfrak{M} of subsets of X (which are called measurable sets) and a non-negative countably additive set function μ (which is called a measure), defined on \mathfrak{M} .

If, in addition, $X \in \mathfrak{M}$, then X is said to be a *measurable space*.

For instance, we can take $X = R^p$, \mathfrak{M} the collection of all Lebesgue-measurable subsets of R^p , and μ Lebesgue measure.

Or, let X be the set of all positive integers, \mathfrak{M} the collection of all subsets of X , and $\mu(E)$ the number of elements of E .

Another example is provided by probability theory, where events may be considered as sets, and the probability of the occurrence of events is an additive (or countably additive) set function.

In the following sections we shall always deal with measurable spaces. It should be emphasized that the integration theory which we shall soon discuss would not become simpler in any respect if we sacrificed the generality we have now attained and restricted ourselves to Lebesgue measure, say, on an interval of the real line. In fact, the essential features of the theory are brought out with much greater clarity in the more general situation, where it is seen that everything depends only on the countable additivity of μ on a σ -ring.

It will be convenient to introduce the notation

$$(41) \quad \{x|P\}$$

for the set of all elements x which have the property P .

MEASURABLE FUNCTIONS

11.13 Definition Let f be a function defined on the measurable space X , with values in the extended real number system. The function f is said to be *measurable* if the set

$$(42) \quad \{x|f(x) > a\}$$

is measurable for every real a .

11.14 Example If $X = R^p$ and $\mathfrak{M} = \mathfrak{M}(\mu)$ as defined in Definition 11.9, every continuous f is measurable, since then (42) is an open set.

11.15 Theorem *Each of the following four conditions implies the other three:*

(43) $\{x|f(x) > a\}$ *is measurable for every real* a .

(44) $\{x|f(x) \geq a\}$ *is measurable for every real* a .

(45) $\{x|f(x) < a\}$ *is measurable for every real* a .

(46) $\{x|f(x) \leq a\}$ *is measurable for every real* a .

Proof The relations

$$\{x|f(x) \geq a\} = \bigcap_{n=1}^{\infty} \left\{x|f(x) > a - \frac{1}{n}\right\},$$

$$\{x|f(x) < a\} = X - \{x|f(x) \geq a\},$$

$$\{x|f(x) \leq a\} = \bigcap_{n=1}^{\infty} \left\{x|f(x) < a + \frac{1}{n}\right\},$$

$$\{x|f(x) > a\} = X - \{x|f(x) \leq a\}$$

show successively that (43) implies (44), (44) implies (45), (45) implies (46), and (46) implies (43).

Hence any of these conditions may be used instead of (42) to define measurability.

11.16 Theorem *If* f *is measurable, then* $|f|$ *is measurable.*

Proof

$$\{x| |f(x)| < a\} = \{x|f(x) < a\} \cap \{x|f(x) > -a\}.$$

11.17 Theorem *Let* $\{f_n\}$ *be a sequence of measurable functions. For* $x \in X$, *put*

$$g(x) = \sup f_n(x) \quad (n = 1, 2, 3, \dots),$$

$$h(x) = \limsup_{n \rightarrow \infty} f_n(x).$$

Then g *and* h *are measurable.*

The same is of course true of the inf and lim inf.

Proof

$$\{x|g(x) > a\} = \bigcup_{n=1}^{\infty} \{x|f_n(x) > a\},$$

$$h(x) = \inf g_m(x),$$

where $g_m(x) = \sup_{n \geq m} f_n(x)$ ($n \geq m$).

Corollaries

- (a) *If f and g are measurable, then $\max(f, g)$ and $\min(f, g)$ are measurable.*
If
- (47)
$$f^+ = \max(f, 0), \quad f^- = -\min(f, 0),$$
- it follows, in particular, that f^+ and f^- are measurable.*
- (b) *The limit of a convergent sequence of measurable functions is measurable.*

11.18 Theorem *Let f and g be measurable real-valued functions defined on X , let F be real and continuous on R^2 , and put*

$$h(x) = F(f(x), g(x)) \quad (x \in X).$$

Then h is measurable.

In particular, $f + g$ and fg are measurable.

Proof Let

$$G_a = \{(u, v) \mid F(u, v) > a\}.$$

Then G_a is an open subset of R^2 , and we can write

$$G_a = \bigcup_{n=1}^{\infty} I_n,$$

where $\{I_n\}$ is a sequence of open intervals:

$$I_n = \{(u, v) \mid a_n < u < b_n, c_n < v < d_n\}.$$

Since

$$\{x \mid a_n < f(x) < b_n\} = \{x \mid f(x) > a_n\} \cap \{x \mid f(x) < b_n\}$$

is measurable, it follows that the set

$$\{x \mid (f(x), g(x)) \in I_n\} = \{x \mid a_n < f(x) < b_n\} \cap \{x \mid c_n < g(x) < d_n\}$$

is measurable. Hence the same is true of

$$\begin{aligned} \{x \mid h(x) > a\} &= \{x \mid (f(x), g(x)) \in G_a\} \\ &= \bigcup_{n=1}^{\infty} \{x \mid (f(x), g(x)) \in I_n\}. \end{aligned}$$

Summing up, we may say that all ordinary operations of analysis, including limit operations, when applied to measurable functions, lead to measurable functions; in other words, all functions that are ordinarily met with are measurable.

That this is, however, only a rough statement is shown by the following example (based on Lebesgue measure, on the real line): If $h(x) = f(g(x))$, where

f is measurable and g is continuous, then h is not necessarily measurable. (For the details, we refer to McShane, page 241.)

The reader may have noticed that measure has not been mentioned in our discussion of measurable functions. In fact, the class of measurable functions on X depends only on the σ -ring \mathfrak{M} (using the notation of Definition 11.12). For instance, we may speak of *Borel-measurable functions* on R^p , that is, of function f for which

$$\{x | f(x) > a\}$$

is always a Borel set, without reference to any particular measure.

SIMPLE FUNCTIONS

11.19 Definition Let s be a real-valued function defined on X . If the range of s is finite, we say that s is a *simple function*.

Let $E \subset X$, and put

$$(48) \quad K_E(x) = \begin{cases} 1 & (x \in E), \\ 0 & (x \notin E). \end{cases}$$

K_E is called the *characteristic function* of E .

Suppose the range of s consists of the distinct numbers c_1, \dots, c_n . Let

$$E_i = \{x | s(x) = c_i\} \quad (i = 1, \dots, n).$$

Then

$$(49) \quad s = \sum_{i=1}^n c_i K_{E_i},$$

that is, every simple function is a finite linear combination of characteristic functions. It is clear that s is measurable if and only if the sets E_1, \dots, E_n are measurable.

It is of interest that every function can be approximated by simple functions:

11.20 Theorem Let f be a real function on X . There exists a sequence $\{s_n\}$ of simple functions such that $s_n(x) \rightarrow f(x)$ as $n \rightarrow \infty$, for every $x \in X$. If f is measurable, $\{s_n\}$ may be chosen to be a sequence of measurable functions. If $f \geq 0$, $\{s_n\}$ may be chosen to be a monotonically increasing sequence.

Proof If $f \geq 0$, define

$$E_{ni} = \left\{ x \mid \frac{i-1}{2^n} \leq f(x) < \frac{i}{2^n} \right\}, \quad F_n = \{x | f(x) \geq n\}$$

for $n = 1, 2, 3, \dots$, $i = 1, 2, \dots, n2^n$. Put

$$(50) \quad s_n = \sum_{i=1}^{n2^n} \frac{i-1}{2^n} K_{E_{ni}} + nK_{F_n}.$$

In the general case, let $f = f^+ - f^-$, and apply the preceding construction to f^+ and to f^- .

It may be noted that the sequence $\{s_n\}$ given by (50) converges uniformly to f if f is bounded.

INTEGRATION

We shall define integration on a measurable space X , in which \mathfrak{M} is the σ -ring of measurable sets, and μ is the measure. The reader who wishes to visualize a more concrete situation may think of X as the real line, or an interval, and of μ as the Lebesgue measure m .

11.21 Definition Suppose

$$(51) \quad s(x) = \sum_{i=1}^n c_i K_{E_i}(x) \quad (x \in X, c_i > 0)$$

is measurable, and suppose $E \in \mathfrak{M}$. We define

$$(52) \quad I_E(s) = \sum_{i=1}^n c_i \mu(E \cap E_i).$$

If f is measurable and nonnegative, we define

$$(53) \quad \int_E f d\mu = \sup I_E(s),$$

where the sup is taken over all measurable simple functions s such that $0 \leq s \leq f$.

The left member of (53) is called the *Lebesgue integral* of f , with respect to the measure μ , over the set E . It should be noted that the integral may have the value $+\infty$.

It is easily verified that

$$(54) \quad \int_E s d\mu = I_E(s)$$

for every nonnegative simple measurable function s .

11.22 Definition Let f be measurable, and consider the two integrals

$$(55) \quad \int_E f^+ d\mu, \quad \int_E f^- d\mu,$$

where f^+ and f^- are defined as in (47).

If at least one of the integrals (55) is finite, we define

$$(56) \quad \int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu.$$

If both integrals in (55) are finite, then (56) is finite, and we say that f is *integrable* (or *summable*) on E in the Lebesgue sense, with respect to μ ; we write $f \in \mathcal{L}(\mu)$ on E . If $\mu = m$, the usual notation is: $f \in \mathcal{L}$ on E .

This terminology may be a little confusing: If (56) is $+\infty$ or $-\infty$, then the integral of f over E is defined, although f is not integrable in the above sense of the word; f is integrable on E only if its integral over E is finite.

We shall be mainly interested in integrable functions, although in some cases it is desirable to deal with the more general situation.

11.23 Remarks The following properties are evident:

- (a) If f is measurable and bounded on E , and if $\mu(E) < +\infty$, then $f \in \mathcal{L}(\mu)$ on E .
- (b) If $a \leq f(x) \leq b$ for $x \in E$, and $\mu(E) < +\infty$, then

$$a\mu(E) \leq \int_E f d\mu \leq b\mu(E).$$

- (c) If f and $g \in \mathcal{L}(\mu)$ on E , and if $f(x) \leq g(x)$ for $x \in E$, then

$$\int_E f d\mu \leq \int_E g d\mu.$$

- (d) If $f \in \mathcal{L}(\mu)$ on E , then $cf \in \mathcal{L}(\mu)$ on E , for every finite constant c , and

$$\int_E cf d\mu = c \int_E f d\mu.$$

- (e) If $\mu(E) = 0$, and f is measurable, then

$$\int_E f d\mu = 0.$$

- (f) If $f \in \mathcal{L}(\mu)$ on E , $A \in \mathfrak{M}$, and $A \subset E$, then $f \in \mathcal{L}(\mu)$ on A .

11.24 Theorem

- (a) Suppose f is measurable and nonnegative on X . For $A \in \mathfrak{M}$, define

$$(57) \quad \phi(A) = \int_A f d\mu.$$

Then ϕ is countably additive on \mathfrak{M} .

(b) The same conclusion holds if $f \in \mathcal{L}(\mu)$ on X .

Proof It is clear that (b) follows from (a) if we write $f = f^+ - f^-$ and apply (a) to f^+ and to f^- .

To prove (a), we have to show that

$$(58) \quad \phi(A) = \sum_{n=1}^{\infty} \phi(A_n)$$

if $A_n \in \mathfrak{M}$ ($n = 1, 2, 3, \dots$), $A_i \cap A_j = \emptyset$ for $i \neq j$, and $A = \bigcup_{n=1}^{\infty} A_n$.

If f is a characteristic function, then the countable additivity of ϕ is precisely the same as the countable additivity of μ , since

$$\int_A K_E d\mu = \mu(A \cap E).$$

If f is simple, then f is of the form (51), and the conclusion again holds.

In the general case, we have, for every measurable simple function s such that $0 \leq s \leq f$,

$$\int_A s d\mu = \sum_{n=1}^{\infty} \int_{A_n} s d\mu \leq \sum_{n=1}^{\infty} \phi(A_n).$$

Therefore, by (53),

$$(59) \quad \phi(A) \leq \sum_{n=1}^{\infty} \phi(A_n).$$

Now if $\phi(A_n) = +\infty$ for some n , (58) is trivial, since $\phi(A) \geq \phi(A_n)$. Suppose $\phi(A_n) < +\infty$ for every n .

Given $\varepsilon > 0$, we can choose a measurable function s such that $0 \leq s \leq f$, and such that

$$(60) \quad \int_{A_1} s d\mu \geq \int_{A_1} f d\mu - \varepsilon, \quad \int_{A_2} s d\mu \geq \int_{A_2} f d\mu - \varepsilon.$$

Hence

$$\phi(A_1 \cup A_2) \geq \int_{A_1 \cup A_2} s d\mu = \int_{A_1} s d\mu + \int_{A_2} s d\mu \geq \phi(A_1) + \phi(A_2) - 2\varepsilon,$$

so that

$$\phi(A_1 \cup A_2) \geq \phi(A_1) + \phi(A_2).$$

It follows that we have, for every n ,

$$(61) \quad \phi(A_1 \cup \cdots \cup A_n) \geq \phi(A_1) + \cdots + \phi(A_n).$$

Since $A \supset A_1 \cup \cdots \cup A_n$, (61) implies

$$(62) \quad \phi(A) \geq \sum_{n=1}^{\infty} \phi(A_n),$$

and (58) follows from (59) and (62).

Corollary *If $A \in \mathfrak{M}$, $B \in \mathfrak{M}$, $B \subset A$, and $\mu(A - B) = 0$, then*

$$\int_A f d\mu = \int_B f d\mu.$$

Since $A = B \cup (A - B)$, this follows from Remark 11.23(e).

11.25 Remarks The preceding corollary shows that sets of measure zero are negligible in integration.

Let us write $f \sim g$ on E if the set

$$\{x | f(x) \neq g(x)\} \cap E$$

has measure zero.

Then $f \sim f$; $f \sim g$ implies $g \sim f$; and $f \sim g$, $g \sim h$ implies $f \sim h$. That is, the relation \sim is an equivalence relation.

If $f \sim g$ on E , we clearly have

$$\int_A f d\mu = \int_A g d\mu,$$

provided the integrals exist, for every measurable subset A of E .

If a property P holds for every $x \in E - A$, and if $\mu(A) = 0$, it is customary to say that P holds for almost all $x \in E$, or that P holds almost everywhere on E . (This concept of "almost everywhere" depends of course on the particular measure under consideration. In the literature, unless something is said to the contrary, it usually refers to Lebesgue measure.)

If $f \in \mathcal{L}(\mu)$ on E , it is clear that $f(x)$ must be finite almost everywhere on E . In most cases we therefore do not lose any generality if we assume the given functions to be finite-valued from the outset.

11.26 Theorem *If $f \in \mathcal{L}(\mu)$ on E , then $|f| \in \mathcal{L}(\mu)$ on E , and*

$$(63) \quad \left| \int_E f d\mu \right| \leq \int_E |f| d\mu.$$

Proof Write $E = A \cup B$, where $f(x) \geq 0$ on A and $f(x) < 0$ on B . By Theorem 11.24,

$$\int_E |f| d\mu = \int_A |f| d\mu + \int_B |f| d\mu = \int_A f^+ d\mu + \int_B f^- d\mu < +\infty,$$

so that $|f| \in \mathcal{L}(\mu)$. Since $f \leq |f|$ and $-f \leq |f|$, we see that

$$\int_E f d\mu \leq \int_E |f| d\mu, \quad -\int_E f d\mu \leq \int_E |f| d\mu,$$

and (63) follows.

Since the integrability of f implies that of $|f|$, the Lebesgue integral is often called an absolutely convergent integral. It is of course possible to define nonabsolutely convergent integrals, and in the treatment of some problems it is essential to do so. But these integrals lack some of the most useful properties of the Lebesgue integral and play a somewhat less important role in analysis.

11.27 Theorem Suppose f is measurable on E , $|f| \leq g$, and $g \in \mathcal{L}(\mu)$ on E . Then $f \in \mathcal{L}(\mu)$ on E .

Proof We have $f^+ \leq g$ and $f^- \leq g$.

11.28 Lebesgue's monotone convergence theorem Suppose $E \in \mathfrak{M}$. Let $\{f_n\}$ be a sequence of measurable functions such that

$$(64) \quad 0 \leq f_1(x) \leq f_2(x) \leq \cdots \quad (x \in E).$$

Let f be defined by

$$(65) \quad f_n(x) \rightarrow f(x) \quad (x \in E)$$

as $n \rightarrow \infty$. Then

$$(66) \quad \int_E f_n d\mu \rightarrow \int_E f d\mu \quad (n \rightarrow \infty).$$

Proof By (64) it is clear that, as $n \rightarrow \infty$,

$$(67) \quad \int_E f_n d\mu \rightarrow \alpha$$

for some α ; and since $f_n \leq f$, we have

$$(68) \quad \alpha \leq \int_E f d\mu.$$

Choose c such that $0 < c < 1$, and let s be a simple measurable function such that $0 \leq s \leq f$. Put

$$E_n = \{x | f_n(x) \geq cs(x)\} \quad (n = 1, 2, 3, \dots).$$

By (64), $E_1 \subset E_2 \subset E_3 \subset \dots$; and by (65),

$$(69) \quad E = \bigcup_{n=1}^{\infty} E_n.$$

For every n ,

$$(70) \quad \int_E f_n d\mu \geq \int_{E_n} f_n d\mu \geq c \int_{E_n} s d\mu.$$

We let $n \rightarrow \infty$ in (70). Since the integral is a countably additive set function (Theorem 11.24), (69) shows that we may apply Theorem 11.3 to the last integral in (70), and we obtain

$$(71) \quad \alpha \geq c \int_E s d\mu.$$

Letting $c \rightarrow 1$, we see that

$$\alpha \geq \int_E s d\mu,$$

and (53) implies

$$(72) \quad \alpha \geq \int_E f d\mu.$$

The theorem follows from (67), (68), and (72).

11.29 Theorem Suppose $f = f_1 + f_2$, where $f_i \in \mathcal{L}(\mu)$ on E ($i = 1, 2$). Then $f \in \mathcal{L}(\mu)$ on E , and

$$(73) \quad \int_E f d\mu = \int_E f_1 d\mu + \int_E f_2 d\mu.$$

Proof First, suppose $f_1 \geq 0, f_2 \geq 0$. If f_1 and f_2 are simple, (73) follows trivially from (52) and (54). Otherwise, choose monotonically increasing sequences $\{s'_n\}, \{s''_n\}$ of nonnegative measurable simple functions which converge to f_1, f_2 . Theorem 11.20 shows that this is possible. Put $s_n = s'_n + s''_n$. Then

$$\int_E s_n d\mu = \int_E s'_n d\mu + \int_E s''_n d\mu,$$

and (73) follows if we let $n \rightarrow \infty$ and appeal to Theorem 11.28.

Next, suppose $f_1 \geq 0, f_2 \leq 0$. Put

$$A = \{x | f(x) \geq 0\}, \quad B = \{x | f(x) < 0\}.$$

Then f, f_1 , and $-f_2$ are nonnegative on A . Hence

$$(74) \quad \int_A f_1 d\mu = \int_A f d\mu + \int_A (-f_2) d\mu = \int_A f d\mu - \int_A f_2 d\mu.$$

Similarly, $-f, f_1$, and $-f_2$ are nonnegative on B , so that

$$\int_B (-f_2) d\mu = \int_B f_1 d\mu + \int_B (-f) d\mu,$$

or

$$(75) \quad \int_B f_1 d\mu = \int_B f d\mu - \int_B f_2 d\mu,$$

and (73) follows if we add (74) and (75).

In the general case, E can be decomposed into four sets E_i on each of which $f_1(x)$ and $f_2(x)$ are of constant sign. The two cases we have proved so far imply

$$\int_{E_i} f d\mu = \int_{E_i} f_1 d\mu + \int_{E_i} f_2 d\mu \quad (i = 1, 2, 3, 4),$$

and (73) follows by adding these four equations.

We are now in a position to reformulate Theorem 11.28 for series.

11.30 Theorem Suppose $E \in \mathfrak{M}$. If $\{f_n\}$ is a sequence of nonnegative measurable functions and

$$(76) \quad f(x) = \sum_{n=1}^{\infty} f_n(x) \quad (x \in E),$$

then

$$\int_E f d\mu = \sum_{n=1}^{\infty} \int_E f_n d\mu.$$

Proof The partial sums of (76) form a monotonically increasing sequence.

11.31 Fatou's theorem Suppose $E \in \mathfrak{M}$. If $\{f_n\}$ is a sequence of nonnegative measurable functions and

$$f(x) = \liminf_{n \rightarrow \infty} f_n(x) \quad (x \in E),$$

then

$$(77) \quad \int_E f d\mu \leq \liminf_{n \rightarrow \infty} \int_E f_n d\mu.$$

Strict inequality may hold in (77). An example is given in Exercise 5.

Proof For $n = 1, 2, 3, \dots$ and $x \in E$, put

$$g_n(x) = \inf f_i(x) \quad (i \geq n).$$

Then g_n is measurable on E , and

$$(78) \quad 0 \leq g_1(x) \leq g_2(x) \leq \dots,$$

$$(79) \quad g_n(x) \leq f_n(x),$$

$$(80) \quad g_n(x) \rightarrow f(x) \quad (n \rightarrow \infty).$$

By (78), (80), and Theorem 11.28,

$$(81) \quad \int_E g_n d\mu \rightarrow \int_E f d\mu,$$

so that (77) follows from (79) and (81).

11.32 Lebesgue's dominated convergence theorem Suppose $E \in \mathfrak{M}$. Let $\{f_n\}$ be a sequence of measurable functions such that

$$(82) \quad f_n(x) \rightarrow f(x) \quad (x \in E)$$

as $n \rightarrow \infty$. If there exists a function $g \in \mathcal{L}(\mu)$ on E , such that

$$(83) \quad |f_n(x)| \leq g(x) \quad (n = 1, 2, 3, \dots, x \in E),$$

then

$$(84) \quad \lim_{n \rightarrow \infty} \int_E f_n d\mu = \int_E f d\mu.$$

Because of (83), $\{f_n\}$ is said to be dominated by g , and we talk about dominated convergence. By Remark 11.25, the conclusion is the same if (82) holds almost everywhere on E .

Proof First, (83) and Theorem 11.27 imply that $f_n \in \mathcal{L}(\mu)$ and $f \in \mathcal{L}(\mu)$ on E .

Since $f_n + g \geq 0$, Fatou's theorem shows that

$$\int_E (f + g) d\mu \leq \liminf_{n \rightarrow \infty} \int_E (f_n + g) d\mu,$$

or

$$(85) \quad \int_E f d\mu \leq \liminf_{n \rightarrow \infty} \int_E f_n d\mu.$$

Since $g - f_n \geq 0$, we see similarly that

$$\int_E (g - f) d\mu \leq \liminf_{n \rightarrow \infty} \int_E (g - f_n) d\mu,$$

so that

$$-\int_E f d\mu \leq \liminf_{n \rightarrow \infty} \left[-\int_E f_n d\mu \right],$$

which is the same as

$$(86) \quad \int_E f d\mu \geq \limsup_{n \rightarrow \infty} \int_E f_n d\mu.$$

The existence of the limit in (84) and the equality asserted by (84) now follow from (85) and (86).

Corollary *If $\mu(E) < +\infty$, $\{f_n\}$ is uniformly bounded on E , and $f_n(x) \rightarrow f(x)$ on E , then (84) holds.*

A uniformly bounded convergent sequence is often said to be boundedly convergent.

COMPARISON WITH THE RIEMANN INTEGRAL

Our next theorem will show that every function which is Riemann-integrable on an interval is also Lebesgue-integrable, and that Riemann-integrable functions are subject to rather stringent continuity conditions. Quite apart from the fact that the Lebesgue theory therefore enables us to integrate a much larger class of functions, its greatest advantage lies perhaps in the ease with which many limit operations can be handled; from this point of view, Lebesgue's convergence theorems may well be regarded as the core of the Lebesgue theory.

One of the difficulties which is encountered in the Riemann theory is that limits of Riemann-integrable functions (or even continuous functions) may fail to be Riemann-integrable. This difficulty is now almost eliminated, since limits of measurable functions are always measurable.

Let the measure space X be the interval $[a, b]$ of the real line, with $\mu = m$ (the Lebesgue measure), and \mathfrak{M} the family of Lebesgue-measurable subsets of $[a, b]$. Instead of

$$\int_X f dm$$

it is customary to use the familiar notation

$$\int_a^b f dx$$

for the Lebesgue integral of f over $[a, b]$. To distinguish Riemann integrals from Lebesgue integrals, we shall now denote the former by

$$\mathcal{R} \int_a^b f dx.$$

11.33 Theorem

(a) If $f \in \mathcal{R}$ on $[a, b]$, then $f \in \mathcal{L}$ on $[a, b]$, and

$$(87) \quad \int_a^b f dx = \mathcal{R} \int_a^b f dx.$$

(b) Suppose f is bounded on $[a, b]$. Then $f \in \mathcal{R}$ on $[a, b]$ if and only if f is continuous almost everywhere on $[a, b]$.

Proof Suppose f is bounded. By Definition 6.1 and Theorem 6.4 there is a sequence $\{P_k\}$ of partitions of $[a, b]$, such that P_{k+1} is a refinement of P_k , such that the distance between adjacent points of P_k is less than $1/k$, and such that

$$(88) \quad \lim_{k \rightarrow \infty} L(P_k, f) = \mathcal{R} \int_a^b f dx, \quad \lim_{k \rightarrow \infty} U(P_k, f) = \mathcal{R} \int_a^b f dx.$$

(In this proof, all integrals are taken over $[a, b]$.)

If $P_k = \{x_0, x_1, \dots, x_n\}$, with $x_0 = a, x_n = b$, define

$$U_k(a) = L_k(a) = f(a);$$

put $U_k(x) = M_i$ and $L_k(x) = m_i$ for $x_{i-1} < x \leq x_i, 1 \leq i \leq n$, using the notation introduced in Definition 6.1. Then

$$(89) \quad L(P_k, f) = \int L_k dx, \quad U(P_k, f) = \int U_k dx,$$

and

$$(90) \quad L_1(x) \leq L_2(x) \leq \dots \leq f(x) \leq \dots \leq U_2(x) \leq U_1(x)$$

for all $x \in [a, b]$, since P_{k+1} refines P_k . By (90), there exist

$$(91) \quad L(x) = \lim_{k \rightarrow \infty} L_k(x), \quad U(x) = \lim_{k \rightarrow \infty} U_k(x).$$

Observe that L and U are bounded measurable functions on $[a, b]$, that

$$(92) \quad L(x) \leq f(x) \leq U(x) \quad (a \leq x \leq b),$$

and that

$$(93) \quad \int L \, dx = \mathcal{R} \int f \, dx, \quad \int U \, dx = \mathcal{R} \int f \, dx,$$

by (88), (90), and the monotone convergence theorem.

So far, nothing has been assumed about f except that f is a bounded real function on $[a, b]$.

To complete the proof, note that $f \in \mathcal{R}$ if and only if its upper and lower Riemann integrals are equal, hence if and only if

$$(94) \quad \int L \, dx = \int U \, dx;$$

since $L \leq U$, (94) happens if and only if $L(x) = U(x)$ for almost all $x \in [a, b]$ (Exercise 1).

In that case, (92) implies that

$$(95) \quad L(x) = f(x) = U(x)$$

almost everywhere on $[a, b]$, so that f is measurable, and (87) follows from (93) and (95).

Furthermore, if x belongs to no P_k , it is quite easy to see that $U(x) = L(x)$ if and only if f is continuous at x . Since the union of the sets P_k is countable, its measure is 0, and we conclude that f is continuous almost everywhere on $[a, b]$ if and only if $L(x) = U(x)$ almost everywhere, hence (as we saw above) if and only if $f \in \mathcal{R}$.

This completes the proof.

The familiar connection between integration and differentiation is to a large degree carried over into the Lebesgue theory. If $f \in \mathcal{L}$ on $[a, b]$, and

$$(96) \quad F(x) = \int_a^x f \, dt \quad (a \leq x \leq b),$$

then $F'(x) = f(x)$ almost everywhere on $[a, b]$.

Conversely, if F is differentiable at every point of $[a, b]$ ("almost everywhere" is not good enough here!) and if $F' \in \mathcal{L}$ on $[a, b]$, then

$$F(x) - F(a) = \int_a^x F'(t) \, dt \quad (a \leq x \leq b).$$

For the proofs of these two theorems, we refer the reader to any of the works on integration cited in the Bibliography.

INTEGRATION OF COMPLEX FUNCTIONS

Suppose f is a complex-valued function defined on a measure space X , and $f = u + iv$, where u and v are real. We say that f is measurable if and only if both u and v are measurable.

It is easy to verify that sums and products of complex measurable functions are again measurable. Since

$$|f| = (u^2 + v^2)^{1/2},$$

Theorem 11.18 shows that $|f|$ is measurable for every complex measurable f .

Suppose μ is a measure on X , E is a measurable subset of X , and f is a complex function on X . We say that $f \in \mathcal{L}(\mu)$ on E provided that f is measurable and

$$(97) \quad \int_E |f| \, d\mu < +\infty,$$

and we define

$$\int_E f \, d\mu = \int_E u \, d\mu + i \int_E v \, d\mu$$

if (97) holds. Since $|u| \leq |f|$, $|v| \leq |f|$, and $|f| \leq |u| + |v|$, it is clear that (97) holds if and only if $u \in \mathcal{L}(\mu)$ and $v \in \mathcal{L}(\mu)$ on E .

Theorems 11.23(a), (d), (e), (f), 11.24(b), 11.26, 11.27, 11.29, and 11.32 can now be extended to Lebesgue integrals of complex functions. The proofs are quite straightforward. That of Theorem 11.26 is the only one that offers anything of interest:

If $f \in \mathcal{L}(\mu)$ on E , there is a complex number c , $|c| = 1$, such that

$$c \int_E f \, d\mu \geq 0.$$

Put $g = cf = u + iv$, u and v real. Then

$$\left| \int_E f \, d\mu \right| = c \int_E f \, d\mu = \int_E g \, d\mu = \int_E u \, d\mu \leq \int_E |f| \, d\mu.$$

The third of the above equalities holds since the preceding ones show that $\int g \, d\mu$ is real.

FUNCTIONS OF CLASS \mathcal{L}^2

As an application of the Lebesgue theory, we shall now extend the Parseval theorem (which we proved only for Riemann-integrable functions in Chap. 8) and prove the Riesz-Fischer theorem for orthonormal sets of functions.

11.34 Definition Let X be a measurable space. We say that a complex function $f \in \mathcal{L}^2(\mu)$ on X if f is measurable and if

$$\int_X |f|^2 d\mu < +\infty.$$

If μ is Lebesgue measure, we say $f \in \mathcal{L}^2$. For $f \in \mathcal{L}^2(\mu)$ (we shall omit the phrase “on X ” from now on) we define

$$\|f\| = \left\{ \int_X |f|^2 d\mu \right\}^{1/2}$$

and call $\|f\|$ the $\mathcal{L}^2(\mu)$ norm of f .

11.35 Theorem Suppose $f \in \mathcal{L}^2(\mu)$ and $g \in \mathcal{L}^2(\mu)$. Then $fg \in \mathcal{L}(\mu)$, and

$$(98) \quad \int_X |fg| d\mu \leq \|f\| \|g\|.$$

This is the Schwarz inequality, which we have already encountered for series and for Riemann integrals. It follows from the inequality

$$0 \leq \int_X (|f| + \lambda|g|)^2 d\mu = \|f\|^2 + 2\lambda \int_X |fg| d\mu + \lambda^2 \|g\|^2,$$

which holds for every real λ .

11.36 Theorem If $f \in \mathcal{L}^2(\mu)$ and $g \in \mathcal{L}^2(\mu)$, then $f + g \in \mathcal{L}^2(\mu)$, and

$$\|f + g\| \leq \|f\| + \|g\|.$$

Proof The Schwarz inequality shows that

$$\begin{aligned} \|f + g\|^2 &= \int |f|^2 + \int f\bar{g} + \int \bar{f}g + \int |g|^2 \\ &\leq \|f\|^2 + 2\|f\| \|g\| + \|g\|^2 \\ &= (\|f\| + \|g\|)^2. \end{aligned}$$

11.37 Remark If we define the distance between two functions f and g in $\mathcal{L}^2(\mu)$ to be $\|f - g\|$, we see that the conditions of Definition 2.15 are satisfied, except for the fact that $\|f - g\| = 0$ does not imply that $f(x) = g(x)$ for all x , but only for almost all x . Thus, if we identify functions which differ only on a set of measure zero, $\mathcal{L}^2(\mu)$ is a metric space.

We now consider \mathcal{L}^2 on an interval of the real line, with respect to Lebesgue measure.

11.38 Theorem The continuous functions form a dense subset of \mathcal{L}^2 on $[a, b]$.

More explicitly, this means that for any $f \in \mathcal{L}^2$ on $[a, b]$, and any $\varepsilon > 0$, there is a function g , continuous on $[a, b]$, such that

$$\|f - g\| = \left\{ \int_a^b |f - g|^2 dx \right\}^{1/2} < \varepsilon.$$

Proof We shall say that f is approximated in \mathcal{L}^2 by a sequence $\{g_n\}$ if $\|f - g_n\| \rightarrow 0$ as $n \rightarrow \infty$.

Let A be a closed subset of $[a, b]$, and K_A its characteristic function. Put

$$t(x) = \inf_{y \in A} |x - y| \quad (y \in A)$$

and

$$g_n(x) = \frac{1}{1 + nt(x)} \quad (n = 1, 2, 3, \dots).$$

Then g_n is continuous on $[a, b]$, $g_n(x) = 1$ on A , and $g_n(x) \rightarrow 0$ on B , where $B = [a, b] - A$. Hence

$$\|g_n - K_A\| = \left\{ \int_B g_n^2 dx \right\}^{1/2} \rightarrow 0$$

by Theorem 11.32. Thus characteristic functions of closed sets can be approximated in \mathcal{L}^2 by continuous functions.

By (39) the same is true for the characteristic function of any measurable set, and hence also for simple measurable functions.

If $f \geq 0$ and $f \in \mathcal{L}^2$, let $\{s_n\}$ be a monotonically increasing sequence of simple nonnegative measurable functions such that $s_n(x) \rightarrow f(x)$. Since $|f - s_n|^2 \leq f^2$, Theorem 11.32 shows that $\|f - s_n\| \rightarrow 0$.

The general case follows.

11.39 Definition We say that a sequence of complex functions $\{\phi_n\}$ is an *orthonormal* set of functions on a measurable space X if

$$\int_X \phi_n \bar{\phi}_m d\mu = \begin{cases} 0 & (n \neq m), \\ 1 & (n = m). \end{cases}$$

In particular, we must have $\phi_n \in \mathcal{L}^2(\mu)$. If $f \in \mathcal{L}^2(\mu)$ and if

$$c_n = \int_X f \bar{\phi}_n d\mu \quad (n = 1, 2, 3, \dots),$$

we write

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n,$$

as in Definition 8.10.

The definition of a trigonometric Fourier series is extended in the same way to \mathcal{L}^2 (or even to \mathcal{L}) on $[-\pi, \pi]$. Theorems 8.11 and 8.12 (the Bessel inequality) hold for any $f \in \mathcal{L}^2(\mu)$. The proofs are the same, word for word.

We can now prove the Parseval theorem.

11.40 Theorem *Suppose*

$$(99) \quad f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx},$$

where $f \in \mathcal{L}^2$ on $[-\pi, \pi]$. Let s_n be the n th partial sum of (99). Then

$$(100) \quad \lim_{n \rightarrow \infty} \|f - s_n\| = 0,$$

$$(101) \quad \sum_{-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2 dx.$$

Proof Let $\varepsilon > 0$ be given. By Theorem 11.38, there is a continuous function g such that

$$\|f - g\| < \frac{\varepsilon}{2}.$$

Moreover, it is easy to see that we can arrange it so that $g(\pi) = g(-\pi)$. Then g can be extended to a periodic continuous function. By Theorem 8.16, there is a trigonometric polynomial T , of degree N , say, such that

$$\|g - T\| < \frac{\varepsilon}{2}.$$

Hence, by Theorem 8.11 (extended to \mathcal{L}^2), $n \geq N$ implies

$$\|s_n - f\| \leq \|T - f\| < \varepsilon,$$

and (100) follows. Equation (101) is deduced from (100) as in the proof of Theorem 8.16.

Corollary *If $f \in \mathcal{L}^2$ on $[-\pi, \pi]$, and if*

$$\int_{-\pi}^{\pi} f(x) e^{-inx} dx = 0 \quad (n = 0, \pm 1, \pm 2, \dots),$$

then $\|f\| = 0$.

Thus if two functions in \mathcal{L}^2 have the same Fourier series, they differ at most on a set of measure zero.

11.41 Definition Let f and $f_n \in \mathcal{L}^2(\mu)$ ($n = 1, 2, 3, \dots$). We say that $\{f_n\}$ converges to f in $\mathcal{L}^2(\mu)$ if $\|f_n - f\| \rightarrow 0$. We say that $\{f_n\}$ is a Cauchy sequence in $\mathcal{L}^2(\mu)$ if for every $\varepsilon > 0$ there is an integer N such that $n \geq N, m \geq N$ implies $\|f_n - f_m\| \leq \varepsilon$.

11.42 Theorem If $\{f_n\}$ is a Cauchy sequence in $\mathcal{L}^2(\mu)$, then there exists a function $f \in \mathcal{L}^2(\mu)$ such that $\{f_n\}$ converges to f in $\mathcal{L}^2(\mu)$.

This says, in other words, that $\mathcal{L}^2(\mu)$ is a *complete* metric space.

Proof Since $\{f_n\}$ is a Cauchy sequence, we can find a sequence $\{n_k\}$, $k = 1, 2, 3, \dots$, such that

$$\|f_{n_k} - f_{n_{k+1}}\| < \frac{1}{2^k} \quad (k = 1, 2, 3, \dots).$$

Choose a function $g \in \mathcal{L}^2(\mu)$. By the Schwarz inequality,

$$\int_X |g(f_{n_k} - f_{n_{k+1}})| \, d\mu \leq \frac{\|g\|}{2^k}.$$

Hence

$$(102) \quad \sum_{k=1}^{\infty} \int_X |g(f_{n_k} - f_{n_{k+1}})| \, d\mu \leq \|g\|.$$

By Theorem 11.30, we may interchange the summation and integration in (102). It follows that

$$(103) \quad |g(x)| \sum_{k=1}^{\infty} |f_{n_k}(x) - f_{n_{k+1}}(x)| < +\infty$$

almost everywhere on X . Therefore

$$(104) \quad \sum_{k=1}^{\infty} |f_{n_{k+1}}(x) - f_{n_k}(x)| < +\infty$$

almost everywhere on X . For if the series in (104) were divergent on a set E of positive measure, we could take $g(x)$ to be nonzero on a subset of E of positive measure, thus obtaining a contradiction to (103).

Since the k th partial sum of the series

$$\sum_{k=1}^{\infty} (f_{n_{k+1}}(x) - f_{n_k}(x)),$$

which converges almost everywhere on X , is

$$f_{n_{k+1}}(x) - f_{n_1}(x),$$

we see that the equation

$$f(x) = \lim_{k \rightarrow \infty} f_{n_k}(x)$$

defines $f(x)$ for almost all $x \in X$, and it does not matter how we define $f(x)$ at the remaining points of X .

We shall now show that this function f has the desired properties. Let $\varepsilon > 0$ be given, and choose N as indicated in Definition 11.41. If $n_k > N$, Fatou's theorem shows that

$$\|f - f_{n_k}\| \leq \liminf_{i \rightarrow \infty} \|f_{n_i} - f_{n_k}\| \leq \varepsilon.$$

Thus $f - f_{n_k} \in \mathcal{L}^2(\mu)$, and since $f = (f - f_{n_k}) + f_{n_k}$, we see that $f \in \mathcal{L}^2(\mu)$. Also, since ε is arbitrary,

$$\lim_{k \rightarrow \infty} \|f - f_{n_k}\| = 0.$$

Finally, the inequality

$$(105) \quad \|f - f_n\| \leq \|f - f_{n_k}\| + \|f_{n_k} - f_n\|$$

shows that $\{f_n\}$ converges to f in $\mathcal{L}^2(\mu)$; for if we take n and n_k large enough, each of the two terms on the right of (105) can be made arbitrarily small.

11.43 The Riesz-Fischer theorem *Let $\{\phi_n\}$ be orthonormal on X . Suppose $\sum |c_n|^2$ converges, and put $s_n = c_1\phi_1 + \cdots + c_n\phi_n$. Then there exists a function $f \in \mathcal{L}^2(\mu)$ such that $\{s_n\}$ converges to f in $\mathcal{L}^2(\mu)$, and such that*

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n.$$

Proof For $n > m$,

$$\|s_n - s_m\|^2 = |c_{m+1}|^2 + \cdots + |c_n|^2,$$

so that $\{s_n\}$ is a Cauchy sequence in $\mathcal{L}^2(\mu)$. By Theorem 11.42, there is a function $f \in \mathcal{L}^2(\mu)$ such that

$$\lim_{n \rightarrow \infty} \|f - s_n\| = 0.$$

Now, for $n > k$,

$$\int_X f \bar{\phi}_k d\mu - c_k = \int_X f \bar{\phi}_k d\mu - \int_X s_n \bar{\phi}_k d\mu,$$

so that

$$\left| \int_X f \bar{\phi}_k d\mu - c_k \right| \leq \|f - s_n\| \cdot \|\phi_k\| + \|f - s_n\|.$$

Letting $n \rightarrow \infty$, we see that

$$c_k = \int_X f \bar{\phi}_k d\mu \quad (k = 1, 2, 3, \dots),$$

and the proof is complete.

11.44 Definition An orthonormal set $\{\phi_n\}$ is said to be *complete* if, for $f \in \mathcal{L}^2(\mu)$, the equations

$$\int_X f \bar{\phi}_n d\mu = 0 \quad (n = 1, 2, 3, \dots)$$

imply that $\|f\| = 0$.

In the Corollary to Theorem 11.40 we deduced the completeness of the trigonometric system from the Parseval equation (101). Conversely, the Parseval equation holds for every complete orthonormal set:

11.45 Theorem Let $\{\phi_n\}$ be a complete orthonormal set. If $f \in \mathcal{L}^2(\mu)$ and if

$$(106) \quad f \sim \sum_{n=1}^{\infty} c_n \phi_n,$$

then

$$(107) \quad \int_X |f|^2 d\mu = \sum_{n=1}^{\infty} |c_n|^2.$$

Proof By the Bessel inequality, $\sum |c_n|^2$ converges. Putting

$$s_n = c_1 \phi_1 + \cdots + c_n \phi_n,$$

the Riesz-Fischer theorem shows that there is a function $g \in \mathcal{L}^2(\mu)$ such that

$$(108) \quad g \sim \sum_{n=1}^{\infty} c_n \phi_n,$$

and such that $\|g - s_n\| \rightarrow 0$. Hence $\|s_n\| \rightarrow \|g\|$. Since

$$\|s_n\|^2 = |c_1|^2 + \cdots + |c_n|^2,$$

we have

$$(109) \quad \int_X |g|^2 d\mu = \sum_{n=1}^{\infty} |c_n|^2.$$

Now (106), (108), and the completeness of $\{\phi_n\}$ show that $\|f - g\| = 0$, so that (109) implies (107).

Combining Theorems 11.43 and 11.45, we arrive at the very interesting conclusion that every complete orthonormal set induces a 1-1 correspondence between the functions $f \in \mathcal{L}^2(\mu)$ (identifying those which are equal almost everywhere) on the one hand and the sequences $\{c_n\}$ for which $\sum |c_n|^2$ converges, on the other. The representation

$$f \sim \sum_{n=1}^{\infty} c_n \phi_n,$$

together with the Parseval equation, shows that $\mathcal{L}^2(\mu)$ may be regarded as an infinite-dimensional euclidean space (the so-called "Hilbert space"), in which the point f has coordinates c_n , and the functions ϕ_n are the coordinate vectors.

EXERCISES

1. If $f \geq 0$ and $\int_E f d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E . *Hint:* Let E_n be the subset of E on which $f(x) > 1/n$. Write $A = \bigcup E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n .
2. If $\int_A f d\mu = 0$ for every measurable subset A of a measurable set E , then $f(x) = 0$ almost everywhere on E .
3. If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points x at which $\{f_n(x)\}$ converges is measurable.
4. If $f \in \mathcal{L}(\mu)$ on E and g is bounded and measurable on E , then $fg \in \mathcal{L}(\mu)$ on E .
5. Put

$$\begin{aligned} g(x) &= \begin{cases} 0 & (0 \leq x \leq \frac{1}{2}), \\ 1 & (\frac{1}{2} < x \leq 1), \end{cases} \\ f_{2k}(x) &= g(x) & (0 \leq x \leq 1), \\ f_{2k+1}(x) &= g(1-x) & (0 \leq x \leq 1). \end{aligned}$$

Show that

$$\liminf_{n \rightarrow \infty} f_n(x) = 0 \quad (0 \leq x \leq 1),$$

but

$$\int_0^1 f_n(x) dx = \frac{1}{2}.$$

[Compare with (77).]

6. Let

$$f_n(x) = \begin{cases} \frac{1}{n} & (|x| \leq n), \\ 0 & (|x| > n). \end{cases}$$

Then $f_n(x) \rightarrow 0$ uniformly on R^1 , but

$$\int_{-\infty}^{\infty} f_n dx = 2 \quad (n = 1, 2, 3, \dots).$$

(We write $\int_{-\infty}^{\infty}$ in place of \int_{R^1} .) Thus uniform convergence does not imply dominated convergence in the sense of Theorem 11.32. However, on sets of finite measure, uniformly convergent sequences of bounded functions do satisfy Theorem 11.32.

7. Find a necessary and sufficient condition that $f \in \mathcal{R}(\alpha)$ on $[a, b]$. *Hint:* Consider Example 11.6(b) and Theorem 11.33.
8. If $f \in \mathcal{R}$ on $[a, b]$ and if $F(x) = \int_a^x f(t) dt$, prove that $F'(x) = f(x)$ almost everywhere on $[a, b]$.
9. Prove that the function F given by (96) is continuous on $[a, b]$.
10. If $\mu(X) < +\infty$ and $f \in \mathcal{L}^2(\mu)$ on X , prove that $f \in \mathcal{L}(\mu)$ on X . If

$$\mu(X) = +\infty,$$

this is false. For instance, if

$$f(x) = \frac{1}{1 + |x|},$$

then $f \in \mathcal{L}^2$ on R^1 , but $f \notin \mathcal{L}$ on R^1 .

11. If $f, g \in \mathcal{L}(\mu)$ on X , define the distance between f and g by

$$\int_X |f - g| d\mu.$$

Prove that $\mathcal{L}(\mu)$ is a complete metric space.

12. Suppose

- (a) $|f(x, y)| \leq 1$ if $0 \leq x \leq 1, 0 \leq y \leq 1$,
- (b) for fixed $x, f(x, y)$ is a continuous function of y ,
- (c) for fixed $y, f(x, y)$ is a continuous function of x .

Put

$$g(x) = \int_0^1 f(x, y) dy \quad (0 \leq x \leq 1).$$

Is g continuous?

13. Consider the functions

$$f_n(x) = \sin nx \quad (n = 1, 2, 3, \dots, -\pi \leq x \leq \pi)$$

as points of \mathcal{L}^2 . Prove that the set of these points is closed and bounded, but not compact.

14. Prove that a complex function f is measurable if and only if $f^{-1}(V)$ is measurable for every open set V in the plane.
15. Let \mathcal{R} be the ring of all elementary subsets of $(0, 1]$. If $0 < a \leq b \leq 1$, define

$$\phi([a, b]) = \phi([a, b)) = \phi((a, b]) = \phi((a, b)) = b - a,$$

but define

$$\phi((0, b)) = \phi((0, b]) = 1 + b$$

if $0 < b \leq 1$. Show that this gives an additive set function ϕ on \mathcal{R} , which is not regular and which cannot be extended to a countably additive set function on a σ -ring.

16. Suppose $\{n_k\}$ is an increasing sequence of positive integers and E is the set of all $x \in (-\pi, \pi)$ at which $\{\sin n_k x\}$ converges. Prove that $m(E) = 0$. *Hint:* For every $A \subset E$,

$$\int_A \sin n_k x \, dx \rightarrow 0,$$

and

$$2 \int_A (\sin n_k x)^2 \, dx = \int_A (1 - \cos 2n_k x) \, dx \rightarrow m(A) \quad \text{as } k \rightarrow \infty.$$

17. Suppose $E \subset (-\pi, \pi)$, $m(E) > 0$, $\delta > 0$. Use the Bessel inequality to prove that there are at most finitely many integers n such that $\sin nx \geq \delta$ for all $x \in E$.
18. Suppose $f \in \mathcal{L}^2(\mu)$, $g \in \mathcal{L}^2(\mu)$. Prove that

$$\left| \int f \bar{g} \, d\mu \right|^2 = \int |f|^2 \, d\mu \int |g|^2 \, d\mu$$

if and only if there is a constant c such that $g(x) = cf(x)$ almost everywhere. (Compare Theorem 11.35.)

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LIST OF SPECIAL SYMBOLS

The symbols listed below are followed by a brief statement of their meaning and by the number of the page on which they are defined.

\in belongs to	3	$\{x_n\}$ sequence.....	26
\notin does not belong to	3	\bigcup, \cup union	27
\subset, \supset inclusion signs	3	\bigcap, \cap intersection	27
Q rational field	3	(a, b) segment	31
$<, \leq, >, \geq$ inequality signs....	3	$[a, b]$ interval	31
\sup least upper bound.....	4	E^c complement of E	32
\inf greatest lower bound	4	E' limit points of E	35
R real field	8	\bar{E} closure of E	35
$+\infty, -\infty, \infty$ infinities	11, 27	\lim limit.....	47
\bar{z} complex conjugate	14	\rightarrow converges to	47, 98
$\operatorname{Re}(z)$ real part.....	14	\limsup upper limit	56
$\operatorname{Im}(z)$ imaginary part	14	\liminf lower limit	56
$ z $ absolute value	14	$g \circ f$ composition	86
Σ summation sign	15, 59	$f(x+)$ right-hand limit	94
R^k euclidean k -space	16	$f(x-)$ left-hand limit	94
$\mathbf{0}$ null vector	16	$f', f'(\mathbf{x})$ derivatives	103, 112
$\mathbf{x} \cdot \mathbf{y}$ inner product	16	$U(P, f), U(P, f, \alpha), L(P, f), L(P, f, \alpha)$	
$ \mathbf{x} $ norm of vector \mathbf{x}	16	Riemann sums	121, 122

$\mathcal{R}, \mathcal{R}(\alpha)$ classes of Riemann (Stieltjes) integrable functions	121, 122
$\mathcal{C}(X)$ space of continuous functions	150
$\ \cdot \ $ norm	140, 150, 326
exp exponential function	179
D_N Dirichlet kernel	189
$\Gamma(x)$ gamma function	192
$\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ standard basis	205
$L(X), L(X, Y)$ spaces of linear transformations	207
$[A]$ matrix	210
$D_j f$ partial derivative	215
∇f gradient	217
$\mathcal{C}', \mathcal{C}''$ classes of differentiable functions	219, 235
$\det [A]$ determinant	232
$J_f(\mathbf{x})$ Jacobian	234
$\frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)}$ Jacobian	234
I^k k -cell	245
Q^k k -simplex	247
dx_I basic k -form	257
\wedge multiplication symbol	254
d differentiation operator	260
ω_T transform of ω	262
∂ boundary operator	269
$\nabla \times \mathbf{F}$ curl	281
$\nabla \cdot \mathbf{F}$ divergence	281
\mathcal{E} ring of elementary sets	303
m Lebesgue measure	303, 308
μ measure	303, 308
$\mathfrak{M}_f, \mathfrak{M}$ families of measurable sets	305
$\{x P\}$ set with property P	310
f^+, f^- positive (negative) part of f	312
K_E characteristic function	313
$\mathcal{L}, \mathcal{L}(\mu), \mathcal{L}^2, \mathcal{L}^2(\mu)$ classes of Lebesgue-integrable functions	315, 326

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Solutions Manual to Walter
Rudin's *Principles of
Mathematical Analysis*

Roger Cooke, University of Vermont

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Chapter 1

The Real and Complex Number Systems

Exercise 1.1 If r is rational ($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.

Solution. If r and $r + x$ were both rational, then $x = r + x - r$ would also be rational. Similarly if rx were rational, then $x = \frac{rx}{r}$ would also be rational.

Exercise 1.2 Prove that there is no rational number whose square is 12.

First Solution. Since $\sqrt{12} = 2\sqrt{3}$, we can invoke the previous problem and prove that $\sqrt{3}$ is irrational. If m and n are integers having no common factor and such that $m^2 = 3n^2$, then m is divisible by 3 (since if m^2 is divisible by 3, so is m). Let $m = 3k$. Then $m^2 = 9k^2$, and we have $3k^2 = n^2$. It then follows that n is also divisible by 3 contradicting the assumption that m and n have no common factor.

Second Solution. Suppose $m^2 = 12n^2$, where m and n have no common factor. It follows that m must be even, and therefore n must be odd. Let $m = 2r$. Then we have $r^2 = 3n^2$, so that r is also odd. Let $r = 2s + 1$ and $n = 2t + 1$. Then

$$4s^2 + 4s + 1 = 3(4t^2 + 4t + 1) = 12t^2 + 12t + 3,$$

so that

$$4(s^2 + s - 3t^2 - 3t) = 2.$$

But this is absurd, since 2 cannot be a multiple of 4.

Exercise 1.3 Prove Proposition 1.15, i.e., prove the following statements:

- (a) If $x \neq 0$ and $xy = xz$, then $y = z$.
- (b) If $x \neq 0$ and $xy = x$, then $y = 1$.
- (c) If $x \neq 0$ and $xy = 1$, then $y = 1/x$.
- (d) If $x \neq 0$, then $1/(1/x) = x$.

Solution. (a) Suppose $x \neq 0$ and $xy = xz$. By Axiom (M5) there exists an element $1/x$ such that $(1/x)x = 1$. By (M3) and (M4) we have $(1/x)(xy) = ((1/x)x)y = 1y = y$, and similarly $(1/x)(xz) = z$. Hence $y = z$.

(b) Apply (a) with $z = 1$.

(c) Apply (a) with $z = 1/x$.

(d) Apply (a) with x replaced by $1/x$, $y = 1/(1/x)$, and $z = x$.

Exercise 1.4 Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E , and β is an upper bound of E . Prove that $\alpha \leq \beta$.

Solution. Since E is nonempty, there exists $x \in E$. Then by definition of lower and upper bounds we have $\alpha \leq x \leq \beta$, and hence by property *ii* in the definition of an ordering, we have $\alpha < \beta$ unless $\alpha = x = \beta$.

Exercise 1.5 Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A).$$

Solution: We need to prove that $-\sup(-A)$ is the greatest lower bound of A . For brevity, let $\alpha = -\sup(-A)$. We need to show that $\alpha \leq x$ for all $x \in A$ and $\alpha \geq \beta$ if β is any lower bound of A .

Suppose $x \in A$. Then, $-x \in -A$, and, hence $-x \leq \sup(-A)$. It follows that $x \geq -\sup(-A)$, i.e., $\alpha \leq x$. Thus α is a lower bound of A .

Now let β be any lower bound of A . This means $\beta \leq x$ for all x in A . Hence $-\beta \leq -x$ for all $x \in A$, which says $y \leq -\beta$ for all $y \in -A$. This means $-\beta$ is an upper bound of $-A$. Hence $-\beta \geq \sup(-A)$ by definition of \sup , i.e., $\beta \leq -\sup(-A)$, and so $-\sup(-A)$ is the greatest lower bound of A .

Exercise 1.6 Fix $b > 1$.

- (a) If m, n, p, q are integers, $n > 0, q > 0$, and $r = m/n = p/q$, prove that

$$(b^m)^{1/n} = (b^p)^{1/q}.$$

Hence it makes sense to define $b^r = (b^m)^{1/n}$.

- (b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.

(c) If x is real, define $B(x)$ to be the set of all numbers b^t , where t is rational and $t \leq x$. Prove that

$$b^r = \sup B(r)$$

when r is rational. Hence it makes sense to define

$$b^x = \sup B(x)$$

for every real x .

(d) Prove that $b^{x+y} = b^x b^y$ for all real x and y .

Solution. (a) Let $k = mq = np$. Since there is only one positive real number c such that $c^{nq} = b^k$ (Theorem 1.21), if we prove that both $(b^m)^{1/n}$ and $(b^p)^{1/q}$ have this property, it will follow that they are equal. The proof is then a routine computation: $((b^m)^{1/n})^{nq} = (b^m)^q = b^{mq} = b^k$, and similarly for $(b^p)^{1/q}$.

(b) Let $r = \frac{m}{n}$ and $s = \frac{v}{w}$. Then $r + s = \frac{mw + vn}{nw}$, and

$$b^{r+s} = (b^{mw+vn})^{1/nw} = ((b^{mw}b^{vn}))^{1/nw},$$

by the laws of exponents for integer exponents. By the corollary to Theorem 1.21 we then have

$$b^{r+s} = (b^{mw})^{1/nw} (b^{vn})^{1/nw} = b^r b^s,$$

where the last equality follows from part (a).

(c) It will simplify things later on if we amend the definition of $B(x)$ slightly, by defining it as $\{b^t : t \text{ rational}, t < x\}$. It is then slightly more difficult to prove that $b^r = \sup B(r)$ if r is rational, but the technique of Problem 7 comes to our rescue. Here is how: It is obvious that b^r is an upper bound of $B(r)$. We need to show that it is the least upper bound. The inequality $b^{1/n} < t$ if $n > (b-1)/(t-1)$ is proved just as in Problem 7 below. It follows that if $0 < x < b^r$, there exists an integer n with $b^{1/n} < b^r/x$, i.e., $x < b^{r-1/n} \in B(r)$. Hence x is not an upper bound of $B(r)$, and so b^r is the least upper bound.

(d) By definition $b^{x+y} = \sup B(x+y)$, where $B(x+y)$ is the set of all numbers b^t with t rational and $t < x+y$. Now any rational number t that is less than $x+y$ can be written as $r+s$, where r and s are rational, $r < x$, and $s < y$. To do this, let r be any rational number satisfying $t-y < r < x$, and let $s = t-r$. Conversely any pair of rational numbers r, s with $r < x$, $s < y$ gives a rational sum $t = r+s < x+y$. Hence $B(x+y)$ can be described as the set of all numbers $b^r b^s$ with $r < x$, $s < y$, and r and s rational, i.e., $B(x+y)$ is the set of all products uv , where $u \in B(x)$ and $v \in B(y)$.

Since any such product is less than $\sup B(x) \sup B(y)$, we see that the number $M = \sup B(x) \sup B(y)$ is an upper bound for $B(x+y)$. On the other hand, suppose $0 < c < \sup B(x) \sup B(y)$. Then $c/(\sup B(x)) < \sup B(y)$. Let $m = (1/2)(c/(\sup B(x)) + \sup B(y))$. Then $c/\sup B(x) < m < \sup B(y)$, and there exist $u \in B(x)$, $v \in B(y)$ such that $c/m < u$ and $m < v$. Hence we have

$c = (c/m)m < uv \in B(x+y)$, and so c is not an upper bound for $B(x+y)$. It follows that $\sup B(x) \sup B(y)$ is the least upper bound of $B(x+y)$, i.e.,

$$b^{x+y} = b^x b^y,$$

as required.

Exercise 1.7 Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the *logarithm of y to the base b* .)

(a) For any positive integer n , $b^n - 1 \geq n(b - 1)$.

(b) Hence $b - 1 \geq n(b^{1/n} - 1)$.

(c) If $t > 1$ and $n > (b - 1)/(t - 1)$, then $b^{1/n} < t$.

(d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n ; to see this apply part (c) with $t = y \cdot b^{-w}$.

(e) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n .

(f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.

(g) Prove that this x is unique.

Solution. (a) The inequality $b^n - 1 \geq n(b - 1)$ is equality if $n = 1$. Then, by induction $b^{n+1} - 1 = b^{n+1} - b + (b - 1) = b(b^n - 1) + (b - 1) \geq bn(b - 1) + (b - 1) = (bn + 1)(b - 1) \geq (n + 1)(b - 1)$.

(b) Replace b by $b^{1/n}$ in part (a).

(c) The inequality $n > (b - 1)/(t - 1)$ can be rewritten as $n(t - 1) > (b - 1)$, and since $b - 1 \geq n(b^{1/n} - 1)$, we have $n(t - 1) > n(b^{1/n} - 1)$, which implies $t > b^{1/n}$.

(d) The application of part (c) with $t = y \cdot b^{-w} > 1$ is immediate.

(e) The application of part (c) with $t = b^w \cdot (1/y)$ yields the result, as in part (d) above.

(f) There are only three possibilities for the number $x = \sup A$: 1) $b^x < y$; 2) $b^x > y$; 3) $b^x = y$. The first assumption, by part (d), implies that $x + (1/n) \in A$ for large n , contradicting the assumption that x is an upper bound for A . The second, by part (e), implies that $x - (1/n)$ is an upper bound for A if n is large, contradicting the assumption that x is the smallest upper bound. Hence the only remaining possibility is that $b^x = y$.

(g) Suppose $z \neq x$, say $z > x$. Then $b^z = b^{x+(z-x)} = b^x b^{z-x} > b^x = y$. Hence x is unique. (It is easy to see that $b^w > 1$ if $w > 0$, since there is a positive rational number $r = \frac{m}{n}$ with $0 < r < w$, and $b^r = (b^m)^{1/n}$. Then $b^m > 1$ since $b > 1$, and $(b^m)^{1/n} > 1$ since $1^n = 1 < b^m$.)

Exercise 1.8 Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.

Solution. By Part (a) of Proposition 1.18, either i or $-i$ must be positive. Hence $-1 = i^2 = (-i)^2$ must be positive. But then $1 = (-1)^2$, must also be positive, and this contradicts Part (a) of Proposition 1.18, since 1 and -1 cannot both be positive.

Exercise 1.9 Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least upper bound property?

Solution. We need to show that either $z < w$ or $z = w$, or $w < z$. Now since the real numbers are ordered, we have $a < c$ or $a = c$, or $c < a$. In the first case $z < w$; in the third case $w < z$. Now consider the second case. We must have $b < d$ or $b = d$ or $d < b$. In the first of these cases $z < w$, in the third case $w < z$, and in the second case $z = w$.

We also need to show that if $z < w$ and $w < u$, then $z < u$. Let $u = e + fi$. Since $z < w$, we have either $a < c$ or $a = c$ and $b < d$. Since $w < u$ we have either $c < e$ or $c = e$ and $d < f$. Hence there are four possible cases:

Case 1: $a < c$ and $c < e$. Then $a < e$ and so $z < u$, as required.

Case 2: $a < c$ and $c = e$ and $d < f$. Again $a < e$, and $z < u$.

Case 3: $a = c$ and $b < d$ and $c < e$. Once again $a < e$ and so $z < u$.

Case 4: $a = c$ and $b < d$ and $c = e$, and $d < f$. Then $a = e$ and $b < f$, and so $z < u$.

Exercise 1.10 Suppose $z = a + bi$, $w = u + iv$, and

$$a = \left(\frac{|w| + u}{2} \right)^{1/2}, \quad b = \left(\frac{|w| - u}{2} \right)^{1/2}.$$

Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z})^2 = w$ if $v \leq 0$. Conclude that every complex number (with one exception) has two complex square roots.

Solution.

$$z^2 = (a + bi)^2 = (a^2 - b^2) + 2abi.$$

Now

$$a^2 - b^2 = \frac{|w| + u}{2} - \frac{|w| - u}{2} = u,$$

and, since $(xy)^{1/2} = x^{1/2}y^{1/2}$,

$$2ab = 2 \left(\frac{|w| + u}{2} \frac{|w| - u}{2} \right)^{1/2} = 2 \left(\frac{|w|^2 - u^2}{4} \right)^{1/2}.$$

Hence

$$2ab = 2\left(\left(\frac{v}{2}\right)^2\right)^{1/2}$$

Now $(x^2)^{1/2} = x$ if $x \geq 0$ and $(x^2)^{1/2} = -x$ if $x \leq 0$. We conclude that $2ab = v$ if $v \geq 0$ and $2ab = -v$ if $v \leq 0$. Hence $z^2 = w$ if $v \geq 0$. Replacing b by $-b$, we find that $(\bar{z})^2 = w$ if $v \leq 0$.

Hence every non-zero complex number has (at least) two complex square roots.

Exercise 1.11 If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z ?

Solution. If $z = 0$, we take $r = 0$, $w = 1$. (In this case w is not unique.) Otherwise we take $r = |z|$ and $w = z/|z|$, and these choices are unique, since if $z = rw$, we must have $r = r|w| = |rw| = |z|$, z/r .

Exercise 1.12 If z_1, \dots, z_n are complex, prove that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|.$$

Solution. The case $n = 2$ is Part (e) of Theorem 1.33. We can then apply this result and induction on n to get

$$\begin{aligned} |z_1 + z_2 + \cdots + z_n| &= |(z_1 + z_2 + \cdots + z_{n-1}) + z_n| \\ &\leq |z_1 + z_2 + \cdots + z_{n-1}| + |z_n| \\ &\leq |z_1| + |z_2| + \cdots + |z_{n-1}| + |z_n|. \end{aligned}$$

Exercise 1.13 If x, y are complex, prove that

$$||x| - |y|| \leq |x - y|.$$

Solution. Since $x = x - y + y$, the triangle inequality gives

$$|x| \leq |x - y| + |y|,$$

so that $|x| - |y| \leq |x - y|$. Similarly $|y| - |x| \leq |x - y|$. Since $|x| - |y|$ is a real number we have either $||x| - |y|| = |x| - |y|$ or $||x| - |y|| = |y| - |x|$. In either case, we have shown that $||x| - |y|| \leq |x - y|$.

Exercise 1.14 If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute

$$|1 + z|^2 + |1 - z|^2.$$

Solution. $|1 + z|^2 = (1 + z)(1 + \bar{z}) = 1 + \bar{z} + z + z\bar{z} = 2 + z + \bar{z}$. Similarly $|1 - z|^2 = (1 - z)(1 - \bar{z}) = 1 - z - \bar{z} + z\bar{z} = 2 - z - \bar{z}$. Hence

$$|1 + z|^2 + |1 - z|^2 = 4.$$

Exercise 1.15 Under what conditions does equality hold in the Schwarz inequality?

Solution. The proof of Theorem 1.35 shows that equality can hold if $B = 0$ or if $Ba_j - Cb_j = 0$ for all j , i.e., the numbers a_j are proportional to the numbers b_j . (In terms of linear algebra this means the vectors $\mathbf{a} = (a_1, a_2, \dots, a_n)$ and $\mathbf{b} = (b_1, b_2, \dots, b_n)$ in complex n -dimensional space are linearly dependent. Conversely, if these vectors are linearly independent, then strict inequality holds.)

Exercise 1.16 Suppose $k \geq 3$, $\mathbf{x}, \mathbf{y} \in R^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:

(a) If $2r > d$, there are infinitely many $\mathbf{z} \in R^k$ such that

$$|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r.$$

(b) If $2r = d$, there is exactly one such \mathbf{z} .

(c) If $2r < d$, there is no such \mathbf{z} .

How must these statements be modified if k is 2 or 1?

Solution. (a) Let \mathbf{w} be any vector satisfying the following two equations:

$$\begin{aligned}\mathbf{w} \cdot (\mathbf{x} - \mathbf{y}) &= 0, \\ |\mathbf{w}|^2 &= r^2 - \frac{d^2}{4}.\end{aligned}$$

From linear algebra it is known that all but one of the components of a solution \mathbf{w} of the first equation can be arbitrary. The remaining component is then uniquely determined. Also, if \mathbf{w} is any non-zero solution of the first equation, there is a unique positive number t such that $t\mathbf{w}$ satisfies both equations. (For example, if $x_1 \neq y_1$, the first equation is satisfied whenever

$$z_1 = \frac{z_2(x_2 - y_2) + \dots + z_k(x_k - y_k)}{y_1 - x_1}.$$

If (z_1, z_2, \dots, z_k) satisfies this equation, so does $(tz_1, tz_2, \dots, tz_k)$ for any real number t .) Since at least two of these components can vary independently, we can find a solution with these components having any prescribed ratio. This

ratio does not change when we multiply by the positive number t to obtain a solution of both equations. Since there are infinitely many ratios, there are infinitely many distinct solutions. For each such solution \mathbf{w} the vector $\mathbf{z} = \frac{1}{2}\mathbf{x} + \frac{1}{2}\mathbf{y} + \mathbf{w}$ is a solution of the required equation. For

$$\begin{aligned} |\mathbf{z} - \mathbf{x}|^2 &= \left| \frac{\mathbf{y} - \mathbf{x}}{2} + \mathbf{w} \right|^2 \\ &= \left| \frac{\mathbf{y} - \mathbf{x}}{2} \right|^2 + 2\mathbf{w} \cdot \frac{\mathbf{x} - \mathbf{y}}{2} + |\mathbf{w}|^2 \\ &= \frac{d^2}{4} + 0 + r^2 - \frac{d^2}{4} \\ &= r^2, \end{aligned}$$

and a similar relation holds for $|\mathbf{z} - \mathbf{y}|^2$.

(b) The proof of the triangle inequality shows that equality can hold in this inequality only if it holds in the Schwarz inequality, i.e., one of the two vectors is a scalar multiple of the other. Further examination of the proof shows that the scalar must be nonnegative. Now the conditions of this part of the problem show that

$$|\mathbf{x} - \mathbf{y}| = d = |\mathbf{x} - \mathbf{z}| + |\mathbf{z} - \mathbf{y}|.$$

Hence it follows that there is a nonnegative scalar t such that

$$\mathbf{x} - \mathbf{z} = t(\mathbf{z} - \mathbf{y}).$$

However, the hypothesis also shows immediately that $t = 1$, and so \mathbf{z} is uniquely determined as

$$\mathbf{z} = \frac{\mathbf{x} + \mathbf{y}}{2}.$$

(c) If \mathbf{z} were to satisfy this condition, the triangle inequality would be violated, i.e., we would have

$$|\mathbf{x} - \mathbf{y}| = d > 2r = |\mathbf{x} - \mathbf{z}| + |\mathbf{z} - \mathbf{y}|.$$

When $k = 2$, there are precisely 2 solutions in case (a). When $k = 1$, there are no solutions in case (a). The conclusions in cases (b) and (c) do not require modification.

Exercise 1.17 Prove that

$$|\mathbf{x} + \mathbf{y}|^2 + |\mathbf{x} - \mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$$

if $\mathbf{x} \in R^k$ and $\mathbf{y} \in R^k$. Interpret this geometrically as a statement about parallelograms.

Solution. The proof is a routine computation, using the relation

$$|\mathbf{x} \pm \mathbf{y}|^2 = (\mathbf{x} \pm \mathbf{y}) \cdot (\mathbf{x} \pm \mathbf{y}) = |\mathbf{x}|^2 \pm 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2.$$

If \mathbf{x} and \mathbf{y} are the sides of a parallelogram, then $\mathbf{x} + \mathbf{y}$ and $\mathbf{x} - \mathbf{y}$ are its diagonals. Hence this result says that the sum of the squares on the diagonals of a parallelogram equals the sum of the squares on the sides.

Exercise 1.18 If $k \geq 2$ and $\mathbf{x} \in R^k$, prove that there exists $\mathbf{y} \in R^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?

Solution. If \mathbf{x} has any components equal to 0, then \mathbf{y} can be taken to have the corresponding components equal to 1 and all others equal to 0. If all the components of \mathbf{x} are nonzero, \mathbf{y} can be taken as $(-x_2, x_1, 0, \dots, 0)$. This is, of course, not true when $k = 1$, since the product of two nonzero real numbers is nonzero.

Exercise 1.19 Suppose $\mathbf{a} \in R^k$, $\mathbf{b} \in R^k$. Find $\mathbf{c} \in R^k$ and $r > 0$ such that

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

if and only if $|\mathbf{x} - \mathbf{c}| = r$. (*Solution:* $3\mathbf{c} = 4\mathbf{b} - \mathbf{a}$, $3r = 2|\mathbf{b} - \mathbf{a}|$.)

Solution. Since the solution is given to us, all we have to do is verify it, i.e., we need to show that the equation

$$|\mathbf{x} - \mathbf{a}| = 2|\mathbf{x} - \mathbf{b}|$$

is equivalent to $|\mathbf{x} - \mathbf{c}| = r$, which says

$$\left| \mathbf{x} - \frac{4}{3}\mathbf{b} + \frac{1}{3}\mathbf{a} \right| = \frac{2}{3}|\mathbf{b} - \mathbf{a}|.$$

If we square both sides of both equations, we an equivalent pair of equations, the first of which reduces to

$$3|\mathbf{x}|^2 + 2\mathbf{a} \cdot \mathbf{x} - 8\mathbf{b} \cdot \mathbf{x} - |\mathbf{a}|^2 + 4|\mathbf{b}|^2 = 0,$$

and the second of which reduces to this equation divided by 3. Hence these equations are indeed equivalent.

Exercise 1.20 With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4) (with a slightly different zero element!) but that (A5) fails.

Solution. We are now defining a cut to be a proper subset of the rational numbers that contains, along with each of its elements, all smaller rational

numbers. Order is defined by containment. Now given a set A of cuts having an upper bound β , let α be the union of all the cuts in A . Obviously α is properly contained in β , and so is a proper subset of the rationals. It also obviously satisfies the property that if $p \in \alpha$ and $q < p$, then $q \in \alpha$; hence α is a cut. It is further obvious that α contains each elements of A , and so is an upper bound for A . It remains to prove that there is no smaller upper bound.

To that end, suppose, $\gamma < \alpha$, then α contains an element x not in γ . By definition of α , x must belong to some cut δ in A . But then $\gamma < \delta$, and so γ is not an upper bound for A . Thus α is the least upper bound.

The proof given in the text goes over without any change to show that (A1), (A2), and (A3) hold. As for (A4) let $O = \{r : r \leq 0\}$. We claim $O + \alpha = \alpha$. The proof is easy. First, we obviously have $O + \alpha \subseteq \alpha$. For $r + s \leq s$ if $r \leq 0$. Hence $r + s \in \alpha$ if $s \in \alpha$. Conversely $\alpha \subseteq O + \alpha$, since each s in α can be written as $0 + s$.

Unfortunately, if $O' = \{r : r < 0\}$, there is no element α such that $\alpha + O' = O$. For $\alpha + O'$ has no largest element. If $x = r + s \in \alpha + O'$, where $r \in \alpha$ and $s \in O'$, there is an element $t \in O'$ with $t > s$, and so $r + t \in \alpha + O'$ and $r + t > s$. Since O has a largest element (namely 0), these two sets cannot be equal.

Chapter 2

Basic Topology

Exercise 2.1 Prove that the empty set is a subset of every set.

Solution. Let \emptyset denote the empty set, and let E be any set. The statement $\emptyset \subset E$ is equivalent to the statement, “If $x \in \emptyset$, then $x \in E$.” Since the hypothesis of this if-then statement is false, the implication is true, and we are done.

Exercise 2.2 A complex number z is said to be *algebraic* if there are integers a_0, \dots, a_n , not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable. *Hint:* For every positive integer N there are only finitely many equations with

$$n + |a_0| + |a_1| + \dots + |a_n| = N.$$

Solution. Following the hint, we let A_N be the set of numbers satisfying one of the equations just listed with $n + |a_0| + |a_1| + \dots + |a_n| = N$. The set A_N is finite, since each equation has only a finite set of solutions and there are only finitely many equations satisfying this condition. By the corollary to Theorem 2.12 the set of algebraic numbers, which is the union $\bigcup_{N=2}^{\infty} A_N$, is at most countable. Since all rational numbers are algebraic, it follows that the set of algebraic numbers is exactly countable.

Exercise 2.3 Prove that there exist real numbers which are not algebraic.

Solution. By the previous exercise, the set of real algebraic numbers is countable. If every real number were algebraic, the entire set of real numbers would be countable, contradicting the remark after Theorem 2.14.

Exercise 2.4 Is the set of irrational real numbers countable?

Answer. No. If it were, the set of all real numbers, being the union of the rational and irrational numbers, would be countable.

Exercise 2.5 Construct a bounded set of real numbers with exactly three limit points.

Solution. Let E be the set of numbers of the form $a + \frac{1}{n}$, where $a \in \{1, 2, 3\}$ and $n \in \{2, 3, 4, 5, \dots\}$. It is clear that $\{1, 2, 3\} \subseteq E'$, since every deleted neighborhood of 1, 2, or 3, contains a point in E . Conversely, if $x \notin \{1, 2, 3\}$, let $\delta = \min\{|x-1|, |x-2|, |x-3|\}$. Then the set U of y such that $|x-y| < \delta/2$ contains at most a finite number of points of E , since the set $V = (1, 1 + \frac{\delta}{2}) \cup (2, 2 + \frac{\delta}{2}) \cup (3, 3 + \frac{\delta}{2})$ is disjoint from U , and V contains all the points of the set E except possibly the finite set of points $a + \frac{1}{n}$ for which $n \leq \frac{2}{\delta}$. If p_1, \dots, p_r are the points of E in U , let η be the minimum of $\frac{\delta}{2}$ and the $|x - p_j|$ for which $x \neq p_j$. Then the set W of points y such that $|y-x| < \eta$ contains no points of E except possibly x . Hence $x \notin E'$. Thus $E' = \{1, 2, 3\}$.

Exercise 2.6 Let E' be the set of all limit points of a set E . Prove that E' is closed. Prove that E and \overline{E} have the same limit points. (Recall that $\overline{E} = E \cup E'$.) Do E and E' always have the same limit points?

Solution. To show that E' is closed, we shall show that $(E')' \subseteq E'$. In fact, we shall show the even stronger statement that $(\overline{E})' \subseteq E'$. To do this let $x \in (\overline{E})'$, and let $r > 0$. We need to show that $x \in E'$; that is, since $r > 0$ is arbitrary, we need to find a point $z \in E$ with $0 < d(z, x) < r$. There certainly is a point y of \overline{E} such that $0 < d(y, x) < r$. If $y \in E$, we can take $z = y$, and we are done. If $y \notin E$, then $y \in E'$. Let $s = \min(d(x, y), r - d(x, y))$, so that $s > 0$. Since $y \in E'$, there exists $z \in E$ with $0 < d(x, z) < s$. But it then follows that $d(z, x) \geq d(x, y) - d(x, z) > 0$ and $d(z, x) \leq d(x, y) + d(y, z) < d(x, y) + r - d(x, y) = r$, and we are done in any case.

To show that E and \overline{E} have the same limit points, we need only show the converse of the preceding containment. But this is easy. Suppose $x \in E'$. Since every deleted neighborhood of x contains a point of E , *a fortiori* every deleted neighborhood of x contains a point of \overline{E} . Hence $E' \subseteq (\overline{E})'$.

Certainly E and E' may have different sets of limit points. For example if $E = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\}$, then $E' = \{0\}$, while $(E')' = \emptyset$.

Exercise 2.7 Let A_1, A_2, A_3, \dots be subsets of a metric space.

(a) If $B_n = \cup_{i=1}^n A_i$, prove that $\overline{B}_n = \cup_{i=1}^n \overline{A}_i$, for $n = 1, 2, 3, \dots$.

(b) If $B = \cup_{i=1}^{\infty} A_i$, prove that $\overline{B} \supset \cup_{i=1}^{\infty} \overline{A}_i$.

Show, by an example, that this inclusion can be proper.

Solution. We first show that $\overline{E \cup F} = \overline{E} \cup \overline{F}$, which follows from the stronger fact that $(E \cup F)' = E' \cup F'$. To show this, in turn, we note that if $x \in E'$, then certainly $x \in (E \cup F)'$, and similarly if $x \in F'$. Hence $E' \cup F' \subseteq (E \cup F)'$. To show the converse, suppose $x \notin E' \cup F'$. Then there is a positive number r such that there is no element y of E with $0 < d(x, y) < r$, and a positive number s such that there is no element y of F with $0 < d(x, y) < s$. Hence if $t = \min(r, s)$, then $t > 0$, and there is no element y of $E \cup F$ with $0 < d(x, y) < t$. Therefore $x \notin (E \cup F)'$.

The general result of (a) now follows easily by induction on n , since

$$\begin{aligned}\overline{B_n} &= \overline{\bigcup_{i=1}^n A_i} \\ &= \overline{A_1 \cup \bigcup_{i=2}^n A_i} \\ &= \overline{A_1} \cup \overline{\bigcup_{i=2}^n A_i} \\ &= \overline{A_1} \cup \bigcup_{i=2}^n \overline{A_i} \\ &= \bigcup_{i=1}^n \overline{A_i}.\end{aligned}$$

Part (b) amounts to the trivial observation that, since $B \supseteq A_i$ for all i , then $\overline{B} \supseteq \overline{A_i}$ for all i , and so

$$\overline{B} \supseteq \bigcup_{i=1}^{\infty} \overline{A_i}.$$

If we let $A_i = \{r_i\}$, where $\{r_1, r_2, \dots, r_n, \dots\}$ is an enumeration of the rational numbers, then B is the full set of rational numbers. Hence $\overline{B} = R^1$, while $\overline{A_i} = A_i$ for each i , i.e., $\bigcup \overline{A_i}$ is the set of rational numbers.

Exercise 2.8 Is every point of every open set $E \subset R^2$ a limit point of E . Answer the same question for closed sets in R^2 .

Answer. Yes. Every point of an open set E is a limit point of E . To see this, let E be an open set in R^2 , let $(x_1, x_2) \in E$, let s be such that $(y_1, y_2) \in E$ if $\sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2} < s$, and let $r > 0$. Then the point $(z_1, z_2) = (x_1 + \frac{1}{2} \min(r, s), x_2)$ belongs to E and satisfies $0 < \sqrt{(z_1 - x_1)^2 + (z_2 - x_2)^2} < r$.

There are closed sets for which this statement is not true. For example, any finite set E is closed, yet $E' = \emptyset$ for a finite set.

Exercise 2.9 Let E° denote the set of all interior points of a set E .

- Prove that E° is always open.
- Prove that E is open if and only if $E^\circ = E$.
- If $G \subset E$ and G is open, prove that $G \subset E^\circ$.
- Prove that the complement of E° is the closure of the complement of E .
- Do E and \overline{E} always have the same interiors?

(f) Do E and E° always have the same closures?

Solution. (a) Let $x \in E^\circ$. Then there exists $r > 0$ such that $y \in E$ if $d(x, y) < r$. We claim that in fact $y \in E^\circ$ if $d(x, y) < r$, so that $x \in (E^\circ)^\circ$. Indeed if $d(x, y) < r$, let $s = r - d(x, y)$, so that $s > 0$. Then if $d(z, y) < s$, we have (by the triangle inequality) $d(x, z) < r$, and so $z \in E$. By definition this means $y \in E^\circ$. Since y was any point with $d(x, y) < r$, it follows that all such points are in E° , and so $x \in (E^\circ)^\circ$.

(b) By definition E is open if and only if each of its points is an interior point, which says precisely that $E = E^\circ$.

(c) If $G \subset E$ and G is open, then $G = G^\circ \subseteq E^\circ$.

(d) Part (c) shows that E° is the largest open set contained in E , i.e., the union of all open sets contained in E . Hence its complement is the intersection of all closed sets containing the complement of E , and this, by Theorem 2.27 (c), is the closure of the complement of E .

(e) Emphatically not. If E is the rational numbers in the space R^1 , then $E^\circ = \emptyset$, while $\overline{E} = R^1$, so that the interior of \overline{E} is R^1 .

(f) Emphatically not. If E is the rational numbers in the space R^1 , then $\overline{E} = R^1$, while $E^\circ = \emptyset$, so that $\overline{E^\circ} = \emptyset$.

Exercise 2.10 Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p, q) = \begin{cases} 1, & (\text{if } p \neq q), \\ 0, & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?

Solution. It is obvious that $d(p, q) > 0$ if $p \neq q$ and $d(p, p) = 0$; likewise it is obvious that $d(p, q) = d(q, p)$. To show the triangle inequality $d(x, z) \leq d(x, y) + d(y, z)$, note that the maximal value of the left-hand side is 1, and can be attained only if $x \neq z$. In that case y cannot be equal to both x and z , so that at least one term on the right-hand side is also 1.

Each one-point set is open in this metric, since $B_{\frac{1}{2}}(x) \subseteq \{x\}$. Therefore every set, being the union of all its one-point subsets, is open. Hence every set, being the complement of its complement, is also closed. Only finite sets are compact, since any infinite subset has an open covering (by the union of its one-point subsets) that cannot be reduced to a finite subcovering.

Exercise 2.11 For $x \in R^1$ and $y \in R^1$, define

$$\begin{aligned} d_1(x, y) &= (x - y)^2, \\ d_2(x, y) &= \sqrt{|x - y|}, \end{aligned}$$

$$\begin{aligned}
d_3(x, y) &= |x^2 - y^2|, \\
d_4(x, y) &= |x - 2y|, \\
d_5(x, y) &= \frac{|x - y|}{1 + |x - y|},
\end{aligned}$$

Determine, for each of these, whether it is a metric or not.

Solution. The function $d_1(x, y)$ fails the triangle inequality condition, since

$$d_1(0, 1) + d_1(1, 2) = 1 + 1 = 2 < 4 = d_1(0, 2).$$

The function $d_2(x, y)$ meets the triangle inequality condition, since

$$\sqrt{|x - z|} \leq \sqrt{|x - y|} + \sqrt{|y - z|},$$

as one can easily see by squaring both sides. Hence d_2 is a metric.

The function $d_3(x, y)$ fails the positivity condition, since $d_2(1, -1) = 0$. (Restricted to $[0, \infty)$, d_3 would be a metric.)

Since $d_4(1, \frac{1}{2}) = 0$, the function $d_4(x, y)$ likewise fails the positivity condition. It also fails the symmetry condition, since $d_4(x, y) \neq d_4(y, x)$ in general.

The function $d_5(x, y)$ is a metric. In fact we can prove more generally that if $d(x, y)$ is a metric, so is $\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}$. It is obvious that ρ meets the nonnegativity and symmetry requirements, and we need only verify the triangle inequality, which in this case says that

$$\frac{d(x, z)}{1 + d(x, z)} \leq \frac{d(x, y)}{1 + d(x, y)} + \frac{d(y, z)}{1 + d(y, z)}.$$

To do this, let $a = d(x, z)$, $b = d(x, y)$, and $c = d(y, z)$. We need to show that if $a \leq b + c$, then

$$\frac{a}{1 + a} \leq \frac{b}{1 + b} + \frac{c}{1 + c}.$$

Clearing out the denominators, we find this inequality to be equivalent to

$$a + ab + ac + abc \leq b + c + ab + ac + 2bc + 2abc,$$

which is clearly true.

Exercise 2.12 Let $K \subset \mathbb{R}^1$ consist of 0 and the numbers $1/n$, for $n = 1, 2, 3, \dots$. Prove that E is compact directly from the definition without using the Heine-Borel theorem.

Solution. Suppose $K \subset U_\alpha$, where U_α is open. Then 0 must be in some set U_{α_0} . Since U_{α_0} is open, there exists $\delta > 0$ such that $(-\delta, \delta) \subset U_{\alpha_0}$. In particular $1/n \in U_{\alpha_0}$ if $n > \frac{1}{\delta}$. Let N be the largest integer in $\frac{1}{\delta}$, and let α_j , $j = 1, \dots, N$, be such that $\frac{1}{j} \in U_{\alpha_j}$. Then $K \subset \bigcup_{j=0}^N U_{\alpha_j}$.

Exercise 2.13 Construct a compact set of real numbers whose limit points form a countable set.

Solution. Let $K = \{0\} \cup \{\frac{1}{n} : n = 1, 2, \dots\} \cup \{\frac{1}{m} + \frac{1}{n} : n = m, m+1, \dots; m = 1, 2, \dots\}$. It is clear that 0 and the points $\frac{1}{m}$ are limit points of K . We need only show that these are all the limit points. Since $x \geq 0$ for all $x \in K$ and for any positive number ε there is only a finite set of numbers in K larger than $1 + \varepsilon$, it is clear that no negative number and no number larger than 1 can be a limit point of K . Hence we need only consider positive numbers x satisfying $0 < x < 1$. If x is such a number and x is not one of the points $\frac{1}{m}$, let p be such that $\frac{1}{p+1} < x < \frac{1}{p}$, and let $\varepsilon = \frac{1}{2} \min(x - \frac{1}{p+1}, \frac{1}{p} - x)$. The intersection of the set K with the interval $(x - \varepsilon, x + \varepsilon)$ is contained in the set of points $\{\frac{1}{p+1} + \frac{1}{k} : p+1 \leq k < \frac{1}{\varepsilon}\} \cup \{\frac{1}{m} + \frac{1}{n} : m \leq n < \frac{1}{p+1} - \frac{1}{p+2}; m = p+2, \dots, 2p+2\}$, which is a finite set. Therefore x cannot be a limit point of K .

Exercise 2.14 Give an example of an open cover of the segment $(0, 1)$ which has no finite subcover.

Solution. Let $A_n = (\frac{1}{n}, \frac{n-1}{n})$, $n = 3, 4, \dots$. If $0 < x < 1$, then $x \in A_n$ if $n > 1/\min(x, 1-x)$, so that $\bigcup_{n=3}^{\infty} A_n$ covers $(0, 1)$. However, the union any finite collection $\{A_1, \dots, A_N\}$ is an interval $(\frac{1}{k}, \frac{k-1}{k})$, which fails to contain the point $\frac{1}{2k}$.

Exercise 2.15 Show that Theorem 2.36 and its Corollary become false (in R^1 , for example) if the word “compact” is replaced by “closed” or “bounded.”

Solution. Theorem 2.36 asserts that if a family of closed subsets has the finite intersection property (any finite collection of the sets has a non-empty intersection), then the entire family has a non-empty intersection. To see why this fails for sets that are merely bounded or merely closed, let $A_n = (0, \frac{1}{n})$ and $B_n = [n, \infty)$. The sets A_n are bounded, and the sets B_n are closed. Any finite intersection of the A'_n s is nonempty, and any finite intersection of the B'_n s is nonempty, yet $\bigcap_{n=1}^{\infty} A_n = \emptyset = \bigcap_{n=1}^{\infty} B_n$.

The corollary asserts that a nested sequence of nonempty compact sets has a nonempty intersection, and the examples just given show that compactness cannot be replaced by either closedness or boundedness.

Exercise 2.16 Regard Q , the set of all rational numbers, as a metric space, with $d(p, q) = |p - q|$. Let E be the set of all $p \in Q$ such that $2 < p^2 < 3$. Show that E is closed and bounded in Q , but that E is not compact. Is E open in Q ?

Solution. Suppose $x \in Q \setminus E$. We claim that x is an interior point of the complement of E (which by definition means E is closed). In fact if $x^2 \leq 2$, then $x^2 < 2$, since there is no rational number whose square is 2. If $x = 0$, let $\delta = 1$; otherwise let $\delta = \min(\sqrt{\frac{2-x^2}{3}}, \frac{2-x^2}{3|x|})$. Then if $y \in (x - \delta, x + \delta)$, we have $y^2 < 2$. This is obvious if $x = 0$ and $\delta = 1$. In the other case let $y = x + h$, where $|h| < \delta$. Then $y^2 = x^2 + 2xh + h^2 < x^2 + 2|x|\delta + \delta^2 < x^2 + \frac{2}{3}(2 - x^2) + \frac{2-x^2}{3} = 2$. Hence x is an interior point of the complement of E .

Similarly suppose $x^2 \geq 3$. Since there is no rational number whose square is 3, we must have $x^2 > 3$. Since $x \neq 0$, we let $\delta = \frac{x^2-3}{2|x|}$. Then if $y \in (x - \delta, x + \delta)$, we have $y^2 > 3$. For since $y = x + h$, with $|h| < \delta$, and so $y^2 = x^2 + 2xh + h^2 > x^2 - 2|x|\delta = 3$. Thus again x is an interior point of the complement of E .

Hence in all cases $Q \setminus E$ is open, so that E is closed.

That E is bounded is obvious, since $E \subset [-2, 2]$.

To show that E is not compact, let $U_n = \{p : 2 < p^2 < 3 - \frac{1}{n}\}$, $n = 2, 3, \dots$. The argument that will be used below to show that E is open shows that U_n is open. The sets U_n cover E , but no finite collection of them covers E . Thus E is not compact.

The set E is also open, since if $2 < x^2 < 3$, we can let δ be the minimum of $\sqrt{\frac{3-x^2}{3}}$, $\frac{3-x^2}{3|x|}$, and $\frac{x^2-2}{2|x|}$. Then if $y \in (x - \delta, x + \delta)$, we must have $2 < y^2 < 3$, by the same set of inequalities that was used above.

Exercise 2.17 Let E be the set of all $x \in [0, 1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0, 1]$? Is E compact? Is E perfect?

Solution. The set E is not countable, since for any hypothetical list of its elements $a_1, a_2, \dots, a_n, \dots$ we can always produce an element a of E not in the list by taking the n th digit of a to be 4 if the n th digit of a_n is 7 and equal to 7 if the n th digit of a_n is 4.

The set E is not dense in $[0, 1]$, since $E \subset [0.4, 0.8]$

The set E is closed and bounded, and therefore compact. To show that E is closed, let $x \in [0, 1] \setminus E$, i.e., the decimal expansion of x contains a digit different from 4 and 7. Let the first such digit occur in the n th place (x_n). Let y be any element of E , and let the first digit in which x and y differ be the m th digit ($m \leq n$, $x_m \neq y_m$). Then $|x - y| \geq 10^{-m} - \varepsilon$, $\varepsilon \leq \sum_{k=m+1}^{\infty} 10^{-k} |x_k - y_k|$. Since $y_k \in \{4, 7\}$ and $x_k \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, it follows that $|x_k - y_k| \leq 7$. Hence $\varepsilon \leq \frac{7}{9} 10^{-m}$, and it follows that $|x - y| \geq \frac{2}{9 \cdot 10^m} \geq \frac{1}{9 \cdot 10^n}$. Thus x is an interior point of $[0, 1] \setminus E$, and so E is closed.

The set E is perfect. For each $x \in E$ and each $\varepsilon > 0$ we can find a point $y \in E$ with $0 < |x - y| < \varepsilon$ by changing the n th digit of x from 4 to 7 or from 7 to 4 in the n th place for any $n > 1 - \log_{10} \varepsilon$. Hence $x \in E'$, i.e., $E \subseteq E'$. Since we already know E is closed, it follows that $E = E'$.

Exercise 2.18 Is there a non-empty perfect set in R^1 which contains no rational number?

Answer. Yes. Let $\{r_1, r_2, \dots, r_n, \dots\}$ be the rational numbers in the interval $[-\pi, \pi]$. Let $E_0 = [-\pi, \pi]$. Now assume that E_k has been chosen for $k < n$ in such a way that E_k is a pairwise disjoint union of at most $2^{k+1} - 1$ closed intervals with irrational endpoints, each of *positive* length at most $(\frac{2}{3})^k \pi$ and that E_k does not contain r_j if $j \leq k$. (All of these conditions hold trivially for $k = 0$.) Define a set F_{k+1} , which is obtained from E_k by removing first the middle third of each of the intervals that constitute E_k . The result is a set of at most $2^{k+2} - 2$ pairwise disjoint intervals having irrational endpoints, each interval being of length at most $(\frac{2}{3})^{k+1} \pi$. If $r_{k+1} \notin F_{k+1}$, let $E_{k+1} = F_{k+1}$. If $r_{k+1} \in F_{k+1}$, then r_{k+1} is not the endpoint of the interval $I = [a, b]$ of F_{k+1} that it belongs to. Hence let δ be an irrational positive number less than the minimum of $r_{k+1} - a$ and $b - r_{k+1}$, and let E_{k+1} be obtained from F_{k+1} by removing the interval $(r_{k+1} - \delta, r_{k+1} + \delta)$ (which has irrational endpoints). Then E_{k+1} consists of at most $2^{k+2} - 1$ pairwise disjoint closed intervals, each of positive length at most $(\frac{2}{3})^{k+1} \pi$, and each having irrational endpoints.

The sets E_k form a nested sequence of nonempty compact sets. Hence the intersection $E = \bigcap_{k=0}^{\infty} E_k$ is a nonempty compact set. By construction it contains no rational numbers. To show that it is perfect, we merely observe that if $x \in E$, then for each k there is a unique interval $I_k = [a_k, b_k]$, among the finite set of closed intervals constituting the set E_k such that $x \in I_k$. Let $y_k = a_k$ if $a_k \neq x$, otherwise let $y_k = b_k$. In either case $y_k \in E$ (since in our construction no endpoint of any E_k is ever removed) and $|y_k - x| < 2 \cdot 3^{-k} \pi$. Therefore $x \in E'$.

Exercise 2.19 (a) if A and B are disjoint closed sets in some metric space X , prove that they are separated.

(b) Prove the same for disjoint open sets.

(c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly with $>$ in place of $<$. Prove that A and B are separated.

(d) Prove that every connected space with at least two points is uncountable. *Hint:* Use (c).

Solution. (a) We are given that $A \cap B = \emptyset$. Since A and B are closed, this means $A \cap \overline{B} = \emptyset = \overline{A} \cap B$, which says that A and B are separated.

(b) Since $X \setminus B$ is a closed set containing A , it follows from Theorem 2.27 (c) that $X \setminus B \supseteq \overline{A}$, i.e., that $\overline{A} \cap B = \emptyset$. Similarly $A \cap \overline{B} = \emptyset$.

(c) The sets A and B are disjoint open sets, hence by part (b) they are separated.

(d) Let $x \in X$ and $y \in X$, and let $d(x, y) = d > 0$. Then for every $\delta \in (0, d)$, there must be a point z such that $d(x, z) = \delta$. (If not, the sets A and B defined in part (c) would separate X .) Hence there is a subset of X that can be placed in one-to-one correspondence with the interval $[0, d]$, and so X is uncountable.

Exercise 2.20 Are closures and interiors of connected sets always connected? (Look at subsets of R^2 .)

Answer. The closure of a connected set is connected. Indeed if E is connected and $E \subseteq F \subseteq \overline{E}$, then F is connected. For, suppose $F = G \cup H$, where G and H are separated, nonempty sets. The set E cannot be contained entirely in G . (If it were, since H is nonempty, H would contain a limit point of E , hence a limit point of G , contrary to hypothesis.) For the same reason E cannot be contained entirely in H . Hence $G_1 = E \cap G$ and $H_1 = E \cap H$ are nonempty separated sets such that $E = G_1 \cup H_1$, and E is not connected.

The interior of a connected set may fail to be connected, as we see by letting E be the union of two closed disks in R^2 that are tangent to each other.

Exercise 2.21 Let A and B be separated subsets of some R^k , suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for $t \in R^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]

- (a) Prove that A_0 and B_0 are separated subsets of R^1 .
- (b) Prove that there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.
- (c) Prove that every convex subset of R^k is connected.

Solution. (a) The definition shows that A_0 and B_0 are disjoint. We need only show that neither contains a limit point of the other. Let x be a limit point of A_0 , and suppose $x \in B_0$. This means that for any $\delta > 0$ there exists $t \in A_0$ with $0 < |x - t| < \delta$, $\mathbf{p}(t) = (1-t)\mathbf{b} + t\mathbf{a} \in A$ and $\mathbf{p}(x) = (1-x)\mathbf{a} + x\mathbf{b} \in B$. Now $d(\mathbf{p}(t), \mathbf{p}(x)) = |\mathbf{p}(t) - \mathbf{p}(x)| = |x - t||\mathbf{a} - \mathbf{b}| \leq |x - t|(|\mathbf{a}| + |\mathbf{b}|) < M\delta$, where $M = |\mathbf{a}| + |\mathbf{b}|$. Since δ is arbitrary, this means that B contains a limit point of A , contrary to hypothesis. This contradiction shows that B_0 contains no limit points of A_0 . Likewise A_0 contains no limit points of B_0 , and so A_0 and B_0 are separated.

(b) If $\mathbf{p}(t) \in A \cup B$ for all $t \in [0, 1]$, then $[0, 1] \subseteq A_0 \cup B_0$. Hence $[0, 1] = G \cup H$, where $G = [0, 1] \cap A_0$ and $H = [0, 1] \cap B_0$ are both nonempty ($0 \in G$ and $1 \in H$) and separated. This would mean $[0, 1]$ is not connected. Therefore $\mathbf{p}(t_0) \notin A \cup B$ for some $t_0 \in [0, 1]$, and necessarily $t_0 \in (0, 1)$, since $\mathbf{p}(0) = \mathbf{a} \in A$ and $\mathbf{p}(1) = \mathbf{b} \in B$.

(c) By definition a convex set C is one for which the mapping \mathbf{p} has the property $\mathbf{p}(t) \in C$ for all $t \in [0, 1]$ provided $\mathbf{p}(0) = \mathbf{a} \in C$ and $\mathbf{p}(1) = \mathbf{b} \in C$. Hence by part (b) there cannot be separated nonempty sets A and B such that $C = A \cup B$.

Exercise 2.22 A metric space is called *separable* if it contains a countable dense subset. Show that R^k is separable. *Hint:* Consider the set of points which have only rational coordinates.

Solution. We need to show that every non-empty open subset E of R^k contains a point with all coordinates rational. Now E contains a ball $B_r(\mathbf{x})$, and this ball contains all points \mathbf{y} such that $(x_j - y_j)^2 < \frac{1}{k}$ for $j = 1, 2, \dots, k$. Each interval $(x_j - \frac{1}{k}, x_j + \frac{1}{k})$ contains a rational number r_j , and so the point $\mathbf{r} = (r_1, \dots, r_k)$ belongs to E . Thus E contains a point with only rational coordinates.

Exercise 2.23 A collection $\{V_\alpha\}$ of open sets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subset X$ such that $x \in G$, we have $x \in V_\alpha \subset G$ for some α . In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.

Prove that every separable metric space has a *countable* base. *Hint:* Take all neighborhoods with rational radius and center in some countable dense subset of X .

Solution. Let $\{x_1, x_2, \dots, x_n, \dots\}$ be a countable dense subset of X . For each positive integer m and each positive rational number r let $V_{m,r} = \{y : d(y, x_m) < r\}$. The collection $V_{m,r}$ is countable.

Let $x \in X$, and let G be any open subset of X with $x \in G$. Then there exists $\delta > 0$ such that $B_\delta(x) \subset G$. The open ball $B_{\frac{\delta}{2}}(x)$ contains a point x_k for some k . Let r be a rational number such that $d(x_k, x) < r < \frac{\delta}{2}$. Then $x \in B_r(x_k) \subset B_\delta(x) \subset G$, and we are done.

Exercise 2.24 Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable. *Hint:* Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, \dots, x_j \in X$, choose $x_{j+1} \in X$; if possible, so that $d(x_j, x_{j+1}) \geq \delta$ for $i = 1, \dots, j$. Show that this process must stop after a finite number of steps, and that X can therefore be covered by finitely many neighborhoods of radius δ . Take $\delta = 1/n$ ($n = 1, 2, 3, \dots$), and consider the centers of the corresponding neighborhoods.

Solution. Following the hint, we observe that if the process of constructing x_j did not terminate, the result would be an infinite set of points x_j , $j = 1, 2, \dots$, such that $d(x_i, x_j) \geq \delta$ for $i \neq j$. It would then follow that for any $x \in X$, the open ball $B_{\frac{\delta}{2}}(x)$ contains at most one point of the infinite set, hence that no point could be a limit point of this set, contrary to hypothesis. Hence X is *totally bounded*, i.e., for each $\delta > 0$ there is a *finite* set x_1, \dots, x_{N_δ} such that

$$X = \bigcup_{j=1}^{N_\delta} B_\delta(x_j).$$

Let $x_{n_1}, \dots, x_{n_{N_n}}$ be such that $X = \bigcup_{j=1}^{N_n} B_{\frac{1}{n}}(x_{n_j})$, $n = 1, 2, \dots$. We claim that $\{x_{n_j} : 1 \leq j \leq N_n; n = 1, 2, \dots\}$ is a countable dense subset of X . Indeed

if $x \in X$ and $\delta > 0$, then $x \in B_{\frac{1}{n}}(x_{nj})$ for some x_{nj} for some $n > \frac{1}{\delta}$, and hence $d(x, x_{nj}) < \delta$. By definition, this means that $\{x_{nj}\}$ is dense in X .

Exercise 2.25 Prove that every compact metric space K has a countable base, and that K is therefore separable. *Hint:* For every positive integer n , there are finitely many neighborhoods of radius $1/n$ whose union covers K .

Solution. It is easier simply to refer to the previous problem. The hint shows that K can be covered by a finite union of neighborhoods of radius $1/n$, and the previous problem shows that this implies that K is separable.

It is not entirely obvious that a metric space with a countable base is separable. To prove this, let $\{V_n\}_{n=1}^{\infty}$ be a countable base, and let $x_n \in V_n$. The points V_n must be dense in X . For if G is any non-empty open set, then G contains V_n for some n , and hence $x_n \in G$. (Thus for a metric space, having a countable base and being separable are equivalent.)

Exercise 2.26 Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact. *Hint:* By Exercises 23 and 24, X has a countable base. It follows that every open cover of X has a *countable* subcover $\{G_n\}_{n=1, 2, 3, \dots}$. If no finite subcollection of $\{G_n\}$ covers X , then the complement F_n of $G_1 \cup \dots \cup G_n$ is nonempty for each n , but $\cap F_n$ is empty. If E is a set which contains a point from each F_n , consider a limit point of E , and obtain a contradiction.

Solution. Following the hint, we consider a set E consisting of one point from the complement of each finite union, i.e., $x_n \notin G_1 \cup \dots \cup G_n$. Since there are infinitely many finite unions and every point is in *some* set of the covering, the set E cannot be finite. (If $\{x_{i_1}, \dots, x_{i_n}\}$ is any finite subset of E , there are sets G_{j_1}, \dots, G_{j_n} such that $x_{i_k} \in G_{j_k}$ for each k . Since E contains a point not in $G_{j_1} \cup \dots \cup G_{j_n}$, it contains a point different from x_1, \dots, x_n . Hence E is not finite.)

Now by hypothesis E must have a limit point z . The point z must belong to some set G_n ; and since G_n is open, there is a number $\delta > 0$ such that $B_\delta(z) \subseteq G_n$. But then $B_\delta(z)$ cannot contain x_m if $m \geq n$, and so z cannot be a limit point of $\{x_m\}$. We have now reached a contradiction.

Exercise 2.27 Define a point p in a metric space X to be a *condensation point* of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E .

Suppose $E \subset R^k$, E is uncountable, and let P be the set of all condensation points of E . Prove that P is perfect and that at most countably many points of E are not in P . In other words, show that $P^c \cap E$ is at most countable. *Hint:*

Let $\{V_n\}$ be a countable base of R^k , let W be the union of those V_n for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

Solution. Following the hint, we see that $E \cap W$ is at most countable, being a countable union of at-most-countable sets. It remains to show that $P = W^c$, and that P is perfect.

If $x \in W^c$, and O is any neighborhood of x , then $x \in V_n \subseteq O$ for some n . Since $x \notin W$, $V_n \cap E$ is uncountable. Hence O contains uncountably many points of E , and so x is a condensation point of E . Thus $x \in P$, i.e., $W^c \subseteq P$.

Conversely if $x \in W$, then $x \in V_n$ for some V_n such that $V_n \cap E$ is countable. Hence x has a neighborhood (any neighborhood contained in V_n) containing at most a countable set of points of E , and so $x \notin P$, i.e., $W \subseteq P^c$. Hence $P = W^c$.

It is clear that P is closed (since its complement W is open), so that we need only show that $P \subseteq P'$. Hence suppose $x \in P$, and O is any neighborhood of x . (By definition of P this means $O \cap E$ is uncountable.) We need to show that there is a point $y \in P \cap (O \setminus \{x\})$. If this is not the case, i.e., if every point y in $O \setminus \{x\}$ is in P^c , then for each such point y there is a set V_n containing y such that $V_n \cap E$ is at most countable. That would mean that $y \in W$, i.e., that $O \setminus \{x\}$ is contained in W . It would follow that $O \cap E \subseteq \{x\} \cup (W \cap E)$, and so $O \cap E$ contains at most a countable set of points, contrary to the hypothesis that $x \in P$. Hence O contains a point of P different from x , and so $P \subseteq P'$. Thus P is perfect.

Remark: This result has now been proved to be true in any separable metric space, not just R^k .

Exercise 2.28 Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (*Corollary:* Every countable closed set in R^k has isolated points.) *Hint:* Use Exercise 27.

Solution. If E is closed, it contains all its limit points, and hence certainly all its condensation points. Thus $E = P \cup (E \setminus P)$, where P is perfect (the set of all condensation points of E), and $E \setminus P$ is at most countable.

Since a perfect set in a separable metric space has the same cardinality as the real numbers, the set P must be empty if E is countable. The at-most-countable set $E \setminus P$ cannot be perfect, hence must have isolated points if it is nonempty.

Exercise 2.29 Prove that every open set in R^1 is the union of an at most countable collection of disjoint segments. *Hint:* Use Exercise 22.

Solution. Let O be open. For each pair of points $x \in O$, $y \in O$, we define an equivalence relation $x \sim y$ by saying $x \sim y$ if and only if $[\min(x, y), \max(x, y)] \subset O$. This is an equivalence relation, since $x \sim x$ ($[x, x] \subset O$ if $x \in O$); if $x \sim y$,

then $y \sim x$ (since $\min(x, y) = \min(y, x)$ and $\max(x, y) = \max(y, x)$); and if $x \sim y$ and $y \sim z$, then $x \sim z$ ($[\min(x, z), \max(x, z)] \subseteq [\min(x, y), \max(x, y)] \cup [\min(y, z), \max(y, z)] \subseteq O$). In fact it is easy to prove that

$$\min(x, z) \geq \min(\min(x, y), \min(y, z))$$

and

$$\max(x, z) \leq \max(\max(x, y), \max(y, z)).$$

It follows that O can be written as a disjoint union of pairwise disjoint equivalence classes. We claim that each equivalence class is an open interval.

To show this, for each $x \in O$; let $A = \{z : [z, x] \subseteq O\}$ and $B = \{z : [x, z] \subseteq O\}$, and let $a = \inf A$, $b = \sup B$. We claim that $(a, b) \subset O$. Indeed if $a < z < b$, there exists $c \in A$ with $c < z$ and $d \in B$ with $d > z$. Then $z \in [c, x] \cup [x, d] \subseteq O$. We now claim that (a, b) is the equivalence class containing x . It is clear that each element of (a, b) is equivalent to x by the way in which a and b were chosen. We need to show that if $z \notin (a, b)$, then z is not equivalent to x . Suppose that $z < a$. If z were equivalent to x , then $[z, x]$ would be contained in O , and so we would have $z \in A$. Hence a would not be a lower bound for A . Similarly if $z > b$ and $z \sim x$, then b could not be an upper bound for B .

We have now established that O is a union of pairwise disjoint open intervals. Such a union must be at most countable, since each open interval contains a rational number not in any other interval.

Exercise 2.30 Imitate the proof of Theorem 2.43 to obtain the following result:

If $R^k = \bigcup_1^\infty F_n$, where each F_n is a closed subset of R^k , then at least one F_n has a nonempty interior.

Equivalent statement: If G_n is a dense open subset of R^k , for $n = 1, 2, 3, \dots$, then $\bigcap_1^\infty G_n$ is not empty (in fact, it is dense in R^k).

(This is a special case of Baire's theorem; see Exercise 22, Chap. 3, for the general case.)

Solution. The equivalence of the two statements is easily established. Suppose the first statement is true, and G_n is a dense open subset of R^k for $n = 1, 2, 3, \dots$. Let $F_n = R^k \setminus G_n$. Then F_n is a closed subset of R^k having empty interior (if the interior of F_n were non-empty, G_n would not be dense). Hence by the first statement, the union of the set F_n cannot be all of R^k , and hence the intersection of their complements is not empty.

Conversely, if the second statement holds and F_n are closed subsets of R^k whose union is all of R^k , let G_n be the complement of F_n . Since the intersection of the G_n 's is empty, at least one of them must fail to be dense in R^k , which means that its complement contains a non-empty open set.

We now prove the second statement, including the parenthetical remark. Let G_n be a sequence of dense open sets in R^k , and let O be any non-empty

open set in R^k . Since O is an open set and G_1 is dense, it must intersect G_1 in a non-empty open set O_1 . Let $x_1 \in O_1$, and choose $r_1 > 0$ such that the *closed* ball $\overline{B_{r_1}(x_1)}$ is contained in O_1 . Then the *open* ball $B_{r_1}(x_1)$ is non-empty, and hence must intersect G_2 in a non-empty open set O_2 . Let $x_2 \in O_2$, and choose $r_2 > 0$ such that the *closed* ball $\overline{B_{r_2}(x_2)}$ is contained in O_2 . In this way we obtain a nested sequence of nonempty compact sets (closed balls) $\overline{B_1} \supseteq \overline{B_2} \supseteq \cdots \supseteq \overline{B_n} \supseteq \cdots$. If $x \in \cap \overline{B_n}$, then $x \in O_n$ for each n , and hence $x \in O \cap G_n$ for each n . Thus $\cap G_n$ intersects each non-empty open set O in at least one point, which says precisely that $\cap G_n$ is dense in R^k . Notice that the whole proof works exactly the same way if R^k is replaced by O , since $G_n \cap O$ is dense in O .

Remark: The stronger form of the second statement that we have proved shows that the first statement can also be strengthened. If $\{F_n\}$ is a sequence of closed sets whose union is all of R^k and O is any non-empty open set, then the interior of $F_n \cap O$ is non-empty for at least one n . (Simply apply the original statement with R^k replaced by O and F_k by $F_k \cap O$.)

Chapter 3

Numerical Sequences and Series

Exercise 3.1 Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?

Solution. Let $\varepsilon > 0$. Since the sequence $\{s_n\}$ is a Cauchy sequence, there exists N such that $|s_m - s_n| < \varepsilon$ for all $m > N$ and $n > N$. We then have $||s_m| - |s_n|| \leq |s_m - s_n| < \varepsilon$ for all $m > N$ and $n > N$. Hence the sequence $\{|s_n|\}$ is also a Cauchy sequence, and therefore must converge.

The converse is not true, as shown by the sequence $\{s_n\}$ with $s_n = (-1)^n$.

Exercise 3.2 Calculate $\lim_{n \rightarrow \infty} (\sqrt{n^2 + n} - n)$.

Solution. Multiplying and dividing by $\sqrt{n^2 + n} + n$ yields

$$\sqrt{n^2 + n} - n = \frac{n}{\sqrt{n^2 + n} + n} = \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}.$$

It follows that the limit is $\frac{1}{2}$.

Exercise 3.3 If $s_1 = \sqrt{2}$ and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \quad (n = 1, 2, 3, \dots),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \dots$.

Solution. Since $\sqrt{2} < 2$, it is manifest that if $s_n < 2$, then $s_{n+1} < \sqrt{2+2} = 2$. Hence it follows by induction that $\sqrt{2} < s_n < 2$ for all n . In view of this fact,

it also follows that $(s_n - 2)(s_n + 1) < 0$ for all $n > 1$, i.e., $s_n > s_n^2 - 2 = s_{n-1}$. Hence the sequence is an increasing sequence that is bounded above (by 2) and so converges. Since the limit s satisfies $s^2 - s - 2 = 0$, it follows that the limit is 2.

Exercise 3.4 Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \quad s_{2m} = \frac{s_{2m-1}}{2}; \quad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Solution. We shall prove by induction that

$$s_{2m} = \frac{1}{2} - \frac{1}{2^m} \quad \text{and} \quad s_{2m+1} = 1 - \frac{1}{2^m}$$

for $m = 1, 2, \dots$. The second of these equalities is a direct consequence of the first, and so we need only prove the first. Immediate computation shows that $s_2 = 0$ and $s_3 = \frac{1}{2}$. Hence assume that both formulas hold for $m \leq r$. Then

$$s_{2r+2} = \frac{1}{2}s_{2r+1} = \frac{1}{2}\left(1 - \frac{1}{2^r}\right) = \frac{1}{2} - \frac{1}{2^{r+1}}.$$

This completes the induction. We thus have $\limsup_{n \rightarrow \infty} s_n = 1$ and $\liminf_{n \rightarrow \infty} s_n = \frac{1}{2}$.

Exercise 3.5 For any two real sequences $\{a_n\}$, $\{b_n\}$ prove that

$$\limsup_{n \rightarrow \infty} (a_n + b_n) \leq \limsup_{n \rightarrow \infty} a_n + \limsup_{n \rightarrow \infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

Solution. Since the case when $\limsup_{n \rightarrow \infty} a_n = +\infty$ and $\limsup_{n \rightarrow \infty} b_n = -\infty$ has been excluded from consideration, we note that the inequality is obvious if $\limsup_{n \rightarrow \infty} a_n = +\infty$. Hence we shall assume that $\{a_n\}$ is bounded above.

Let $\{n_k\}$ be a subsequence of the positive integers such that $\lim_{k \rightarrow \infty} (a_{n_k} + b_{n_k}) = \limsup_{n \rightarrow \infty} (a_n + b_n)$. Then choose a subsequence of the positive integers $\{k_m\}$ such that

$$\lim_{m \rightarrow \infty} a_{n_{k_m}} = \limsup_{k \rightarrow \infty} a_{n_k}.$$

The subsequence $a_{n_{k_m}} + b_{n_{k_m}}$ still converges to the same limit as $a_{n_k} + b_{n_k}$, i.e., to $\limsup_{n \rightarrow \infty} (a_n + b_n)$. Hence, since a_{n_k} is bounded above (so that $\limsup_{k \rightarrow \infty} a_{n_k}$ is finite), it follows that $b_{n_{k_m}}$ converges to the difference

$$\lim_{m \rightarrow \infty} b_{n_{k_m}} = \lim_{m \rightarrow \infty} (a_{n_{k_m}} + b_{n_{k_m}}) - \lim_{m \rightarrow \infty} a_{n_{k_m}}.$$

Thus we have proved that there exist subsequences $\{a_{n_{k_m}}\}$ and $\{b_{n_{k_m}}\}$ which converge to limits a and b respectively such that $a + b = \limsup_{n \rightarrow \infty} (a_n + b_n)$. Since a is the limit of a subsequence of $\{a_n\}$ and b is the limit of a subsequence of $\{b_n\}$, it follows that $a \leq \limsup_{n \rightarrow \infty} a_n$ and $b \leq \limsup_{n \rightarrow \infty} b_n$, from which the desired inequality follows.

Exercise 3.6 Investigate the behavior (convergence or divergence) of $\sum a_n$ if

- (a) $a_n = \sqrt{n+1} - \sqrt{n}$;
 (b) $a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n}$;
 (c) $a_n = (\sqrt[n]{n} - 1)^n$;
 (d) $a_n = \frac{1}{1+z^n}$ for complex values of z .

Solution. (a) Multiplying and dividing a_n by $\sqrt{n+1} + \sqrt{n}$, we find that $a_n = \frac{1}{\sqrt{n+1} + \sqrt{n}}$, which is larger than $\frac{1}{2\sqrt{n+1}}$. The series $\sum a_n$ therefore diverges by comparison with the p series ($p = \frac{1}{2}$).

Alternatively, since the sum telescopes, the n th partial sum is $\sqrt{n+1} - 1$, which obviously tends to infinity.

(b) Using the same trick as in part (a), we find that $a_n = \frac{1}{n(\sqrt{n+1} + \sqrt{n})}$, which is less than $\frac{1}{n^{3/2}}$. Hence the series converges by comparison with the p series ($p = \frac{3}{2}$).

(c) Using the root test, we find that $a_n^{\frac{1}{n}} = \sqrt[n]{n} - 1$, which tends to zero as $n \rightarrow \infty$. Hence the series converges. (Alternatively, since by part (c) of Theorem 3.20 $\sqrt[n]{n}$ tends to 1 as $n \rightarrow \infty$, we have $a_n \leq 2^{-n}$ for all large n , and the series converges by comparison with a geometric series.)

(d) If $|z| \leq 1$, then $|a_n| \geq \frac{1}{2}$, so that a_n does not tend to zero. Hence the series diverges. If $|z| > 1$, the series converges by comparison with a geometric series with $r = \frac{1}{|z|} < 1$.

Exercise 3.7 Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

Solution. Since $(\sqrt{a_n} - \frac{1}{n})^2 \geq 0$, it follows that

$$\frac{\sqrt{a_n}}{n} \leq \frac{1}{2} \left(a_n^2 + \frac{1}{n^2} \right).$$

Now Σa_n^2 converges by comparison with Σa_n (since Σa_n converges, we have $a_n < 1$ for large n , and hence $a_n^2 < a_n$). Since $\Sigma \frac{1}{n^2}$ also converges (p series, $p = 2$), it follows that $\Sigma \frac{\sqrt{a_n}}{n}$ converges.

Exercise 3.8 If Σa_n converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\Sigma a_n b_n$ converges.

Solution. We shall show that the partial sums of this series form a Cauchy sequence, i.e., given $\varepsilon > 0$ there exists N such that $\left| \sum_{k=m+1}^n a_k b_k \right| < \varepsilon$ if $n > m \geq N$. To do this, let $S_n = \sum_{k=1}^n a_k$ ($S_0 = 0$), so that $a_k = S_k - S_{k-1}$ for $k = 1, 2, \dots$. Let M be an upper bound for both $|b_n|$ and $|S_n|$, and let $S = \sum a_n$ and $b = \lim b_n$. Choose N so large that the following three inequalities hold for all $m > N$ and $n > N$:

$$|b_n S_n - bS| < \frac{\varepsilon}{3}; \quad |b_m S_m - bS| < \frac{\varepsilon}{3}; \quad |b_m - b_n| < \frac{\varepsilon}{3M}.$$

Then if $n > m > N$, we have, from the formula for summation by parts,

$$\sum_{k=m+1}^n a_k b_k = b_n S_n - b_m S_m + \sum_{k=m}^{n-1} (b_k - b_{k+1}) S_k.$$

Our assumptions yield immediately that $|b_n S_n - b_m S_m| < \frac{2\varepsilon}{3}$, and

$$\left| \sum_{k=m}^{n-1} (b_k - b_{k+1}) S_k \right| \leq M \sum_{k=m}^{n-1} |b_k - b_{k+1}|.$$

Since the sequence $\{b_n\}$ is monotonic, we have

$$\sum_{k=m}^{n-1} |b_k - b_{k+1}| = \left| \sum_{k=m}^{n-1} (b_k - b_{k+1}) \right| = |b_m - b_n| < \frac{\varepsilon}{3M},$$

from which the desired inequality follows.

Exercise 3.9 Find the radius of convergence of each of the following power series

$$(a) \sum n^3 z^n, \quad (b) \sum \frac{2^n}{n!} z^n,$$

$$(c) \sum \frac{2^n}{n^2} z^n, \quad (d) \sum \frac{n^3}{3^n} z^n.$$

Solution. (a) The radius of convergence is 1, since $a_n = n^3$ satisfies $\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = 1$.

(b) The radius of convergence is infinite, since $a_n = \frac{2^n}{n!}$ satisfies $\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = \lim_{n \rightarrow \infty} \frac{n+1}{2} = \infty$.

(c) The radius of convergence is $\frac{1}{2}$, since $a_n = \frac{2^n}{n^2}$ satisfies

$$\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = \lim_{n \rightarrow \infty} \frac{1}{2} \left(1 + \frac{1}{n}\right)^2 = \frac{1}{2}.$$

(d) The radius of convergence is 3, since $a_n = \frac{n^3}{3^n}$ satisfies

$$\lim_{n \rightarrow \infty} \frac{a_n}{a_{n+1}} = \lim_{n \rightarrow \infty} 3 \left(\frac{n}{n+1}\right)^3 = 3.$$

Exercise 3.10 Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

Solution. The series diverges if $|z| > 1$, since its general term does not tend to zero. (Infinitely many terms are larger than 1 in absolute value.)

Exercise 3.11 Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

(d) What can be said about

$$\sum \frac{a_n}{1 + na_n} \quad \text{and} \quad \sum \frac{a_n}{1_n^2 a_n}?$$

Solution. (a) If a_n does not remain bounded, then $\frac{a_n}{1+a_n}$ does not tend to zero, and hence the series $\sum \frac{a_n}{1+a_n}$ diverges. If $a_n \leq M$ for all n , then $\frac{a_n}{1+a_n} \geq \frac{1}{1+M} a_n$, and hence again the series is divergent.

(b) Replacing each denominator on the left by s_{N+k} , we have

$$\begin{aligned} \frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}} &\geq \frac{1}{s_{N+k}}(a_{N+1} + a_{N+2} + \cdots + a_{N+k}) = \\ &= \frac{1}{s_{N+k}}(s_{N+k} - s_N) = 1 - \frac{s_N}{s_{N+k}}. \end{aligned}$$

It follows that the partial sums of the series $\sum \frac{a_n}{s_n}$ do not form a Cauchy sequence. For, no matter how large N is taken, if N is held fixed, the right hand side can be made larger than $\frac{1}{2}$ by taking k sufficiently large (since $s_{N+k} \rightarrow \infty$).

(c) We observe that if $n \geq 2$, then

$$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_{n-1}s_n} = \frac{a_n}{s_{n-1}s_n} \geq \frac{a_n}{s_n^2}.$$

Since the series $\sum_{n=2}^{\infty} \frac{1}{s_{n-1}} - \frac{1}{s_n}$ converges to $\frac{1}{a_1}$, it follows by comparison that $\sum \frac{a_n}{s_n^2}$ converges.

(d) The series $\sum \frac{a_n}{1+na_n}$ may be either convergent or divergent. If the sequence $\{na_n\}$ is bounded above or has a positive lower bound, it definitely diverges. Thus if $na_n \leq M$, each term is at least $\frac{1}{1+M}a_n$, and so the series diverges. If $na_n \geq \varepsilon > 0$ for all n , then each term is at least $\frac{\varepsilon}{1+\varepsilon} \frac{1}{n}$, and once again the series is divergent.

In general, however, the series $\sum \frac{a_n}{1+na_n}$ may converge. For example let $a_n = \frac{1}{n^2}$ if n is not a perfect square and $a_n = \frac{1}{\sqrt{n}}$ if n is a perfect square. The sum of $\frac{a_n}{1+na_n}$ over the nonsquares obviously converges by comparison with the p series, $p = 2$. As for the sum over the square integers it is $\sum \frac{1}{n+n^2}$, which converges by comparison with the p series, $p = 2$.

Finally, the series $\sum \frac{a_n}{1+n^2a_n}$ is obviously majorized by the p series with $p = 2$, hence converges.

Exercise 3.12 Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) Prove that

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Solution. (a) Replacing all the denominators on the left-hand side by the largest one (r_m), we find

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{a_m + \cdots + a_n}{r_m} = \frac{r_m - r_{n+1}}{r_m} > 1 - \frac{r_n}{r_m},$$

since $r_n > r_{n+1}$.

As in the previous problem, this keeps the partial sums of the series $\sum \frac{a_n}{r_n}$ from forming a Cauchy sequence. No matter how large m is taken, one can choose n larger so that the difference $\sum_{k=m}^n \frac{a_k}{r_k}$ is at least $\frac{1}{2}$, since $r_n \rightarrow 0$ as $n \rightarrow \infty$.

(b) We have

$$\frac{a_n}{\sqrt{r_n}}(\sqrt{r_n} + \sqrt{r_{n+1}}) = a_n + a_n \frac{\sqrt{r_{n+1}}}{\sqrt{r_n}} < 2a_n = 2(r_n - r_{n+1}).$$

Dividing both sides by $\sqrt{r_n} + \sqrt{r_{n+1}}$ now yields the desired inequality.

Since the series $\sum(\sqrt{r_n} - \sqrt{r_{n+1}})$ converges to $\sqrt{r_1}$, it follows by comparison that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Exercise 3.13 Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Solution. Since both the hypothesis and conclusion refer to absolute convergence, we may assume both series consist of nonnegative terms. We let $S_n = \sum_{k=0}^n a_k$, $T_n = \sum_{k=0}^n b_k$, and $U_n = \sum_{k=0}^n \sum_{l=0}^k a_l b_{k-l}$. We need to show that U_n remains bounded, given that S_n and T_n are bounded. To do this we make the convention that $a_{-1} = T_{-1} = 0$, in order to save ourselves from having to separate off the first and last terms when we sum by parts. We then have

$$\begin{aligned} U_n &= \sum_{k=0}^n \sum_{l=0}^k a_l b_{k-l} \\ &= \sum_{k=0}^n \sum_{l=0}^k a_l (T_{k-l} - T_{k-l-1}) \\ &= \sum_{k=0}^n \sum_{j=0}^k a_{k-j} (T_j - T_{j-1}) \\ &= \sum_{k=0}^n \sum_{j=0}^k (a_{k-j} - a_{k-j-1}) T_j \\ &= \sum_{j=0}^n \sum_{k=j}^n (a_{k-j} - a_{k-j-1}) T_j \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^n a_{n-j} T_j \\
&\leq T \sum_{m=0}^n a_m \\
&= TS_n \\
&\leq ST.
\end{aligned}$$

Thus U_n is bounded, and hence approaches a finite limit.

Exercise 3.14 If $\{s_n\}$ is a complex sequence, define its arithmetic mean σ_n by

$$\sigma_n = \frac{s_0 + s_1 + \cdots + s_n}{n+1} \quad (n = 0, 1, 2, \dots).$$

- (a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.
 (b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.
 (c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, even though $\lim \sigma_n = 0$?
 (d) Put $a_n = s_n - s_{n-1}$ for $n \geq 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^n k a_k.$$

Assume that $\lim(na_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges. [This gives a converse of (a), but under the additional assumption that $na_n \rightarrow 0$.]

- (e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|na_n| \leq M$ for all n , and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$ by completing the following outline:

If $m < n$, then

$$s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{i=m+1}^n (s_n - s_i).$$

For these i ,

$$|s_n - s_i| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}.$$

Fix $\varepsilon > 0$ and associate with each n the integer m that satisfies

$$m \leq \frac{n-\varepsilon}{1+\varepsilon} < m+1.$$

then $(m+1)/(n-m) \leq 1/\varepsilon$ and $|s_n - s_i| < M\varepsilon$. Hence

$$\limsup_{n \rightarrow \infty} |s_n - \sigma| \leq M\varepsilon.$$

Since ε was arbitrary, $\lim s_n = \sigma$.

Solution. Let $\varepsilon > 0$. Let $M = \sup\{|s_n|\}$, and let N_0 be the first integer such that $|s_n - s| < \frac{\varepsilon}{2}$ for all $n > N_0$. Let $N = \max\left(N_0, \left\lceil \frac{2(N_0 + 1)(M + |s|)}{\varepsilon} \right\rceil\right)$. Then if $n > N$, we have

$$\begin{aligned} |\sigma_n - s| &= \left| \frac{(s_0 - s) + (s_1 - s) + \cdots + (s_n - s)}{n + 1} \right| \\ &\leq \left| \frac{(s_0 - s) + \cdots + (s_{N_0} - s)}{n + 1} \right| + \\ &\quad + \left| \frac{(s_{N_0+1} - s) + \cdots + (s_n - s)}{n + 1} \right|. \end{aligned}$$

The first sum on the right-hand side here is at most $\frac{(N_0+1)(M+|s|)}{n+1}$, and since $n + 1 > \frac{2(N_0+1)(M+|s|)}{\varepsilon}$, this sum is at most $\frac{\varepsilon}{2}$. The second sum is at most $\frac{(n-N_0)\frac{\varepsilon}{2}}{n+1}$, which is at most $\frac{\varepsilon}{2}$. Thus $|\sigma_n - s| < \varepsilon$ if $n > N$, which was to be proved.

(b) Let $s_n = (-1)^n$. Here σ_n is 0 if n is odd and $\frac{1}{n+1}$ if n is even. Thus $\sigma_n \rightarrow 0$, though s_n has no limit.

(c) Let $s_n = \frac{1}{n}$ if n is not a perfect cube and $s_n = \sqrt[3]{n}$ if n is a perfect cube. Then if $k^3 \leq n < (k+1)^3$ we have

$$\begin{aligned} \sigma_n &\leq \frac{1}{n+1} \sum_{m=1}^n \frac{1}{m} + \frac{1}{n+1} \sum_{j=1}^k j \\ &= \frac{1}{n+1} \left(\sum_{m=1}^n \frac{1}{m} \right) + \frac{1}{n+1} \cdot \frac{k(k+1)}{2}. \end{aligned}$$

The first sum on the right tends to zero by part (a) applied to the sequence $s_0 = 0$, $s_n = \frac{1}{n}$ for $n \geq 1$. As for the last term, since $n \geq k^3$, it is less than $\frac{1}{2k} + \frac{1}{2k^2}$, which tends to zero as $k \rightarrow \infty$. Since $(k+1)^3 > n$, it follows that k tends to infinity as n tends to infinity, and hence we have $\sigma_n \rightarrow 0$, even though $s_{n^3} \rightarrow \infty$.

(d) If we set $a_0 = s_0$, we have $s_n = \sum_{k=0}^n a_k$. Then

$$\begin{aligned} s_n - \sigma_n &= s_n - \frac{s_0 + s_1 + \cdots + s_n}{n + 1} \\ &= (a_0 + a_1 + \cdots + a_{n-1} + a_n) - \\ &\quad \frac{(n+1)a_0 + na_1 + \cdots + 2a_{n-1} + a_n}{n + 1} \\ &= \frac{a_1 + 2a_2 + \cdots + (n-1)a_{n-1} + na_n}{n + 1}, \end{aligned}$$

which was to be proved. If $na_n \rightarrow 0$, then the expression on the right-hand side tends to zero by part (a) with s_n replaced by na_n . Hence $s_n - \sigma_n \rightarrow 0$.

(e) If $m < n$ we have

$$\begin{aligned}
 \sigma_n - \sigma_m &= \frac{s_0 + \cdots + s_n}{n+1} - \frac{s_0 + \cdots + s_m}{m+1} \\
 &= (s_0 + \cdots + s_n) \left(\frac{1}{n+1} - \frac{1}{m+1} \right) + \sum_{i=m+1}^n \frac{s_i}{m+1} \\
 &= \frac{m-n}{m+1} \sigma_n + \frac{1}{m+1} \sum_{i=m+1}^n s_i.
 \end{aligned}$$

If we multiply both sides of this equation by $\frac{m+1}{m-n}$, and then transpose the left-hand side to the right and the term σ_n to the left, we obtain

$$-\sigma_n = \frac{m+1}{n-m} (\sigma_n - \sigma_m) - \frac{1}{n-m} \sum_{i=m+1}^n s_i.$$

Adding $s_n = \frac{1}{n-m} \sum_{i=m+1}^n s_n$ to both sides then yields the result.

We then have

$$|s_n - s_i| = |a_{i+1} + \cdots + a_n| \leq M \left(\frac{1}{i+1} + \cdots + \frac{1}{n} \right) \leq \frac{(n-i)M}{i+1}.$$

Since the function $\frac{n-x}{x+1} = \frac{n+1}{x+1} - 1$ is decreasing, the maximal value of the right-hand side here is reached with $i = m+1$, so that $|s_n - s_i| \leq \frac{(n-m-1)M}{m+2}$, as asserted.

When we choose m to be the largest integer in $\frac{n-\varepsilon}{1+\varepsilon}$, we clearly have $m < n$. Since ε is fixed, we can assume $m > \varepsilon$. The inequality $\frac{n-\varepsilon}{1+\varepsilon} < m+1$ can easily be converted to $\frac{n-m-1}{m+2} < \varepsilon$, and the inequality $m \leq \frac{n-\varepsilon}{1+\varepsilon}$ likewise becomes $\frac{m+1}{n-m} \leq \frac{1}{\varepsilon}$. The first of these implies that $m \rightarrow \infty$ as $n \rightarrow \infty$, and we have

$$|s_n - \sigma_n| \leq \frac{1}{\varepsilon} |\sigma_n - \sigma_m| + M\varepsilon$$

for all n . This implies that the limit of any subsequence of $|s_n - \sigma_n|$ is at most $M\varepsilon$, and since ε is arbitrary, every convergent subsequence of $|s_n - \sigma_n|$ converges to zero. This, of course, implies that $s_n - \sigma_n$ tends to zero, so that if $\sigma_n \rightarrow s$, then $s_n \rightarrow s$.

Exercise 3.15 Definition 3.21 can be extended to the case in which the a_n lie in some fixed R^k . Absolute convergence is defined as convergence of $\sum |a_n|$. Show that Theorems 3.22, 3.23, 3.25(a), 3.33, 3.34, 3.42, 3.45, 3.47, and 3.55 are true in this more general setting. (Only slight modifications are required in any of the proofs.)

Solution. (Theorem 3.22). $\sum \mathbf{a}_n$ converges if and only if for every $\varepsilon > 0$ there is an integer N such that

$$\left| \sum_{k=n}^m \mathbf{a}_k \right| \leq \varepsilon$$

if $m \geq n \geq N$.

It is a trivial remark that, since $|a_j - b_j| \leq |\mathbf{a} - \mathbf{b}| \leq |a_1 - b_1| + \cdots + |a_k - b_k|$, the sequence $\{\mathbf{a}_n\}$ converges if and only if each sequence of components $\{a_{nj}\}$ converges, $j = 1, \dots, k$. Hence the sequence of vector-valued functions converges if and only if each sequence of its components is a Cauchy sequence, and by the same inequalities, this is equivalent to saying that the vector-valued sequence is a Cauchy sequence.

(Theorem 3.23) If $\sum \mathbf{a}_n$ converges, then $\lim_{n \rightarrow \infty} \mathbf{a}_n = \mathbf{0}$.

Using the remark made in the previous paragraph, if $\sum \mathbf{a}_n$ converges, then each sum of components $\sum a_{nj}$ converges. Hence for each j we have $a_{nj} \rightarrow 0$, which, again by the remark, means $\mathbf{a}_n \rightarrow \mathbf{0}$.

(Theorem 3.25 (a)) If $|\mathbf{a}_n| \leq c_n$ for $n \geq N_0$, where N_0 is some fixed integer, and if $\sum c_n$ converges, then $\sum \mathbf{a}_n$ converges.

Again, the hypothesis implies that $|a_{nj}| \leq c_n$ for $n \geq N_0$, so that $\sum a_{nj}$ converges for each $j = 1, 2, \dots, k$. Once again, by the remark, this means that $\sum \mathbf{a}_n$ converges.

(Theorem 3.33) Given $\sum \mathbf{a}_n$, put $\alpha = \limsup_{n \rightarrow \infty} \sqrt[n]{|\mathbf{a}_n|}$. Then

- (a) if $\alpha < 1$, $\sum \mathbf{a}_n$ converges;
- (b) if $\alpha > 1$, $\sum \mathbf{a}_n$ diverges;
- (c) if $\alpha = 1$, the test gives no information.

Part (a) follows from the remarks made above, since $\sqrt[n]{|a_{nj}|} \leq \sqrt[n]{|\mathbf{a}_n|}$. (If $\alpha < 1$, then each component series converges.)

As for part (b), if $\alpha > 1$, then $|\mathbf{a}_n| > 1$ for infinitely many n , and hence the series diverges.

(Theorem 3.34) The series $\sum \mathbf{a}_n$

- (a) converges if $\limsup_{n \rightarrow \infty} \frac{|\mathbf{a}_{n+1}|}{|\mathbf{a}_n|} < 1$,
- (b) diverges if $\frac{|\mathbf{a}_{n+1}|}{|\mathbf{a}_n|} \geq 1$ for $n \geq n_0$, where n_0 is some fixed integer.

(a) The inequality implies that for some constant A and some fixed $r < 1$ we have $|\mathbf{a}_n| < Ar^n$, so that $\sum |\mathbf{a}_n|$ converges. Therefore by 3.25 the series $\sum \mathbf{a}_n$ also converges.

(b) As in the numerical case, this inequality implies that \mathbf{a}_n does not tend to zero, so that the series must diverge.

(Theorem 3.42) Suppose

- (a) the partial sums \mathbf{A}_n of $\sum \mathbf{a}_n$ form a bounded sequence;
 (b) $b_0 \geq b_1 \geq b_2 \geq \cdots$;
 (c) $\lim_{n \rightarrow \infty} b_n = 0$.

Then $\sum b_n \mathbf{a}_n$ converges.

We reduce this to Theorem 3.22 by showing that the partial sums of the series $\sum b_n \mathbf{a}_n$ form a Cauchy sequence. In fact

$$\begin{aligned} \left| \sum_{n=p}^q b_n \mathbf{a}_n \right| &= \left| \sum_{n=p}^{q-1} (b_n - b_{n+1}) \mathbf{A}_n + b_q \mathbf{A}_q - b_p \mathbf{A}_{p-1} \right| \\ &\leq M \left(\sum_{n=p}^{q-1} |b_n - b_{n+1}| + b_q + b_p \right) \\ &\leq 2Mb_p. \end{aligned}$$

Now, given $\varepsilon > 0$ choose N so large that $b_p < \frac{\varepsilon}{2M}$ for all $p > N$. Then if $q \geq p > N$, we have

$$\left| \sum_{n=p}^q b_n \mathbf{a}_n \right| \leq 2Mb_p < \varepsilon.$$

This proves that the partial sums form a Cauchy sequence, as required.

(Theorem 3.45) If $\sum \mathbf{a}_n$ converges absolutely, then $\sum \mathbf{a}_n$ converges.

Again this is a consequence of 3.25, with $c_n = |\mathbf{a}_n|$.

(Theorem 3.47) If $\sum \mathbf{a}_n = \mathbf{A}$ and $\sum \mathbf{b}_n = \mathbf{B}$, then $\sum (\mathbf{a}_n + \mathbf{b}_n) = \mathbf{A} + \mathbf{B}$ and $\sum c\mathbf{a}_n = c\mathbf{A}$ for any fixed c .

This theorem holds for each component of the vectors involved, hence it holds for the vectors themselves.

(Theorem 3.55) If $\sum \mathbf{a}_n$ is a series of vectors which converges absolutely, then every rearrangement of $\sum \mathbf{a}_n$ converges, and they all converge to the same sum.

Let \mathbf{A} be the sum of the series in its original arrangement, and let $\varepsilon > 0$. Choose N so large that $\sum_{k=m}^n |\mathbf{a}_k| < \frac{\varepsilon}{2}$ if $n \geq m > N$. Then of course $\left| \sum_{k=1}^n \mathbf{a}_k - \mathbf{A} \right| \leq \frac{\varepsilon}{2}$ if $n > N$. For any arrangement of the series $\sum \mathbf{a}_{n_k}$, Choose N_1 so large that $\{1, 2, \dots, N\} \subseteq \{n_1, n_2, \dots, n_{N_1}\}$. Then if $m > N_1$ and N_2 is such that $\{n_1, \dots, n_m\} \subseteq \{1, \dots, N_2\}$ have,

$$\begin{aligned} \left| \sum_{k=1}^m \mathbf{a}_{n_k} - \mathbf{A} \right| &\leq \left| \sum_{k=1}^m \mathbf{a}_{n_k} - \sum_{k=1}^m \mathbf{a}_k \right| + \left| \sum_{k=1}^m \mathbf{a}_k - \mathbf{A} \right| \\ &\leq \sum_{k=N+1}^m |\mathbf{a}_k| + \frac{\varepsilon}{2} \\ &< \varepsilon \end{aligned}$$

Exercise 3.16 Fix a positive number α . Choose $x_1 > \sqrt{\alpha}$, and define x_1, x_2, x_3, \dots , by the recursion formula

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{\alpha}{x_n} \right).$$

(a) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.

(b) Put $\varepsilon = x_n - \sqrt{\alpha}$, and show that

$$\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2x_n} < \frac{\varepsilon_n^2}{2\sqrt{\alpha}}$$

so that, setting $\beta = 2\sqrt{\alpha}$,

$$\varepsilon_{n+1} < \beta \left(\frac{\varepsilon_1}{\beta} \right)^{2^n} \quad (n = 1, 2, 3, \dots).$$

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\varepsilon_1/\beta < \frac{1}{10}$, and that therefore

$$\varepsilon_5 < 4 \cdot 10^{-16}, \quad \varepsilon_6 < 4 \cdot 10^{-32}.$$

Solution. (a) We note that x_n is always positive, and that if $x_n > \sqrt{\alpha}$, then $x_{n+1}^2 - \alpha = \frac{1}{4} \left(x_n - \frac{\alpha}{x_n} \right)^2 > 0$. Thus $x_n > \sqrt{\alpha}$ for all n . Since $x_n > \sqrt{\alpha}$, it follows that $\frac{\alpha}{x_n} < \sqrt{\alpha} < x_n$. Hence $x_n - x_{n+1} = \frac{1}{2} \left(x_n - \frac{\alpha}{x_n} \right) > 0$, and so $\{x_n\}$ decreases to a limit $\lambda \geq \sqrt{\alpha}$, which must satisfy $\lambda = \frac{\alpha}{\lambda}$, i.e., $\lambda = \sqrt{\alpha}$.

(b) We have $\frac{\varepsilon_n^2}{2x_n} = \frac{x_n^2 - 2x_n\sqrt{\alpha} + \alpha}{2x_n} = \frac{1}{2} \left(x_n + \frac{\alpha}{x_n} \right) - \sqrt{\alpha} = x_{n+1} - \sqrt{\alpha} = \varepsilon_{n+1}$. The inequality then results from the simple fact that $x_n > \sqrt{\alpha}$. Thus $\varepsilon_2 < \frac{\varepsilon_1^2}{\beta} = \beta \left(\frac{\varepsilon_1}{\beta} \right)^2$. By induction, if we suppose that $\varepsilon_n < \beta \left(\frac{\varepsilon_1}{\beta} \right)^{2^{n-1}}$, we find $\varepsilon_{n+1} < \frac{\varepsilon_n^2}{\beta} < \beta \left(\frac{\varepsilon_1}{\beta} \right)^{2^n}$.

(d) Taking $x_1 = 2$, $\alpha = 3$, we certainly have $\beta < 4$. And, since $\sqrt{3} > \frac{5}{3}$, we deduce that $12\sqrt{3} > 20$, so that $2\sqrt{3} > 10(2 - \sqrt{3})$, i.e., $\varepsilon_1 = 2 - \sqrt{3}$ and $\beta = 2\sqrt{3}$ satisfy $\varepsilon_1/\beta < \frac{1}{10}$, as asserted. It follows that $\varepsilon_n < 4 \cdot 10^{-2^{n-1}}$. In particular $\varepsilon_5 < 4 \cdot 10^{-16}$ and $\varepsilon_6 < 4 \cdot 10^{-32}$.

Exercise 3.17 Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}.$$

(a) Prove that $x_1 > x_3 > x_5 > \dots$.

(b) Prove that $x_2 < x_4 < x_6 < \cdots$.

(c) Prove that $\lim x_n = \sqrt{\alpha}$.

(d) Compare the rapidity of convergence of this process with the one described in Exercise 16.

Solution. Most of the work in this problem is done by the following three identities, whose proofs are routine computations:

$$(1 + x_n)(1 + x_{n+1}) = 2(1 + x_n) + (\alpha - 1),$$

$$x_{n+1}^2 - \alpha = -\left[\frac{(\alpha - 1)}{(1 + x_n)^2}\right](x_n^2 - \alpha),$$

$$\begin{aligned} x_{n+1}^2 - \alpha &= \frac{(\alpha - 1)^2}{(1 + x_n)^2(1 + x_{n-1})^2}(x_{n-1}^2 - \alpha) = \\ &= \left[\frac{\alpha - 1}{(\alpha - 1) + 2(1 + x_{n-1})}\right]^2(x_{n-1}^2 - \alpha). \end{aligned}$$

The second of these identities shows that x_n and x_{n+1} lie on opposite sides of $\sqrt{\alpha}$. The third shows that x_{n+1} is closer to $\sqrt{\alpha}$ than x_{n-1} . Hence, since $x_1 > \sqrt{\alpha}$ by hypothesis, parts (a) and (b) are proved. As for (c), the third relation shows that $|x_{n+1}^2 - \alpha| \leq r^2|x_{n-1}^2 - \alpha|$, where $r = \frac{\alpha-1}{2+\alpha-1} < 1$. It follows that $|x_{n+2k}^2 - \alpha| \leq r^{2k}|x_n^2 - \alpha|$, and the right-hand side of this expression tends to zero as $k \rightarrow \infty$. Thus $\lim_{k \rightarrow \infty} x_{n+2k} = \sqrt{\alpha}$ whether n is odd or even, and so

$$\lim_{n \rightarrow \infty} x_n = \sqrt{\alpha}.$$

The convergence in this case is geometric, but not quadratically geometric, as in Exercise 16. The rate of convergence will depend on the size of α . For $1 < \alpha \leq 2$ we certainly have $x_n \geq \alpha - 2$ for all n , and so in this case $r < \frac{1}{3}$, i.e., $|x_{n+1}^2 - \alpha| < \frac{1}{9}|x_{n-1}^2 - \alpha|$. This implies that $|x_{n+1} - \sqrt{\alpha}| < \frac{1}{9} \frac{x_{n-1} + \sqrt{\alpha}}{x_{n+1} + \sqrt{\alpha}} |x_{n-1} - \sqrt{\alpha}|$. If n is odd, we have $x_{n-1} < x_{n+1}$, and so $|x_{n+1} - \sqrt{\alpha}| < \frac{1}{9}|x_{n-1} - \sqrt{\alpha}|$. If n is even, we can at least assume $x_1 < 1.5$ (since $\alpha \leq 2$), and so $\frac{x_{n-1} + \sqrt{\alpha}}{x_{n+1} + \sqrt{\alpha}} < 1.5$, so that $|x_{n+1} - \sqrt{\alpha}| < \frac{1.5}{9}|x_{n-1} - \sqrt{\alpha}|$.

Exercise 3.18 Replace the recursion formula of Exercise 16 by

$$x_{n+1} = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1},$$

where p is a fixed positive integer, and describe the behavior of the resulting sequences $\{x_n\}$.

Solution. (Exercise 16 is the case $p = 2$, of course.) The main work is done by the following easily derived formulas, which hold if $x_n > \alpha^{\frac{1}{p}}$.

$$x_{n+1} - \alpha^{\frac{1}{p}} = (x_n - \alpha^{\frac{1}{p}}) \left[\left(\frac{p-1}{p} \right) - \frac{1}{p} \left(\left(\frac{\alpha^{\frac{1}{p}}}{x_n} \right) + \cdots + \left(\frac{\alpha^{\frac{1}{p}}}{x_n} \right)^{p-1} \right) \right]$$

$$\begin{aligned}
&< (x_n - \alpha^{\frac{1}{p}}) \left(\frac{p-1}{p} \right) \left(1 - \left(\frac{\alpha^{\frac{1}{p}}}{x_n} \right)^{p-1} \right) \\
&= (x_n - \alpha^{\frac{1}{p}}) \left(\frac{p-1}{px_n^{p-1}} \right) (x_n^{p-1} - (\alpha^{\frac{1}{p}})^{p-1}) \\
&= (x_n - \alpha^{\frac{1}{p}})^2 \cdot \frac{p-1}{px_n^{p-1}} \cdot [x_n^{p-2} + x_n^{p-3} \alpha^{\frac{1}{p}} + \cdots + \alpha^{\frac{p-2}{p}}] \\
&< (x_n - \alpha^{\frac{1}{p}})^2 \cdot \frac{(p-1)^2}{px_n} \\
&< (x_n - \alpha^{\frac{1}{p}})^2 \cdot \frac{(p-1)^2}{p\alpha^{\frac{1}{p}}}.
\end{aligned}$$

Thus we can guarantee quadratic-geometric convergence if we start with $x_1 - \alpha^{\frac{1}{p}} = \varepsilon_1 < \beta = \frac{p\alpha^{\frac{1}{p}}}{(p-1)^2}$. In that case we obtain the same inequalities as in Exercise 16, and $x_n \rightarrow \alpha^{\frac{1}{p}}$.

Exercise 3.19 Associate to each sequence $a = \{\alpha_n\}$, in which α_n is 0 or 2, the real number

$$x(a) = \sum_{n=1}^{\infty} \frac{\alpha_n}{3^n}.$$

Prove that the set of all $x(a)$ is precisely the Cantor set described in Sec. 2.44.

Solution. We note that the open middle third removed at the first stage of the construction is precisely the set of points whose ternary expansions *must* have a 1 as their first digit. (The numbers $\frac{1}{3}$ and $\frac{2}{3}$ *can* be written with a 1 in this place, since

$$\begin{aligned}
\frac{1}{3} &= \frac{1}{3} + \frac{0}{9} + \cdots + \frac{0}{3^n} + \cdots \\
\frac{2}{3} &= \frac{1}{3} + \frac{2}{9} + \cdots + \frac{2}{3^n} + \cdots.
\end{aligned}$$

However, these numbers can also be written as

$$\begin{aligned}
\frac{1}{3} &= \frac{0}{3} + \frac{2}{9} + \cdots + \frac{2}{3^n} + \cdots \\
\frac{2}{3} &= \frac{2}{3} + \frac{0}{9} + \cdots + \frac{0}{3^n} + \cdots.
\end{aligned}$$

Thus the points retained in the Cantor set after the first dissection are precisely those whose ternary expansions *may* be written without a 1 in the first digit. The same argument shows that the points retained in the Cantor set after the n th dissection are precisely those whose ternary expansions *may* be written without using a 1 in any of the first n digits. It then follows that the Cantor set is the set of points in $[0, 1]$ whose ternary expansions can be written without using any 1's, i.e., it is precisely the set of numbers $x(a)$ just described.

Exercise 3.20 Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X , and some subsequence $\{p_{n_k}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p .

Solution. Let $\varepsilon > 0$. Choose N_1 so large that $d(p_m, p_n) < \frac{\varepsilon}{2}$ if $m > N_1$ and $n > N_1$. Then choose $N \geq N_1$ so large that $d(p_{n_k}, p) < \frac{\varepsilon}{2}$ if $k > N$. Then if $n > N$, we have

$$d(p_n, p) \leq d(p_n, p_{n_{N+1}}) + d(p_{n_{N+1}}, p) < \varepsilon.$$

For the first term on the right is less than $\frac{\varepsilon}{2}$ since $n > N_1$ and $n_{N+1} > N + 1 > N_1$. The second term is less than $\frac{\varepsilon}{2}$ by the choice of N .

Exercise 3.21 Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed and bounded sets in a *complete* metric space X , if $E_n \supset E_{n+1}$, and if

$$\lim_{n \rightarrow \infty} \text{diam } E_n = 0,$$

then $\bigcap_1^\infty E_n$ consists of exactly one point.

Solution. Choose $x_n \in E_n$. (We use the axiom of choice here.) The sequence $\{x_n\}$ is a Cauchy sequence, since the diameter of E_n tends to zero as n tends to infinity and E_n contains E_{n+1} . Since the metric space X is complete, the sequence x_n converges to a point x , which must belong to E_n for all n , since E_n is closed and contains x_m for all $m \geq n$. There cannot be a second point y in all of the E_n , since for any point $y \neq x$ the diameter of E_n is less than $d(x, y)$ for large n .

Exercise 3.22 Suppose X is a complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X . Prove Baire's theorem, namely that $\bigcap_1^\infty G_n$ is not empty. (In fact, it is dense in X .) *Hint:* Find a shrinking sequence of neighborhoods E_n such that $\overline{E_n} \subset G_n$, and apply Exercise 21.

Solution. Let F_n be the complement of G_n , so that F_n is closed and contains no open sets. We shall prove that any nonempty open set U contains a point not in any F_n , hence in all G_n . To this end, we note that U is not contained in F_1 , so that there is a point $x_1 \in U \setminus F_1$. Since $U \setminus F_1$ is open, there exists $r_1 > 0$ such that B_1 , defined as the open ball of radius r_1 about x_1 , is contained in $U \setminus F_1$. Let E_1 be the open ball of radius $\frac{r_1}{2}$ about x_1 , so that the closure of E_1 is contained in B_1 . Now F_2 does not contain E_1 , and so we can find a point $x_2 \in E_1 \setminus F_2$. Since $E_1 \setminus F_2$ is an open set, there exists a positive number r_2 such that B_2 , the open ball of radius R_2 about x_2 , is contained in $E_1 \setminus F_2$, which in turn is contained in $U \setminus (F_1 \cup F_2)$. We let E_2 be the open ball of radius $\frac{r_2}{2}$ about x_2 , so that $\overline{E_2} \subseteq B_2$. Proceeding in this way, we construct a sequence of open balls E_j such that $E_j \supseteq \overline{E_{j+1}}$, and the diameter of E_j tends to zero. By the previous exercise, there is a point x belonging to all the sets $\overline{E_j}$, hence to all the sets $U \setminus (F_1 \cup F_2 \cup \cdots \cup F_n)$. Thus the point x belongs to $U \cap \left(\bigcap_1^\infty G_n \right)$.

Exercise 3.23 Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X . Show that the sequence $\{d(p_n, q_n)\}$ converges. *Hint:* For any m, n ,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n);$$

it follows that

$$|d(p_n, q_n) - d(p_m, q_m)|$$

is small if m and n are large.

Solution. The inequality in the hint, which is an extension of the triangle inequality, shows that

$$d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_m, q_n);$$

and since the same inequality holds with m and n reversed, it follows that

$$|d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_m, q_n).$$

Now if $\varepsilon > 0$, choose N_1 and N_2 so that $d(p_n, p_m) < \frac{\varepsilon}{2}$ if $m > N_1, n > N_1$, and $d(q_n, q_m) < \frac{\varepsilon}{2}$ if $m > N_2, n > N_2$. Then let $N = \max(N_1, N_2)$. It follows immediately that $|d(p_n, q_n) - d(p_m, q_m)| < \varepsilon$ if $m > N$ and $n > N$. Since the real numbers are a complete metric space, it follows that $\{d(p_n, q_n)\}$ converges.

Exercise 3.24 Let X be a metric space.

(a) Call two Cauchy sequences $\{p_n\}, \{q_n\}$ in X *equivalent* if

$$\lim_{n \rightarrow \infty} d(p_n, q_n) = 0.$$

Prove that this is an equivalence relation.

(b) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*$ and $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define

$$\Delta(P, Q) = \lim_{n \rightarrow \infty} d(p_n, q_n);$$

by Exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^* .

(c) Prove that the resulting metric space X^* is complete.

(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p ; let P_p be the element of X^* which contains this sequence. Prove that

$$\Delta(P_p, P_q) = d(p, q)$$

for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry (i.e., a distance-preserving mapping) of X into X^* .

(e) Prove that $\varphi(X)$ is dense in X , and that $\varphi(X) = X^*$ if X is complete. By (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^* . We call X^* the *completion* of X .

Solution. (a) We need to show that: 1) $\{p_n\}$ is equivalent to itself; 2) if $\{p_n\}$ is equivalent to $\{q_n\}$, then $\{q_n\}$ is equivalent to $\{p_n\}$; and 3) if $\{p_n\}$ is equivalent to $\{q_n\}$ and $\{q_n\}$ is equivalent to $\{r_n\}$, then $\{p_n\}$ is equivalent to $\{r_n\}$. These follow from the properties of any metric. Thus 1) follows, since $d(p_n, p_n) = 0$ for all n ; 2) follows since $d(p_n, q_n) = d(q_n, p_n)$; and 3) follows from the triangle inequality, i.e., $d(p_n, r_n) \leq d(p_n, q_n) + d(q_n, r_n)$, so that if $d(p_n, q_n) \rightarrow 0$ and $d(q_n, r_n) \rightarrow 0$, then $d(p_n, r_n) \rightarrow 0$.

(b) Let $\{p_n\}$ be equivalent to $\{p'_n\}$ and $\{q_n\}$ equivalent to $\{q'_n\}$. Then, since we know in advance that all the limits exist, we have

$$\lim_{n \rightarrow \infty} d(p'_n, q'_n) \leq \lim_{n \rightarrow \infty} (d(p'_n, p_n) + d(p_n, q_n) + d(q_n, q'_n)) = \lim_{n \rightarrow \infty} d(p_n, q_n).$$

By symmetry, however, we must also have the opposite inequality, so that the two limits are actually equal.

Now X^* is a metric space; for $\Delta(P, Q) \geq 0$, by definition $\Delta(P, Q) = 0$ means $P = Q$, and symmetry and the triangle inequality on X^* follow from the same properties on X .

(c) Suppose $\{P_k\}$ is a Cauchy sequence in X^* . Choose Cauchy sequences $\{p_{kn}\}$ in X such that $\{p_{kn}\} \in P_k$, $k = 1, 2, \dots$. For each k , let N_k be the first positive integer such that $d(p_{kn}, p_{km}) < 2^{-k}$ if $m \geq N_k$ and $n \geq N_k$. Let $p_k = p_{kN_k}$. Observe that $d(p_k, p_{kn}) < 2^{-k}$ for any $n \geq N_k$, so that $\lim_{n \rightarrow \infty} d(p_k, p_{kn}) \leq 2^{-k}$. (This limit exists since the sequence all of whose terms equal p_k is a Cauchy sequence.) Also, for any k, l , and n we have

$$d(p_k, p_l) \leq d(p_k, p_{kn}) + d(p_{kn}, p_{ln}) + d(p_{ln}, p_l).$$

Hence, taking n sufficiently large and assuming $k < l$, we obtain

$$d(p_k, p_l) \leq 2^{-k} + \Delta(P_k, P_l) + 2^{-k} + 2^{-l} < 3 \cdot 2^{-k} + \Delta(P_k, P_l).$$

It follows that $\{p_k\}$ is a Cauchy sequence. Let P be the element of X^* containing $\{p_k\}$. We claim $P_k \rightarrow P$ in X^* . For

$$\begin{aligned} \Delta(P_k, P) &= \lim_{n \rightarrow \infty} d(p_{kn}, p_n) \\ &\leq \lim_{n \rightarrow \infty} (d(p_{kn}, p_k) + d(p_k, p_n)) \\ &\leq 2^{-k} + \limsup_{n \rightarrow \infty} \Delta(P_k, P_n) + 3 \cdot 2^{-k}. \end{aligned}$$

Thus if $\varepsilon > 0$, choose $N_1 = 2 + \left\lceil \frac{-\log \varepsilon}{\log 2} \right\rceil$, and N_2 such that $\Delta(P_k, P_l) < \frac{\varepsilon}{2}$ if $k > N_2$ and $l > N_2$. Let $N = \max(N_1, N_2)$. We claim that if $k > N$, then $d(P_k, P) < \varepsilon$. Indeed this follows, since we then have $2^{-k+2} < \frac{\varepsilon}{2}$ and $\limsup_{n \rightarrow \infty} \Delta(P_k, P_n) \leq \frac{\varepsilon}{2}$. We have thus finally proved that X^* is complete.

(d) The assertion $\Delta(P_p, P_q) = d(p, q)$ is the trivial assertion that if $p_n = p$ and $q_n = q$ for all n , then

$$\lim_{n \rightarrow \infty} d(p_n, q_n) = d(p, q).$$

(e) Let P be any element of X^* , and let $\varepsilon > 0$. We shall find $p \in X$ such that $\Delta(P, P_p) < \varepsilon$. To this end, let $\{p_n\} \in P$ and let N be such that $d(p_n, p_m) < \frac{\varepsilon}{2}$ if $n > N$ and $m > N$. Let $p = p_{N+1}$. Then $\Delta(P, P_p) = \lim d(p_n, p) \leq \frac{\varepsilon}{2}$, and we are done.

If X is already complete, then for each $P \in X^*$ and $\{p_n\} \in P$ there exists $p \in X$ such that $p_n \rightarrow p$. This p is obviously the same for any sequence equivalent to $\{p_n\}$, and it is clear that $P = P_p$. Hence $\varphi(X) = X^*$ when X is complete.

It should be remarked that X^* is unique, in the sense that if Y and Z are any two complete metric spaces, each containing a dense subset isometric to X , then Y is isometric to Z . Indeed let φ and ψ be isometries of X into Y and Z respectively, such that $\varphi(X)$ is dense in Y and $\psi(X)$ is dense in Z . We construct an isometry of Y onto Z as follows. For each $y \in Y$, there is a sequence $\{x_n\} \subset X$ such that $\varphi(x_n) \rightarrow y$. The sequence $\{x_n\}$ is a Cauchy sequence in X , and hence $\{\psi(x_n)\}$ is a Cauchy sequence in Z (since ψ preserves distance). Since Z is complete, there is an element z such that $\psi(x_n) \rightarrow z$. We define $\theta(y) = z$. We claim first of all that this definition is unambiguous. For if y is given and some other sequence $\{x'_n\}$ in X is such that $\{\varphi(x'_n)\}$ converges to y , then $d_Z(\psi(x_n), \psi(x'_n)) = d_X(x_n, x'_n) = d_Y(\varphi(x_n), \varphi(x'_n)) \rightarrow 0$, and hence $\psi(x'_n) \rightarrow z$ also. The mapping θ is an isometry, since if $y_1 = \lim \varphi(x_{1n})$ and $y_2 = \lim \varphi(x_{2n})$, then

$$\begin{aligned} d_Z(\theta(y_1), \theta(y_2)) &= \lim d_Z(\psi(x_{1n}), \psi(x_{2n})) \\ &= \lim d_X(x_{1n}, x_{2n}) \\ &= \lim d_Y(\varphi(x_{1n}), \varphi(x_{2n})) \\ &= d_Y(y_1, y_2). \end{aligned}$$

(Here we have used the fact that if $p_n \rightarrow p$ and $q_n \rightarrow q$, then $d(p_n, q_n) \rightarrow d(p, q)$, which in turn follows from the inequality

$$|d(p, q) - d(p_n, q_n)| \leq d(p, p_n) + d(q, q_n)$$

proved in Exercise 23 above.)

Finally $\theta(Y) = Z$, since one can easily define an inverse mapping $\eta : Z \rightarrow Y$ by merely reversing the steps used to define θ .

Exercise 3.25 Let X be the metric space whose points are the rational numbers, with the metric $d(x, y) = |x - y|$. What is the completion of this space? (Compare Exercise 24.)

Answer. By the remarks at the end of Exercise 24, the completion of a metric space X is any complete metric space containing a dense subset isometric to the space X . Since the real numbers have this property, the completion of the rational numbers is the real numbers. A Cauchy sequence of rational numbers converges to a unique real number, of course, and two sequences are equivalent if and only if they converge to the same real number. Hence we have also a more direct reason for claiming that the completion of the rational numbers is the real numbers.

Chapter 4

Continuity

Exercise 4.1 Suppose f is a real function defined on R^1 which satisfies

$$\lim_{h \rightarrow 0} [f(x+h) - f(x-h)] = 0$$

for every $x \in R^1$. Does this imply that f is continuous?

Solution. No. In fact even the stronger statement

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{h^n} = 0$$

for every $x \in R^1$, where n is an arbitrary positive number, does not imply that f is continuous, since this property is possessed by the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is an integer,} \\ 0 & \text{if } x \text{ is not an integer.} \end{cases}$$

(If x is an integer, then $f(x+h) - f(x-h) \equiv 0$ for all h ; while if x is not an integer, $f(x+h) - f(x-h) = 0$ for $|h| < \min(x - [x], 1 + [x] - x)$).

Exercise 4.2 If f is a continuous mapping of a metric space X into a metric space Y , prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. (\overline{E} denotes the closure of E .) Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

Solution. Let $x \in \overline{E}$. We need to show that $f(x) \in \overline{f(E)}$. To this end, let O be any neighborhood of $f(x)$. Since f is continuous, $f^{-1}(O)$ contains (is) a neighborhood of x . Since $x \in \overline{E}$, there is a point u of E in $f^{-1}(O)$. Hence $f(u) \in O \cap f(E)$. Since O was any neighborhood of $f(x)$, it follows that $f(x) \in \overline{f(E)}$.

Consider $f : R^1 \rightarrow R^1$ given by $f(x) = \frac{x}{1+x^2}$, and let $E = \overline{E} = [1, \infty)$, so that $f(E) = f(\overline{E}) = (0, \frac{1}{2}]$, yet $\overline{f(E)} = [0, \frac{1}{2}]$.

Exercise 4.3 Let f be a continuous real function on a metric space X . Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

Solution. $Z(f) = f^{-1}(\{0\})$, which is the inverse image of a closed set. Hence $Z(f)$ is closed.

Exercise 4.4 Let f and g be continuous mappings of a metric space X into a metric space Y , and let E be a dense subset of X . Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)

Solution. To prove that $f(E)$ is dense in $f(X)$, simply use Exercise 2 above: $f(X) = f(\overline{E}) \subseteq \overline{f(E)}$.

The function $\varphi : X \rightarrow R^1$ given by

$$\varphi(p) = d_Y(f(p), g(p))$$

is continuous, since

$$|d_Y(f(p), g(p)) - d_Y(f(q), g(q))| \leq d_Y(f(p), f(q)) + d_Y(g(p), g(q)).$$

(This inequality follows from the triangle inequality, since

$$d_Y(f(p), g(p)) \leq d_Y(f(p), f(q)) + d_Y(f(q), g(q)) + d_Y(g(q), g(p)),$$

and the same inequality holds with p and q interchanged. The absolute value $|d_Y(f(p), g(p)) - d_Y(f(q), g(q))|$ must be either $d_Y(f(p), g(p)) - d_Y(f(q), g(q))$ or $d_Y(f(q), g(q)) - d_Y(f(p), g(p))$, and the triangle inequality shows that both of these numbers are at most $d_Y(f(p), f(q)) + d_Y(g(p), g(q))$.)

By the previous problem, the zero set of φ is closed. But by definition

$$Z(\varphi) = \{p : f(p) = g(p)\}.$$

Hence the set of p for which $f(p) = g(p)$ is closed. Since by hypothesis it is dense, it must be X .

Exercise 4.5 If f is a real continuous function defined on a closed set $E \subset R^1$, prove that there exist continuous real functions g on R^1 such that $g(x) = f(x)$ for all $x \in E$. (Such functions g are called *continuous extensions* of f from E to R^1 .) Show that the result becomes false if the word “closed” is omitted. Extend the result to vector-valued functions. *Hint:* Let the graph of g be a straight line on each of the segments which constitute the complement of E (compare Exercise 29, Chap. 2). The result remains true if R^1 is replaced by any metric space, but the proof is not so simple.

Solution. Following the hint, let the complement of E consist of a countable collection of finite open intervals (a_k, b_k) together with possibly one or both of the semi-infinite intervals $(b, +\infty)$ and $(-\infty, a)$. The function $f(x)$ is already defined at a_k and b_k , as well as at a and b (if these last two points exist). Define $g(x)$ to be $f(b)$ for $x > b$ and $f(a)$ for $x < a$ if a and b exist. On the interval (a_k, b_k) let

$$g(x) = f(a_k) + \frac{x - a_k}{b_k - a_k} (f(b_k) - f(a_k)).$$

Of course we let $g(x) = f(x)$ for $x \in E$. It is now fairly clear that $g(x)$ is continuous. A rigorous proof proceeds as follows. Let $\varepsilon > 0$. To choose $\delta > 0$ such that $|x - u| < \delta$ implies $|g(x) - g(u)| < \varepsilon$, we consider three cases.

i. If $x > b$, let $\delta = x - b$. Then if $|x - u| < \delta$, it follows that $u > b$ also, so that $g(u) = f(b) = g(x)$, and $|g(u) - g(x)| = 0 < \varepsilon$. Similarly if $x < a$, let $\delta = a - x$.

ii. If $a_k < x < b_k$ and $f(a_k) = f(b_k)$, let $\delta = \min(x - a_k, b_k - x)$. Since $|x - u| < \delta$ implies $a_k < u < b_k$, so that $g(u) = f(a_k) = f(b_k) = g(x)$, we again have $|g(x) - g(u)| = 0 < \varepsilon$. If $a_k < x < b_k$ and $f(a_k) \neq f(b_k)$, let $\delta = \min\left(x - a_k, b_k - x, \frac{(b_k - a_k)\varepsilon}{|f(b_k) - f(a_k)|}\right)$. Then if $|x - u| < \delta$, we again have $a_k < u < b_k$ and so

$$|g(x) - g(u)| = \frac{|x - u|}{b_k - a_k} |f(b_k) - f(a_k)| < \varepsilon.$$

iii. If $x \in E$, let δ_1 be such that $|f(u) - f(x)| < \varepsilon$ if $u \in E$ and $|x - u| < \delta_1$.

(Subcase a). If there are points $x_1 \in E \cap (x - \delta_1, x)$ and $x_2 \in E \cap (x, x + \delta_1)$, let $\delta = \min(x - x_1, x_2 - x)$. If $|u - x| < \delta$ and $u \in E$, then $|f(u) - f(x)| < \varepsilon$ by definition of δ_1 . If $u \notin E$, then, since x_1, x , and x_2 are all in E , it follows that $u \in (a_k, b_k)$, where $a_k \in E$, $b_k \in E$, and $|a_k - x| < \delta$ and $|b_k - x| < \delta$, so that $|f(a_k) - f(x)| < \varepsilon$ and $|f(b_k) - f(x)| < \varepsilon$. If $f(a_k) = f(b_k)$, then $f(u) = f(a_k)$ also, and we have $|f(u) - f(x)| < \varepsilon$. If $f(a_k) \neq f(b_k)$, then

$$\begin{aligned} |f(u) - f(x)| &= \left| f(a_k) - f(x) + \frac{u - a_k}{b_k - a_k} (f(b_k) - f(a_k)) \right| \\ &= \left| \frac{b_k - u}{b_k - a_k} (f(a_k) - f(x)) + \frac{u - a_k}{b_k - a_k} (f(b_k) - f(x)) \right| \\ &< \frac{b_k - u}{b_k - a_k} \varepsilon + \frac{u - a_k}{b_k - a_k} \varepsilon \\ &= \varepsilon \end{aligned}$$

(Subcase b). Suppose x_2 does not exist, i.e., either $x = a_k$ or $x = b_k$ and $b_k > a_k + \delta_1$. Let us consider the second of these cases and show how to get $|f(u) - f(x)| < \varepsilon$ for $x < u < x + \delta$. If $f(a_k) = f(b_k)$, let $\delta = \delta_1$. If $u > x$ we have $a_k < u < b_k$ and $f(u) = f(a_k) = f(x)$. If $f(a_k) \neq f(b_k)$, let $\delta = \min\left(\delta_1, \frac{(b_k - a_k)\varepsilon}{|f(b_k) - f(a_k)|}\right)$. Then, just as in Subcase a, we have $|f(u) - f(x)| < \varepsilon$.

The case when $x = b_k$ for some k and $a_k < x - \delta_1$ is handled in exactly the same way.

If $x = b$, let $\delta = \delta_1$. If $u > x$ we have $f(x) - f(u)$; and if $u < x$ and $u \notin E$, we use the same argument as in Subcases a and b.

The case $x = a$ is handled similarly.

The extension of this result to vector-valued functions is immediate: Simply extend each component of the function. A vector-valued function is continuous if and only if each of its components is continuous.

Exercise 4.6 If f is defined on E , the *graph* of f is the set of points $(x, f(x))$ for $x \in E$. In particular, if E is the set of real numbers and f is real-valued, the graph of f is a subset of the plane.

Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.

Solution. Let Y be the co-domain of the function f . We invent a new metric space $E \times Y$ as the set of pairs of points (x, y) , $x \in E$, $y \in Y$, with the metric $\rho((x_1, y_1), (x_2, y_2)) = d_E(x_1, x_2) + d_Y(y_1, y_2)$. The function $\varphi(x) = (x, f(x))$ is then a mapping of E into $E \times Y$.

We claim that the mapping φ is continuous if f is continuous. Indeed, let $x \in X$ and $\varepsilon > 0$ be given. Choose $\eta > 0$ so that $d_Y(f(x), f(u)) < \frac{\varepsilon}{2}$ if $d_E(x, u) < \eta$. Then let $\delta = \min\left(\eta, \frac{\varepsilon}{2}\right)$. It is easy to see that $\rho(\varphi(x), \varphi(u)) < \varepsilon$ if $d_E(x, u) < \delta$. Conversely if φ is continuous, it is obvious from the inequality $\rho(\varphi(x), \varphi(u)) \geq d_Y(f(x), f(u))$ that f is continuous.

From these facts we deduce immediately that the graph of a continuous function f on a compact set E is compact, being the image of E under the continuous mapping φ . Conversely, if f is not continuous at some point x , there is a sequence of points x_n converging to x such that $f(x_n)$ does not converge to $f(x)$. If no subsequence of $f(x_n)$ converges, then the sequence $\{(x_n, f(x_n))\}_{n=1}^{\infty}$ has no convergent subsequence, and so the graph is not compact. If some subsequence of $f(x_n)$ converges, say $f(x_{n_k}) \rightarrow z$, but $z \neq f(x)$, then the graph of f fails to contain the limit point (x, z) , and hence is not closed. A fortiori it is not compact.

Exercise 4.7 If $E \subset X$ and if f is a function defined on X , the *restriction* of f to E is the function g whose domain of definition is E , such that $g(p) = f(p)$ for $p \in E$. Define f and g on R^2 by $f(0, 0) = g(0, 0) = 0$, $f(x, y) = xy^2/(x^2 + y^4)$, $g(x, y) = xy^2/(x^2 + y^6)$ if $(x, y) \neq (0, 0)$. Prove that f is bounded on R^2 , that g is unbounded in every neighborhood of $(0, 0)$, and that f is not continuous at $(0, 0)$; nevertheless, the restrictions of both f and g to every straight line in R^2 are continuous!

Solution. The fact that $|f(x, y)| \leq \frac{1}{2}$ is an easy consequence of the inequality $(x - y^2)^2 \geq 0$. The fact that $\lim_{y \rightarrow 0} g(y^3, y) = \lim_{y \rightarrow 0} \frac{y^5}{2y^6} = \lim_{y \rightarrow 0} \frac{1}{2y} = \infty$ shows that g is unbounded on every neighborhood of infinity. The fact that $\lim_{y \rightarrow 0} f(y^2, y) = \lim_{y \rightarrow 0} \frac{y^4}{2y^4} = \frac{1}{2}$ shows that f is not continuous at $(0, 0)$.

Since f and g are continuous except at $(0, 0)$, it is obvious that their restrictions to any line that does not pass through $(0, 0)$ are continuous. Now a line that *does* pass through $(0, 0)$ has an equation that is either $x = 0$ or $y = ax$ for some a . Both f and g are constantly 0 on the first of these, and on the second we have $f(x, ax) = a^2x^3/(x^2 + a^4x^4) = a^2x/(1 + a^4x^2)$, while $g(x, ax) = a^2x^3/(x^2 + a^6x^6) = a^2x/(1 + a^6x^4)$. Both of the latter are obviously continuous functions.

Exercise 4.8 Let f be a real uniformly continuous function on the bounded set E in R^1 . Prove that f is bounded on E .

Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

Let $a = \inf E$ and $b = \sup E$, and let $\delta > 0$ be such that $|f(x) - f(y)| < 1$ if $x, y \in E$ and $|x - y| < \delta$. Now choose a positive integer N larger than $(b - a)/\delta$, and consider the N intervals $I_k = \left[a + \frac{k-1}{b-a}, a + \frac{k}{b-a}\right]$, $k = 1, 2, \dots, N$. For each k such that $I_k \cap E \neq \emptyset$ let $x_k \in E \cap I_k$. Then let $M = 1 + \max\{|f(x_k)|\}$. If $x \in E$, we have $|x - x_k| < \delta$ for some k , and hence $|f(x)| < M$.

The function $f(x) = x$ is uniformly continuous on the entire line, but not bounded.

Exercise 4.9 Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets: To every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\text{diam } f(E) < \varepsilon$ for all $E \subset X$ with $\text{diam } E < \delta$.

Solution. Suppose f is uniformly continuous and $\varepsilon > 0$ is given. Choose any positive number α smaller than ε . Then there exists $\delta > 0$ such that $d_Y(f(x), f(u)) < \alpha$ if $d_X(x, u) < \delta$. Hence if E is any set of diameter less than δ and x and u are any two points in E we have $d_Y(f(x), f(u)) < \alpha$, so that $\text{diam } f(E) \leq \alpha < \varepsilon$.

Conversely if f satisfies the condition stated in the problem, it is obvious that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(x), f(u)) < \varepsilon$ whenever $d_X(x, u) < \delta$. (Choose $\delta > 0$ corresponding to ε in the condition of the problem and then let E be the two-point set $\{x, u\}$.)

Exercise 4.10 Complete the details of the following alternate proof of Theorem 4.19: If f is not uniformly continuous, then for some $\varepsilon > 0$ there are sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) \rightarrow 0$ but $d_Y(f(p_n), f(q_n)) > \varepsilon$. Use Theorem 2.37 to obtain a contradiction.

Solution. Theorem 4.19 asserts that a continuous function on a compact set is uniformly continuous. By Theorem 2.37 there are subsequences $\{p_{n_k}\}$ and $\{q_{n_k}\}$ that converge to points p and q respectively. Since $d_X(p_n, q_n) \rightarrow 0$, it follows that $p = q$. However, since f is continuous, it follows from Theorem 4.2 that $f(p_{n_k})$ and $f(q_{n_k})$ converge to $f(p)$, which, since $d_Y(f(p_{n_k}), f(q_{n_k})) \leq d_Y(f(p_{n_k}), f(p)) + d_Y(f(p), f(q_{n_k}))$, implies that $d_Y(f(p_{n_k}), f(q_{n_k})) \rightarrow 0$, contradicting the inequality $d_Y(f(p_{n_k}), f(q_{n_k})) > \varepsilon$.

Exercise 4.11 Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X . Use this result to give an alternative proof of the theorem stated in Exercise 13.

Solution. Suppose $\{x_n\}$ is a Cauchy sequence in X . Let $\varepsilon > 0$ be given. Let $\delta > 0$ be such that $d_Y(f(x), f(u)) < \varepsilon$ if $d_X(x, u) < \delta$. Then choose N so that $d_X(x_n, x_m) < \delta$ if $n, m > N$. Obviously $d_Y(f(x_n), f(x_m)) < \varepsilon$ if $m, n > N$, showing that $\{f(x_n)\}$ is a Cauchy sequence.

Now let f be a uniformly continuous function defined on a dense subset E of X , mapping E into a complete metric space Y (for example, Y could be the real numbers). To prove that f has a unique continuous extension to all of X , proceed as follows. For each $x \in X \setminus E$ let $\{x_n\}$ be a sequence of points in E converging to x . Define $f(x)$ to be the limit of the Cauchy sequence $\{f(x_n)\}$. This definition is unambiguous; for if $\{u_n\}$ also converges to x , then the sequence $\{y_n\}$ defined by

$$y_n = \begin{cases} x_{n/2} & \text{if } n \text{ is even,} \\ u_{(n+1)/2} & \text{if } n \text{ is odd,} \end{cases}$$

also converges to x . Hence $\{f(y_n)\}$ is a Cauchy sequence in Y , and so all subsequences of $\{f(y_n)\}$ converge to the same limit. In particular $\{f(x_n)\}$ and $\{f(u_n)\}$ both converge to the same value.

The extended function is also uniformly continuous. For if $\varepsilon > 0$, let $\delta > 0$ be such that $d_Y(f(x), f(u)) < \frac{\varepsilon}{3}$ if $x, u \in E$ and $d_X(x, u) < \delta$. Then if $x \in E$, $u \in X \setminus E$, and $d_X(x, u) < \delta$, choose $v \in E$ with $d_X(v, u) < \delta - d_X(x, u)$ and $d_Y(f(v), f(u)) < \frac{\varepsilon}{3}$ (this is possible because of the definition of $f(u)$). We then have $d_X(x, v) \leq d_X(x, u) + d_X(u, v) < \delta$, and so

$$d_Y(f(x), f(u)) \leq d_Y(f(x), f(v)) + d_Y(f(v), f(u)) < \frac{2\varepsilon}{3} < \varepsilon.$$

Similarly if $x \in X \setminus E$, $u \in X \setminus E$, and $d_X(x, u) < \delta$, choose $v, w \in E$ with $d_X(v, u) < \frac{1}{2}(\delta - d_X(x, u))$, $d_X(x, w) < \frac{1}{2}(\delta - d_X(x, u))$, $d_Y(f(v), f(u)) < \frac{\varepsilon}{3}$,

and $d_Y(f(w), f(x)) < \frac{\varepsilon}{3}$. We then have

$$d_X(v, w) \leq d_X(v, u) + d_X(u, x) + d_X(x, w) < \delta$$

and hence

$$d_Y(f(x), f(u)) \leq d_Y(f(x), f(w)) + d_Y(f(w), f(v)) + d_Y(f(v), f(u)) < \varepsilon.$$

The uniqueness of this extension follows from Exercise 4 above.

Exercise 4.12 A uniformly continuous function of a uniformly continuous function is uniformly continuous.

State this more precisely and prove it.

Solution. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be uniformly continuous. Then $g \circ f : X \rightarrow Z$ is uniformly continuous, where $g \circ f(x) = g(f(x))$ for all $x \in X$.

To prove this fact, let $\varepsilon > 0$ be given. Then, since g is uniformly continuous, there exists $\eta > 0$ such that $d_Z(g(u), g(v)) < \varepsilon$ if $d_Y(u, v) < \eta$. Since f is uniformly continuous, there exists $\delta > 0$ such that $d_Y(f(x), f(y)) < \eta$ if $d_X(x, y) < \delta$.

It is then obvious that $d_Z(g(f(x)), g(f(y))) < \varepsilon$ if $d_X(x, y) < \delta$, so that $g \circ f$ is uniformly continuous.

Exercise 4.13 Let E be a dense subset of a metric space X , and let f be a uniformly continuous *real* function defined on E . Prove that f has a continuous extension from E to X (see Exercise 5 for terminology). (Uniqueness follows from Exercise 4.) *Hint:* For each $p \in X$ and each positive integer n , let $V_n(p)$ be the set of all $q \in E$ with $d(p, q) < 1/n$. Use Exercise 9 to show that the intersection of the closures of the sets $f(V_1(p)), f(V_2(p)), \dots$, consists of a single point, say $g(p)$, of R^1 . Prove that the function g so defined is the desired extension of f .

Could the range space R^1 be replaced by R^n . By any compact metric space? By any complete metric space? By any metric space?

Solution. We shall carry out the proof in the context of any complete metric space, showing that the range space could be R^n or any compact metric space.

The diameter of the closure of $f(V_i(p))$ is the same as the diameter of $f(V_i(p))$ itself. Hence by Exercise 9 above these diameters tend to zero. Since they form a nested sequence of nonempty closed sets, their intersection must consist of a single point, which can be defined to be $g(p)$. If $p \in E$, the intersection of these sets is just $f(p)$ (since $f(p)$ is in all the sets, and only one point belongs to all of them), so that g coincides with f on E . It remains to show that g is continuous. This proof is identical to the proof given in Exercise 11 above, which depends only on the fact that for $u \in X \setminus E$ and $\varepsilon > 0$, $\delta > 0$ there is a point $v \in E$

with $d_X(v, u) < \delta$ and $d_Y(f(v), f(u)) < \varepsilon$. This condition clearly holds in the present context as well.

In general this theorem fails on an incomplete metric space. For example, take X to be the real numbers, Y and E the rational numbers, and let $f : E \rightarrow Y$ be given by $f(x) = x$. There is no possible extension of f to a mapping from X into Y . (There is a unique extension of f to a mapping from X into X , but its range is not contained in Y . If there were an extension of f to a mapping from X into Y , there would be two extensions of f to mappings from X into X , contradicting the uniqueness of the extension.)

Exercise 4.14 Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I . Prove that $f(x) = x$ for at least one $x \in I$.

Solution. If $f(0) = 0$ or $f(1) = 1$, we are done. If not, then $0 < f(0)$ and $f(1) < 1$. Hence the continuous function $g(x) = x - f(x)$ satisfies $g(0) < 0 < g(1)$. By the intermediate value theorem, there must be a point $x \in (0, 1)$ where $g(x) = 0$.

Exercise 4.15 Call a mapping from X into Y *open* if $f(V)$ is an open set in Y whenever V is an open set in X .

Prove that every continuous open mapping of R^1 into R^1 is monotonic.

Solution. Suppose f is continuous and not monotonic, say there exist points $a < b < c$ with $f(a) < f(b)$, and $f(c) < f(b)$. Then the maximum value of f on the closed interval $[a, c]$ is assumed at a point u in the open interval (a, c) . If there is also a point v in the open interval (a, c) where f assumes its minimum value on $[a, c]$, then $f(a, c) = [f(v), f(u)]$. If no such point v exists, then $f(a, c) = (d, f(u)]$, where $d = \min(f(a), f(c))$. In either case, the image of (a, c) is not open.

Exercise 4.16 Let $[x]$ denote the largest integer contained in x , that is $[x]$ is the integer such that $x - 1 < [x] \leq x$; and let $(x) = x - [x]$ denote the fractional part of x . What discontinuities do the functions $[x]$ and (x) have?

Solution. The two functions have the same discontinuities, since each can be written as the difference of the continuous function $f(x) = x$ and the other function. Now the function $[x]$ is constant on each open interval $(k, k + 1)$; hence its only possible discontinuities are the integers. These are of course real discontinuities, since if $\varepsilon = 1$, there is no $\delta > 0$ such that $|[x] - [k]| < \varepsilon$ whenever $|x - k| < \delta$. (For if any δ is given, let $\eta = \min(1, \delta)$. Then $[k] - [k - \frac{\eta}{2}] = 1$.)

Exercise 4.17 Let f be a real function defined on (a, b) . Prove that the set of points at which f has a simple discontinuity is at most countable. *Hint:* Let E be the set on which $f(x-) < f(x+)$. With each point x of E associate a triple (p, q, r) of rational numbers such that

- (a) $f(x-) < p < f(x+)$,
- (b) $a < q < t < x$ implies $f(t) < p$,
- (c) $x < t < r < b$ implies $f(t) > p$.

The set of such triples is countable. Show that each triple is associated with at most one point of E . Deal similarly with the other possible types of simple discontinuities.

Solution. The existence of three such rational numbers (p, q, r) for each simple discontinuity of this type follows from the assumption $f(x-) < f(x+)$, and the definition of $f(x-)$ and $f(x+)$. We need to show that a given triple (p, q, r) cannot be associated with any other discontinuity of this type. To that end, suppose $y > x$ and $f(y-) < f(y+)$. If we do not have $f(y-) < p < f(y+)$, then the triple chosen for y will differ from (p, q, r) in its first element. Hence suppose $f(y-) < p < f(y+)$. In this case we definitely cannot have $r > y$, since there are points $t \in (x, y)$ such that $f(t) < p$ (if there weren't, we would have $f(y-) \geq p$).

We have thus shown that the set of points $x \in (a, b)$ at which $f(x-) < f(x+)$ is at most countable. The proof that the set of points at which $f(x-) > f(x+)$ is at most countable is, of course, nearly identical.

Now consider the set of points x at which $\lim_{t \rightarrow x} f(t)$ exists, but is not equal to $f(x)$. For each point $x \in (a, b)$ such that $\lim_{t \rightarrow x} f(t) < f(x)$, we take a triple (p, q, r) of rational numbers such that

- (a) $\lim_{t \rightarrow x} f(t) < p < f(x)$,
- (b) $a < q < t < x$ or $x < t < r < b$ implies $f(t) < p$.

As before, if $y > x$ and $\lim_{t \rightarrow y} f(t) < f(y)$, the triple associated with y will be different from that associated with x . For even if $\lim_{t \rightarrow y} f(t) < p < f(y)$, we cannot have $r > y$, since $f(y) > p$ and $x < y$.

The proof that the set of points $x \in (a, b)$ at which $\lim_{t \rightarrow x} f(t) > f(x)$ is countable is nearly identical.

Hence, the number of points in $[a, b]$ at which f has a discontinuity of first kind is countable.

Exercise 4.18 Every rational x can be written in the form $x = m/n$, where $n > 0$ and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on R^1 by

$$f(x) = \begin{cases} 0 & (x \text{ irrational}), \\ \frac{1}{n} & \left(x = \frac{m}{n}\right). \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

Solution. We shall show that $\lim_{t \rightarrow x} f(t) = 0$ for every t . Both assertions follow immediately from this fact. To this end, let $\varepsilon > 0$ be given, and let x be any real number. Let N be the unique positive integer such that $N \leq 1/\varepsilon < N+1$, and for each positive integer $n = 1, 2, \dots, N$, let k_n be the unique integer such that

$$\frac{k_n}{n} \leq x < \frac{k_n + 1}{n}$$

Then for each such n let $\delta_n = \frac{1}{n}$ if $x = \frac{k_n}{n}$, otherwise let $\delta_n = \min\left(x - \frac{k_n}{n}, \frac{k_n + 1}{n} - x\right)$. Finally let $\delta = \min(\delta_1, \dots, \delta_N)$. We claim that $|f(t)| < \varepsilon$ if $0 < |x - t| < \delta$. This is obvious if t is irrational, while if t is rational and $t = \frac{m}{n}$, we necessarily have $n > N$ by the choice of the numbers δ_n for $n \leq N$. Hence if t is rational, then $f(t) \leq \frac{1}{N+1} < \varepsilon$. The proof is now complete.

Exercise 4.19 Suppose f is a real function with domain R^1 which has the intermediate-value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b .

Suppose also, for every rational r , that the set of all x with $f(x) = r$ is closed.

Prove that f is continuous.

Hint: If $x_n \rightarrow x_0$ but $f(x_n) > r > f(x_0)$ for some r and all n , then $f(t_n) = r$ for some t_n between x_0 and x_n ; thus $t_n \rightarrow x_0$. Find a contradiction. (N. M. Fine, *Amer. Math. Monthly*, vol. 73, 1966, p. 782.)

Solution. The contradiction is evidently that x_0 is a limit point of the set of t such that $f(t) = r$, yet, x_0 does not belong to this set. This contradicts the hypothesis that the set is closed.

Exercise 4.20 If E is a nonempty subset of a metric space X , define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

(a) Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.

(b) Prove that ρ_E is a uniformly continuous function on X by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

for all $x \in X$ and $y \in X$.

Hint: $\rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z)$, so that

$$\rho_E(x) \leq d(x, y) + \rho_E(y).$$

Solution. (a) For each positive integer n , let $z_n \in E$ be such that $\rho_E(x) \leq d(x, z_n) < \rho_E(x) + \frac{1}{n}$. It follows that $d(x, z_n) \rightarrow \rho_E(x)$. If $\rho_E(x) = 0$, this means $z_n \rightarrow x$, i.e., $x \in \overline{E}$. Conversely, if $x \in \overline{E}$, there exists a sequence $\{z_n\}_{n=1}^{\infty} \subseteq E$ such that $z_n \rightarrow x$, and this means $d(z_n, x) \rightarrow 0$, so that $\rho_E(x) = 0$.

(b) The last inequality given in the hint follows from the first by taking the infimum over z on the right-hand side. This inequality immediately implies that

$$\rho_E(x) - \rho_E(y) \leq d(x, y).$$

By interchanging x and y , we also obtain

$$\rho_E(y) - \rho_E(x) \leq d(y, x) = d(x, y).$$

Since $|\rho_E(x) - \rho_E(y)|$ must be either $\rho_E(x) - \rho_E(y)$ or $\rho_E(y) - \rho_E(x)$, it follows that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y).$$

Exercise 4.21 Suppose K and F are disjoint sets in a metric space X , K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K$, $q \in F$. *Hint:* ρ_F is a continuous positive function on K .

Show that the conclusion may fail for two disjoint closed sets if neither is compact.

Solution. Following the hint, we observe that $\rho_F(x)$ must attain its minimum value on K , i.e., there is some point $r \in K$ such that

$$\rho_F(r) = \min_{q \in K} \rho_F(q).$$

Since F is closed and $r \notin F$, it follows from Exercise 4.20 that $\rho_F(r) > 0$. Let δ be any positive number smaller than $\rho_F(r)$. Then for any $p \in F$, $q \in K$, we have

$$d(p, q) \geq \rho_F(q) \geq \rho_F(r) > \delta.$$

This proves the positive assertion.

As for closed sets in general, one could let $F = \{1, 2, 3, \dots\}$ and $K = \{1 + \frac{1}{2}, 2 + \frac{1}{3}, 3 + \frac{1}{4}, \dots\}$ in R^1 , or one could let $F = \{(x, y) : y = 0\}$ and $K = \{(x, y) : y = \frac{1}{1+x^2}\}$ in R^2 . In both cases there are sequences of points $p_n \in F$, $q_n \in K$ such that $d(p_n, q_n) \rightarrow 0$.

Exercise 4.22 Let A and B be disjoint nonempty closed sets in a metric space X , and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)} \quad (p \in X).$$

Show that f is a continuous function on X whose range lies in $[0, 1]$, that $f(p) = 0$ precisely on A and $f(p) = 1$ precisely on B . This establishes a converse of Exercise 3: Every closed set $A \subset X$ is $Z(f)$ for some continuous real f on X . Setting

$$V = f^{-1}\left(\left[0, \frac{1}{2}\right)\right), \quad W = f^{-1}\left(\left(\frac{1}{2}, 1\right]\right),$$

show that V and W are open and disjoint, and that $A \subset V$, $B \subset W$. (Thus pairs of disjoint closed sets in a metric space can be covered by pairs of disjoint open sets. This property of metric spaces is called *normality*.)

Solution. The continuity of f follows from the fact that the quotient of two continuous real-valued continuous functions is continuous wherever the denominator is non-zero. Now the denominator of the fraction that defines f cannot be zero, since the first term is zero only on A and the second is zero only on B , while A and B are disjoint. The fact that $f(p) = 0$ if and only if $p \in A$ follows from Exercise 20 and the fact that A is closed. Likewise the fact that $f(p) = 1$ if and only if $p \in B$ follows from Exercise 20 and the fact that B is closed. The assertion about V and W is immediate, since V and W are the inverse images of disjoint open sets containing 0 and 1 respectively.

Exercise 4.23 A real-valued function f defined in (a, b) is said to be *convex* if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$. Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex. (For example, if f is convex, so is e^f .)

If f is convex in (a, b) and if $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}.$$

Solution. Fix any points c, d with $a < c < d < b$, let $\eta > 0$ be any fixed positive number with $\eta < \frac{d - c}{2}$ and consider any two points x, y satisfying $c + \eta \leq x < y \leq d - \eta$. The inequality in the definition implies that $f(t)$ is bounded above on $[c, d]$. Indeed, if $c < t < d$, taking $\lambda = \frac{t - c}{d - c}$, we have $t = (1 - \lambda)c + \lambda d$, and so, if $M = \max(f(c), f(d))$, we have

$$f(t) \leq (1 - \lambda)f(c) + \lambda f(d) \leq (1 - \lambda)M + \lambda M = M.$$

It is less obvious that f is also bounded below on $[c, d]$. In fact if $\frac{c+d}{2} < t < d$, we have

$$\frac{c+d}{2} = (1-\lambda)c + \lambda t,$$

where $\lambda = \frac{d-c}{2(t-c)}$, so that

$$f\left(\frac{c+d}{2}\right) \leq \left(\frac{2t-(c+d)}{2(t-c)}\right)f(c) + \left(\frac{d-c}{2(t-c)}\right)f(t),$$

which implies

$$f(t) \geq \left(\frac{2(t-c)}{d-c}\right)f\left(\frac{c+d}{2}\right) - \frac{2t-(c+d)}{d-c}f(c) \geq -2\left|f\left(\frac{c+d}{2}\right)\right| - |f(c)|.$$

The proof that f is bounded below on $\left[c, \frac{c+d}{2}\right]$ is similar. Hence there exists M such that $|f(t)| \leq M$ for all $t \in [c, d]$.

We can also write

$$x = (1-\lambda)c + \lambda y,$$

where $\lambda = \frac{x-c}{y-c} \in (0, 1)$. Accordingly we have

$$\begin{aligned} f(x) - f(y) &\leq (1-\lambda)(f(c) - f(y)) = \\ &= \frac{y-x}{y-c}(f(c) - f(y)) \leq \frac{y-x}{\eta}|f(c) - f(y)|. \end{aligned}$$

Thus

$$f(x) - f(y) \leq \frac{2M}{\eta}(y-x).$$

Similarly, writing $y = \lambda x + (1-\lambda)d$, where $\lambda = \frac{d-y}{d-x} \in (0, 1)$, we find

$$\begin{aligned} f(y) - f(x) &\leq (1-\lambda)(f(d) - f(x)) = \\ &= \frac{y-x}{d-x}(f(d) - f(x)) \leq \frac{y-x}{\eta}|f(d) - f(x)|. \end{aligned}$$

Hence we also have

$$f(y) - f(x) \leq \frac{2M}{\eta}(y-x).$$

Therefore

$$|f(y) - f(x)| \leq \frac{2M}{\eta}|y-x|$$

for all $x, y \in [c+\eta, d-\eta]$. Since c , d , and η are arbitrary, it follows that f is continuous on (a, b) .

If $f(x)$ is convex on (a, b) , and $g(x)$ is an increasing convex function on $f((a, b))$, we have

$$g(f(\lambda x + (1 - \lambda)y)) \leq g(\lambda f(x) + (1 - \lambda)f(y)) \leq \lambda g(f(x)) + (1 - \lambda)g(f(y)).$$

The inequality

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s}$$

can be rewritten as

$$f(t) \leq \frac{t - s}{u - s} f(u) + \left(1 - \frac{t - s}{u - s}\right) f(s),$$

which is precisely the definition of convexity if we note that

$$t = \lambda u + (1 - \lambda)s$$

when $\lambda = \frac{t - s}{u - s}$.

The other inequality is proved in exactly the same way.

Exercise 4.24 Assume that f is a continuous real function defined in (a, b) such that

$$f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2}$$

for all $x, y \in (a, b)$. Prove that f is convex.

Solution. We shall prove that

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all “dyadic rational” numbers, i.e., all numbers of the form $\lambda = \frac{k}{2^n}$, where k is a nonnegative integer not larger than 2^n . We do this by induction on n . The case $n = 0$ is trivial (since $\lambda = 0$ or $\lambda = 1$). In the case $n = 1$ we have $\lambda = 0$ or $\lambda = 1$ or $\lambda = \frac{1}{2}$. The first two cases are again trivial, and the third is precisely the hypothesis of the theorem. Suppose the result is proved for $n \leq r$, and consider $\lambda = \frac{k}{2^{r+1}}$. If k is even, say $k = 2l$, then $\frac{k}{2^{r+1}} = \frac{l}{2^r}$, and we can appeal to the induction hypothesis. Now suppose k is odd. Then $1 \leq k \leq 2^{r+1} - 1$, and so the numbers $l = \frac{k - 1}{2}$ and $m = \frac{k + 1}{2}$ are integers with $0 \leq l < m \leq 2^r$. We can now write

$$\lambda = \frac{s + t}{2},$$

where $s = \frac{k - 1}{2^{r+1}} = \frac{l}{2^r}$ and $t = \frac{k + 1}{2^{r+1}} = \frac{m}{2^r}$. We then have

$$\lambda x + (1 - \lambda)y = \frac{[sx + (1 - s)y] + [tx + (1 - t)y]}{2}.$$

Hence by the hypothesis of the theorem and the induction hypothesis we have

$$\begin{aligned}
 f(\lambda x + (1 - \lambda)y) &\leq \frac{f(sx + (1 - s)y) + f(tx + (1 - t)y)}{2} \\
 &\leq \frac{sf(x) + (1 - s)f(y) + tf(x) + (1 - t)f(y)}{2} \\
 &= \left(\frac{s+t}{2}\right)f(x) + \left(1 - \frac{s+t}{2}\right)f(y) \\
 &= \lambda f(x) + (1 - \lambda)f(y).
 \end{aligned}$$

This completes the induction.

Now for each fixed x and y both sides of the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

are continuous functions of λ . Hence the set on which this inequality holds (the inverse image of the closed set $[0, \infty)$ under the mapping $\lambda \mapsto \lambda f(x) + (1 - \lambda)f(y) - f(\lambda x + (1 - \lambda)y)$) is a closed set. Since it contains all the points $\frac{k}{2^n}$, $0 \leq k \leq n$, $n = 1, 2, \dots$, it must contain the closure of this set of points, i.e., it must contain all of $[0, 1]$. Thus f is convex.

Exercise 4.25 If $A \subset R^k$ and $B \subset R^k$, define $A + B$ to be the set of all sums $\mathbf{x} + \mathbf{y}$ with $\mathbf{x} \in A$, $\mathbf{y} \in B$.

(a) If K is compact and C is closed in R^k , prove that $K + C$ is closed.

Hint: Take $\mathbf{z} \notin K + C$, put $F = \mathbf{z} - C$, the set of all $\mathbf{z} - \mathbf{y}$ with $\mathbf{y} \in C$. Then K and F are disjoint. Choose δ as in Exercise 21. Show that the open ball with center \mathbf{z} and radius δ does not intersect $K + C$.

(b) Let α be an irrational number. Let C_1 be the set of all integers. Let C_2 be the set of all $n\alpha$ with $n \in C_1$. Show that C_1 and C_2 are closed subsets of R^1 whose sum $C_1 + C_2$ is not closed, by showing that $C_1 + C_2$ is a countable dense subset of R^1 .

Solution. (a) It is clear that the set F defined in the hint is a closed set. It is disjoint from K , since $\mathbf{z} \notin K + C$. Let δ be such that $|\mathbf{p} - \mathbf{q}| > \delta$ if $\mathbf{p} \in F$ and $\mathbf{q} \in K$. We claim that there is no point of $K + C$ inside the ball of radius δ about \mathbf{z} . For suppose \mathbf{w} were such a point. By definition we would have $\mathbf{w} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in K$ and $\mathbf{v} \in C$. But then we would have

$$|\mathbf{u} - (\mathbf{z} - \mathbf{v})| = |\mathbf{w} - \mathbf{z}| < \delta,$$

which is a contradiction, since $\mathbf{u} \in K$ and $\mathbf{z} - \mathbf{v} \in F$. Thus $K + C$ is closed.

(b) Neither of the sets C_1 and C_2 has any limit points; hence both are closed sets. For each fixed integer $N \geq 2$, consider the fractional parts $\beta_1 = \alpha - [\alpha]$, $\beta_2 = 2\alpha - [2\alpha]$, \dots , $\beta_N = N\alpha - [N\alpha]$. There must be some half-open interval

$\left[\frac{k-1}{N-1}, \frac{k}{N-1}\right)$, $k = 1, 2, \dots, N-1$ containing two of the numbers β_1, \dots, β_N , since there are N numbers and only $N-1$ intervals. (Note: No two of these numbers are equal, since $\beta_i = \beta_j$, $i \neq j$, would imply

$$\alpha = \frac{[i\alpha] - [j\alpha]}{i - j},$$

i.e., α would be a rational number.) Now the inequalities

$$0 < (i\alpha - [i\alpha]) - (j\alpha - [j\alpha]) < \frac{1}{N-1}$$

say that $(i-j)\alpha + ([j\alpha] - [i\alpha]) \in \left(0, \frac{1}{N-1}\right)$, that is, there is certainly a point of $C_1 + C_2$ in $\left(0, \frac{1}{N-1}\right)$ for any $N \geq 2$. We shall now prove that there is a point of $C_1 + C_2$ in $\left(\frac{k}{n}, \frac{k+1}{n}\right)$ for any integer k and any positive integer n . To do so, fix the integer q such that $qn \leq k < (q+1)n$, and choose $y \in C_1 + C_2$ such that $0 < y < \frac{1}{n}$. Then $x = ny \in C_1 + C_2$ and $0 < x < 1$. Hence there is a positive integer p such that $k < px + qn < k+1$. This says precisely that

$$\frac{k}{n} < py + q < \frac{k+1}{n},$$

and certainly $py + q \in C_1 + C_2$. Now let O be any nonempty open subset of R^1 . Then O contains an interval (a, b) . If $n > \frac{2}{b-a}$, there is an integer k such that $\left(\frac{k}{n}, \frac{k+1}{n}\right) \subset (a, b)$. This interval, as just shown, contains a point of $C_1 + C_2$, and hence O contains such a point. Therefore $C_1 + C_2$ is dense in R^1 . Since it is a countable set, it is not all of R^1 , and hence not closed.

Exercise 4.26 Suppose X, Y, Z are metric spaces and Y is compact. Let f map X into Y , let g be a continuous one-to-one mapping of Y into Z , and put $h(x) = g(f(x))$ for $x \in X$.

Prove that f is uniformly continuous if h is uniformly continuous.

Hint: g^{-1} has compact domain $g(Y)$, and $f(x) = g^{-1}(h(x))$.

Prove also that f is continuous if h is continuous.

Show (by modifying Example 4.21, or by finding a different example) that the compactness of Y cannot be omitted from the hypotheses, even when X and Z are compact.

Solution. Theorem 4.17 asserts that g^{-1} is continuous, and since its domain is compact, it is uniformly continuous. Exercise 12 above then implies that f is uniformly continuous. The same argument, with the word “uniformly” omitted, shows that f is continuous if h is continuous.

To get a counterexample when Y is not compact, let $X = [0, 1] = Z$, $Y = \{0\} \cup [1, \infty)$, and let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be given by

$$f(x) = \begin{cases} \frac{1}{x}, & 0 < x \leq 1, \\ 0, & x = 0, \end{cases}$$

$$g(y) = \begin{cases} \frac{1}{y}, & 1 \leq y < \infty, \\ 0, & y = 0. \end{cases}$$

Then $h(x) = g(f(x)) = x$, so that h is uniformly continuous, and g is continuous and one-to-one, yet f is not even continuous.

Chapter 5

Differentiation

Exercise 5.1 Let f be defined for all real x , and suppose that

$$|f(x) - f(y)| \leq (x - y)^2$$

for all real x and y . Prove that f is constant.

Solution. Dividing by $x - y$, and letting $x \rightarrow y$, we find that $f'(y) = 0$ for all y . Hence f is constant.

Exercise 5.2 Suppose $f'(x) > 0$ in (a, b) . Prove that f is strictly increasing in (a, b) , and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \quad (a < x < b).$$

Solution. For any c, d with $a < c < d < b$ there exists a point $p \in (c, d)$ such that $f(d) - f(c) = f'(p)(d - c) > 0$. Hence $f(c) < f(d)$.

We know from Theorem 4.17 that the inverse function g is continuous. (Its restriction to each closed subinterval $[c, d]$ is continuous, and that is sufficient.) Now observe that if $f(x) = y$ and $f(x + h) = y + k$, we have

$$\frac{g(y + k) - g(y)}{k} - \frac{1}{f'(x)} = \frac{1}{\frac{f(x+h)-f(x)}{h}} - \frac{1}{f'(x)}.$$

Since we know $\lim_{\varphi(t)} \frac{1}{\varphi(t)} = \frac{1}{\lim \varphi(t)}$ provided $\lim \varphi(t) \neq 0$, it follows that for any $\varepsilon > 0$ there exists $\eta > 0$ such that

$$\left| \frac{1}{\frac{f(x+h)-f(x)}{h}} - \frac{1}{f'(x)} \right| < \varepsilon$$

if $0 < |h| < \eta$. Since $h = g(y + k) - g(y)$, there exists $\delta > 0$ such that $0 < |h| < \eta$ if $0 < |k| < \delta$. The proof is now complete.

Exercise 5.3 Suppose g is a real function on R^1 with bounded derivative (say $|g'| \leq M$). Fix $\varepsilon > 0$, and define $f(x) = x + \varepsilon g(x)$. Prove that f is one-to-one if ε is small enough. (A set of admissible values of ε can be determined which depends only on M .)

Solution. If $0 < \varepsilon < \frac{1}{M}$, we certainly have

$$f'(x) \geq 1 - \varepsilon M > 0,$$

and this implies that $f(x)$ is one-to-one, by the preceding problem.

Exercise 5.4 If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where C_0, \dots, C_n are real constants, prove that the equation

$$C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n = 0$$

has at least one real root between 0 and 1.

Solution. Consider the polynomial

$$p(x) = C_0x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1},$$

whose derivative is

$$p'(x) = C_0 + C_1x + \cdots + C_{n-1}x^{n-1} + C_nx^n.$$

It is obvious that $p(0) = 0$, and the hypothesis of the problem is that $p(1) = 0$. Hence Rolle's theorem implies that $p'(x) = 0$ for some x between 0 and 1.

Exercise 5.5 Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \rightarrow 0$ as $x \rightarrow +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \rightarrow 0$ as $x \rightarrow +\infty$.

Solution. Let $\varepsilon > 0$. Choose x_0 such that $|f'(x)| < \varepsilon$ if $x > x_0$. Then for any $x \geq x_0$ there exists $x_1 \in (x, x+1)$ such that

$$f(x+1) - f(x) = f'(x_1).$$

Since $|f'(x_1)| < \varepsilon$, it follows that $|f(x+1) - f(x)| < \varepsilon$, as required.

Exercise 5.6 Suppose

- (a) f is continuous for $x \geq 0$,
- (b) $f'(x)$ exists for $x > 0$,
- (c) $f(0) = 0$,
- (d) f' is monotonically increasing.

Put

$$g(x) = \frac{f(x)}{x} \quad (x > 0)$$

and prove that g is monotonically increasing.

Solution. By the mean-value theorem

$$f(x) = f(x) - f(0) = f'(c)x,$$

for some $c \in (0, x)$. Since f' is monotonically increasing, this result implies that $f(x) < xf'(x)$. It therefore follows that

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} > 0,$$

so that g is also monotonically increasing.

Exercise 5.7 Suppose $f'(x)$ and $g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \rightarrow x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

(This holds also for complex functions.)

Solution. Since $f(x) = g(x) = 0$, we have

$$\begin{aligned} \lim_{t \rightarrow x} \frac{f(t)}{g(t)} &= \lim_{t \rightarrow x} \frac{\frac{f(t) - f(x)}{t - x}}{\frac{g(t) - g(x)}{t - x}} \\ &= \frac{\lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x}}{\lim_{t \rightarrow x} \frac{g(t) - g(x)}{t - x}} \\ &= \frac{f'(x)}{g'(x)}. \end{aligned}$$

Exercise 5.8 Suppose f' is continuous on $[a, b]$ and $\varepsilon > 0$. Prove that there exists $\delta > 0$ such that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \varepsilon$$

whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying that f is *uniformly differentiable* on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?

Solution. Let δ be such that $|f'(x) - f'(u)| < \varepsilon$ for all $x, u \in [a, b]$ with $|x - u| < \delta$. Then if $0 < |t - x| < \delta$ there exists u between t and x such that

$$\frac{f(t) - f(x)}{t - x} = f'(u),$$

and hence, since $|u - x| < \delta$,

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| = |f'(u) - f'(x)| < \varepsilon.$$

Since this result holds for each component of a vector-valued function $\mathbf{f}(x)$, it must hold also for \mathbf{f} .

Exercise 5.9 Let f be a continuous real function on R^1 , of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \rightarrow 3$ as $x \rightarrow 0$. Does it follow that $f'(0)$ exists?

Solution. Yes. By L'Hospital's rule

$$\lim_{t \rightarrow 0} \frac{f(t) - f(0)}{t} = \lim_{t \rightarrow 0} f'(t) = 3,$$

and this by definition means that $f'(0) = 3$.

Exercise 5.10 Suppose f and g are complex differentiable functions on $(0, 1)$, $f(x) \rightarrow 0$, $g(x) \rightarrow 0$, $f'(x) \rightarrow A$, $g'(x) \rightarrow B$ as $x \rightarrow 0$, where A and B are complex numbers, $B \neq 0$. Prove that

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$

Compare with Example 5.18. *Hint:*

$$\frac{f(x)}{g(x)} = \left\{ \frac{f(x)}{x} - A \right\} \cdot \frac{x}{g(x)} + A \cdot \frac{x}{g(x)}.$$

Apply Theorem 5.13 to the real and imaginary parts of $f(x)/x$ and $g(x)/x$.

Solution. We can make f and g continuous on $[0, 1)$ by simply defining $f(0) = 0 = g(0)$. Then Exercise 9 applied to the real and imaginary parts of f and g show that $f'(0) = A$ and $g'(0) = B$. (These are one-sided derivatives, since f and g are not defined for negative values of x ; however, we could extend them as odd functions, since both are 0 at 0). We could then apply Exercise 7, whose proof does not use anything but the definition of the derivative and some general facts about limits. In this way we get the result without resorting to the combinatorial trick referred to in the hint. This result shows that many of the facts ordinarily proved for real functions by use of the mean-value theorem and L'Hospital's rule remain true for complex-valued functions, even though, as Example 5.18 shows, these theorems are not true for complex-valued functions.

Exercise 5.11 Suppose f is defined in a neighborhood of x , and suppose $f''(x)$ exists. Show that

$$\lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$

Solution. For a real-valued function this is a routine application of L'Hospital's rule:

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} &= \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x-h)}{2h} \\ &= \frac{1}{2} \lim_{h \rightarrow 0} \frac{f'(x+h) - f'(x)}{h} + \\ &\quad + \frac{f'(x) - f'(x-h)}{h} \\ &= f''(x). \end{aligned}$$

For complex-valued functions the result follows from separate consideration of real and imaginary parts.

The limit will be zero at $x = 0$ for any odd function f whatsoever, even if the function is not continuous. For example we could take $f(x) = \operatorname{sgn}(x)$, which is $+1$ for $x > 0$, 0 for $x = 0$, and -1 for $x < 0$.

Exercise 5.12 If $f(x) = |x|^3$, compute $f'(x)$, $f''(x)$ for all real x , and show that $f^{(3)}(0)$ does not exist.

Solution. For $x > 0$ we have $f'(x) = 3x^2$, $f''(x) = 6x$, and for $x < 0$ $f'(x) = -6x^2$, $f''(x) = -6x$, i.e., $f'(x) = 3x|x|$, and $f''(x) = 6|x|$ for $x \neq 0$. By Exercise 9, it therefore follows that $f'(0)$ exists and equals 0, and then another application of Exercise 9 shows that $f''(0)$ also exists and equals 0. However

$$\frac{f''(x) - f''(0)}{x} = 6\operatorname{sgn}(x),$$

which has no limit at 0. Hence $f^{(3)}(0)$ does not exist.

Exercise 5.13 Suppose a and c are real numbers, $c > 0$, and f is defined on $[-1, 1]$ by

$$f(x) = \begin{cases} x^a \sin(x^{-c}) & (\text{if } x \neq 0), \\ 0 & (\text{if } x = 0). \end{cases}$$

Prove the following statements:

- (a) f is continuous if and only if $a > 0$.
- (b) $f'(0)$ exists if and only if $a > 1$.
- (c) f' is bounded if and only if $a \geq 1 + c$.
- (d) f' is continuous if and only if $a > 1 + c$.
- (e) $f''(0)$ exists if and only if $a > 2 + c$.
- (f) f'' is bounded if and only if $a \geq 2 + 2c$.
- (g) f'' is continuous if and only if $a > 2 + 2c$.

Solution. We remark editorially that there are two difficulties with this problem. One is that we haven't yet introduced the function \sin . To overcome that problem we can rely on our intuitive notion or use the Taylor series if we have to. The second problem is more serious, however: What do x^a and x^{-c} mean when $x < 0$? In general these will be complex-valued functions. It might be better to use absolute values in both cases. Thus we shall amend the problem by defining $f(x) = |x|^a \sin(|x|^{-c})$ when $x \neq 0$.

(a) Since f is infinitely differentiable except at $x = 0$, the only question of continuity is at $x = 0$. Let $t_n = 2\pi(n + \frac{1}{8})$, $x_n = t_n^{-\frac{1}{c}}$ and $y_n = \frac{1}{\sqrt{2}} t_n^{-\frac{a}{c}}$. Notice that $f(x_n) = y_n$ and that y_n tends to $\frac{1}{\sqrt{2}}$ if $a = 0$ and to $+\infty$ if $a < 0$. Hence the function cannot be continuous if $a \leq 0$. On the other hand, we have

$$|f(x) - f(0)| = |f(x)| \leq |x|^a,$$

so that if $a > 0$ and ε is given, we can choose $\delta = \varepsilon^{\frac{1}{a}}$, and then $|x - 0| < \delta$ implies $|f(x) - f(0)| < \varepsilon$, i.e., $f(x)$ is continuous at $x = 0$.

(b) If $f'(0)$ exists, then f is continuous at 0, so that $a > 0$. Notice that

$$\frac{f(x_n) - f(0)}{x_n} = \frac{y_n}{x_n} = \frac{1}{\sqrt{2}} t_n^{\frac{1-a}{c}}.$$

which tends to $\frac{1}{\sqrt{2}}$ if $a = 1$ and to $+\infty$ if $0 < a < 1$. Hence $f'(0)$ does not exist if $a \leq 1$. On the other hand if $a > 1$ we have

$$0 \leq \frac{f(x) - f(0)}{x} < |x|^{a-1} \rightarrow 0,$$

and so $f'(0) = 0$.

(c) For $x \neq 0$ we have

$$f'(x) = \operatorname{sgn}(x) |x|^{a-1} [a \sin(|x|^{-c}) - c |x|^{-c} \cos(|x|^{-c})].$$

Hence $f'(x_n) = \frac{1}{\sqrt{2}}[ax_n^{a-1} - cx_n^{-c+a-1}]$, which tends to $-\infty$ if $a < 1 + c$. On the other hand we have

$$|f'(x)| \leq |a||x|^{a-1} + c|x|^{a-1-c},$$

which is certainly bounded on $[-1, 1]$ if $a \geq 1 + c$.

(d) If f' is continuous, it is bounded, and so $a \geq 1 + c$. However if $a = 1 + c$, then

$$f'(x_n) = \frac{1}{\sqrt{2}}[(1+c)t_n^{-1} - c]$$

which tends to $-\frac{c}{\sqrt{2}}$ as $n \rightarrow \infty$, while $x_n \rightarrow 0$. Hence f' is not continuous at 0 unless $a > 1 + c$. If $a > 1 + c$, the inequality

$$|f'(x)| \leq |a||x|^{a-1} + c|x|^{a-1-c},$$

implies that $f(x) \rightarrow 0$ as $x \rightarrow 0$, and so f' is continuous.

(e) If $f''(0)$ exists, then f' must be continuous at 0, and so $a \geq 1 + c$. Now for $x \neq 0$

$$\frac{f'(x) - f'(0)}{x} = \operatorname{sgn}(x)[a|x|^{a-2} \sin(|x|^{-c}) - c|x|^{a-c-2} \cos(|x|^{-c})].$$

Taking $x = x_n$, we find that this difference quotient equals

$$\frac{1}{\sqrt{2}}[at_n^{\frac{2-a}{c}} - ct_n^{\frac{c+2-a}{c}}],$$

which tends to $\frac{1}{\sqrt{2}}$ if $a = c + 2$ and to $-\infty$ if $a < c + 2$. Hence $f''(0)$ exists only if $a > c + 2$.

On the other hand, if $a > c + 2$, we have the inequality

$$\left| \frac{f'(x) - f'(0)}{x} \right| \leq a|x|^{a-2} + c|x|^{a-c-2},$$

from which it follows immediately that $f''(0) = 0$.

(f) For $x \neq 0$ we have

$$f''(x) = \operatorname{sgn}(x)[a(a-1)|x|^{a-2} - c^2|x|^{a-2c-2}] \sin(|x|^{-c}) - c(2a-c-1)|x|^{a-c-1} \cos(|x|^{-c}).$$

In particular

$$f''(x_n) = \frac{1}{\sqrt{2}}[a(a-1)t_n^{\frac{2-a}{c}} - c^2t_n^{\frac{2+2c-a}{c}} - c(2a-c-1)t_n^{\frac{c+1-a}{c}}],$$

which tends to $-\infty$ if $a < 2 + 2c$. On the other hand, we have the inequality

$$|f''(x)| \leq |a||a-1||x|^{a-2} + c^2|x|^{a-2c-2} + c|2a-c-1||x|^{a-c-1},$$

and the right-hand side is certainly bounded if $a \geq 2 + 2c$.

(g) If f'' is continuous, then it is bounded, and hence $a \geq 2 + 2c$. If $a = 2 + 2c$, we have

$$f''(x_n) = \frac{1}{\sqrt{2}}[(2c+2)(2c+1)t_n^{-2} - c^2 - c(3+3c)t_n^{\frac{-c-1}{c}}],$$

which tends to $-\frac{c^2}{\sqrt{2}}$, so that f'' is not continuous at 0. On the other hand, if $a > 2 + 2c$, the inequality

$$|f''(x)| \leq |a||a-1||x|^{a-2} + c^2|x|^{a-2c-2} + c|2a-c-1||x|^{a-c-1},$$

shows that $f''(x) \rightarrow 0$ as $x \rightarrow 0$, and hence f'' is continuous.

Exercise 5.14 Let f be a differentiable real function defined in (a, b) . Prove that f is convex if and only if f' is monotonically increasing. Assume next that $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.

Suppose first that f' is monotonically increasing, and that $x < y$. We wish to show that if $0 < \lambda < 1$, then

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y).$$

Letting $z = \lambda x + (1-\lambda)y$, we have $\lambda = \frac{y-z}{y-x}$, $1-\lambda = \frac{z-x}{y-x}$, and $x < z < y$. Now the required inequality can be written

$$(1-\lambda)[f(y) - f(z)] \geq \lambda[f(z) - f(x)],$$

which, when we insert the values of λ and $1-\lambda$, and multiply by the positive number $\frac{y-x}{(z-x)(y-z)}$, becomes

$$\frac{f(y) - f(z)}{y-z} \geq \frac{f(z) - f(x)}{z-x}.$$

Since the left-hand side is $f'(d)$ for some $d \in (z, y)$, the right-hand side is $f'(c)$ for some $c \in (x, z)$, and f' is nondecreasing, we have the required inequality.

By Exercise 23 of Chapter 4 we know that if f is convex on (a, b) and $a < c < d < p < q < b$, then

$$\frac{f(d) - f(c)}{d-c} \leq \frac{f(p) - f(d)}{p-d} \leq \frac{f(q) - f(p)}{q-p}.$$

Hence, if f' exists, letting $d \rightarrow c$ and $q \rightarrow p$, we find

$$f'(c) \leq f'(p),$$

so that f' is nondecreasing.

Finally if f'' exists, we know that f' is nondecreasing if and only if $f''(x) \geq 0$ for all $x \in (a, b)$. Hence f is convex if and only if $f''(x) \geq 0$ for all $x \in (a, b)$.

Exercise 5.15 Suppose $a \in \mathbb{R}^1$, f is a twice-differentiable real function on (a, ∞) , and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively, on (a, ∞) . Prove that

$$M_1^2 \leq 4M_0M_2.$$

Hint: If $h > 0$, Taylor's theorem shows that

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi)$$

for some $\xi \in (x, x+2h)$. Hence

$$|f(x)| \leq hM_2 + \frac{M_0}{h}.$$

To show that $M_1^2 = 4M_0M_2$ can actually happen, take $a = -1$, define

$$f(x) = \begin{cases} 2x^2 - 1, & (-1 < x < 0), \\ \frac{x^2 - 1}{x^2 + 1}, & (0 \leq x < \infty), \end{cases}$$

and show that $M_0 = 1, M_1 = 4, M_2 = 4$.

Does $M_1^2 \leq 4M_0M_2$ hold for vector-valued functions too?

Solution. The inequality is obvious if $M_0 = +\infty$ or $M_2 = +\infty$, so we shall assume that M_0 and M_2 are both finite. We need to show that

$$|f'(x)| \leq 2\sqrt{M_0M_2}$$

for all $x > a$. We note that this is obvious if $M_2 = 0$, since in that case $f'(x)$ is constant, $f(x)$ is a linear function, and the only bounded linear function is a constant, whose derivative is zero. Hence we shall assume from now on that $0 < M_2 < +\infty$ and $0 < M_0 < +\infty$.

Following the hint, we need only choose $h = \sqrt{\frac{M_0}{M_2}}$, and we obtain

$$|f'(x)| \leq 2\sqrt{M_0M_2},$$

which is precisely the desired inequality.

The case of equality follows, since the example proposed satisfies

$$f(x) = 1 - \frac{2}{x^2 + 1}$$

for $x \geq 0$. We see easily that $|f(x)| \leq 1$ for all $x > -1$. Now $f'(x) = \frac{4x}{(x^2 + 1)^2}$ for $x > 0$ and $f'(x) = 4x$ for $x < 0$. It thus follows from Exercise 9 above that $f'(0) = 0$, and that $f'(x)$ is continuous. Likewise $f''(x) = 4$ for $x < 0$

and $f''(x) = \frac{4-4x^2}{(x^2+1)^3} = -4\frac{x^2-1}{(x^2+1)^3}$. This shows that $|f''(x)| < 4$ for $x > 0$ and also that $\lim_{x \rightarrow 0} f''(x) = 4$. Hence Exercise 9 again implies that $f''(x)$ is continuous and $f''(0) = 4$.

On n -dimensional space let $\mathbf{f}(x) = (f_1(x), \dots, f_n(x))$, $M_0 = \sup |\mathbf{f}(x)|$, $M_1 = \sup |\mathbf{f}'(x)|$, and $M_2 = \sup |\mathbf{f}''(x)|$. Just as in the numerical case, there is nothing to prove if $M_2 = 0$ or $M_0 = +\infty$ or $M_2 = +\infty$, and so we assume $0 < M_0 < +\infty$ and $0 < M_2 < \infty$. Let a be any positive number less than M_1 , let x_0 be such that $|\mathbf{f}'(x_0)| > a$, and let $\mathbf{u} = \frac{1}{|\mathbf{f}'(x_0)|} \mathbf{f}'(x_0)$. Consider the real-valued function $\varphi(x) = \mathbf{u} \cdot \mathbf{f}(x)$. Let N_0 , N_1 , and N_2 be the suprema of $|\varphi(x)|$, $|\varphi'(x)|$, and $|\varphi''(x)|$ respectively. By the Schwarz inequality we have (since $|\mathbf{u}| = 1$) $N_0 \leq M_0$ and $N_2 \leq M_2$, while $N_1 \geq \varphi'(x_0) = |\mathbf{f}'(x_0)| > a$. We therefore have $a^2 < 4N_0N_2 \leq 4M_0M_2$. Since a was any positive number less than M_1 , we have $M_1^2 \leq 4M_0M_2$, i.e., the result holds also for vector-valued functions.

Equality can hold on any R^n , as we see by taking $\mathbf{f}(x) = (f(x), 0, \dots, 0)$ or $\mathbf{f}(x) = (f(x), f(x), \dots, f(x))$, where $f(x)$ is a real-valued function for which equality holds.

Exercise 5.16 Suppose f is twice-differentiable on $(0, \infty)$, f'' is bounded on $(0, \infty)$, and $f(x) \rightarrow 0$ as $x \rightarrow \infty$. Prove that $f'(x) \rightarrow 0$ as $x \rightarrow \infty$.

Solution. We shall prove an even stronger statement. If $f(x) \rightarrow L$ as $x \rightarrow \infty$ and $f'(x)$ is uniformly continuous on $(0, \infty)$, then $f'(x) \rightarrow 0$ as $x \rightarrow \infty$.

For, if not, let $x_n \rightarrow \infty$ be a sequence such that $f(x_n) \geq \varepsilon > 0$ for all n . (We can assume $f(x_n)$ is positive by replacing f with $-f$ if necessary.) Let δ be such that $|f'(x) - f'(y)| < \frac{\varepsilon}{2}$ if $|x - y| < \delta$. We then have $f'(y) > \frac{\varepsilon}{2}$ if $|y - x_n| < \delta$, and so

$$|f(x_n + \delta) - f(x_n - \delta)| \geq 2\delta \cdot \frac{\varepsilon}{2} = \delta\varepsilon.$$

But, since $\delta\varepsilon > 0$, there exists X such that

$$|f(x) - L| < \frac{1}{2}\delta\varepsilon$$

for all $x > X$. Hence for all large n we have

$$|f(x_n + \delta) - f(x_n - \delta)| \leq |f(x_n + \delta) - L| + |L - f(x_n - \delta)| < \delta\varepsilon,$$

and we have reached a contradiction.

The problem follows from this result, since if f'' is bounded, say $|f''(x)| \leq M$, then $|f'(x) - f'(y)| \leq M|x - y|$, and f' is certainly uniformly continuous.

Exercise 5.17 Suppose f is a real, three times differentiable function on $[-1, 1]$, such that

$$f(-1) = 0, \quad f(0) = 0, \quad f(1) = 1, \quad f'(0) = 0.$$

Prove that $f^{(3)}(x) \geq 3$ for some $x \in (-1, 1)$.

Note that equality holds for $\frac{1}{2}(x^3 + x^2)$.

Hint: Use Theorem 5.15 with $\alpha = 1$ and $\beta = \pm 1$, to show that there are $s \in (0, 1)$ and $t \in (-1, 0)$ such that

$$f^{(3)}(s) + f^{(3)}(t) = 6.$$

Solution. Following the hint, we observe that Theorem 5.15 (Taylor's formula with remainder) implies that

$$\begin{aligned} f(1) &= f(0) + f'(0) + \frac{1}{2}f''(0) + \frac{1}{6}f^{(3)}(s) \\ f(-1) &= f(0) - f'(0) + \frac{1}{2}f''(0) - \frac{1}{6}f^{(3)}(t) \end{aligned}$$

for some $s \in (0, 1)$, $t \in (-1, 0)$. By subtracting the second equation from the first and using the given values of $f(1)$, $f(-1)$, and $f'(0)$, we obtain

$$1 = \frac{1}{6}(f^{(3)}(s) + f^{(3)}(t)),$$

which is the desired result. Note that we made no use of the hypothesis $f(0) = 0$.

Exercise 5.18 Suppose f is a real function on $[a, b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let α , β , and P be as in Taylor's theorem (5.15). Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a, b]$, $t \neq \beta$, differentiate

$$f(t) - f(\beta) = (t - \beta)Q(t)$$

$n - 1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{n-1}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

Solution. The function $Q(t)$ is differentiable $n - 1$ times except possibly at $t = \beta$, so we don't have to worry when differentiating $n - 1$ times at $t = \alpha$. It is easy to prove by induction that

$$f^{(k)}(t) = (t - \beta)Q^{(k)}(t) + kQ^{(k-1)}(t)$$

for $0 < k \leq n-1$. Hence

$$\frac{1}{k!} f^{(k)}(\alpha)(\beta - \alpha)^k = -\frac{(\beta - \alpha)^{k+1}}{k!} Q^{(k)}(\alpha) + \frac{(\beta - \alpha)^k}{(k-1)!} Q^{(k-1)}(\alpha).$$

Then, because the sum telescopes, we find

$$\sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k = f(\beta) - \frac{Q^{(n-1)}(\alpha)}{(n-1)!} (\beta - \alpha)^n,$$

which can be rewritten as

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!} (\beta - \alpha)^n.$$

Exercise 5.19 Suppose f is defined in $(-1, 1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \rightarrow 0$, and $\beta_n \rightarrow 0$ as $n \rightarrow \infty$. Define the difference quotients

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}.$$

Prove the following statements:

- (a) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$.
- (b) If $0 < \alpha_n < \beta_n$ and $\beta_n/(\beta_n - \alpha_n)$ is bounded, then $\lim D_n = f'(0)$.
- (c) if f' is continuous in $(-1, 1)$, then $\lim D_n = f'(0)$.

Give an example in which f is differentiable in $(-1, 1)$ (but f' is not continuous at 0) and in which α_n, β_n tends to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.

Solution. We assume that $\alpha_n \beta_n \neq 0$ throughout, i.e., that neither α_n nor β_n is zero.

(a) Write

$$\begin{aligned} D_n &= \frac{f(\beta_n) - f(0)}{\beta_n - \alpha_n} + \frac{f(0) - f(\alpha_n)}{\beta_n - \alpha_n} \\ &= \frac{\beta_n}{\beta_n - \alpha_n} \frac{f(\beta_n) - f(0)}{\beta_n} + \frac{-\alpha_n}{\beta_n - \alpha_n} \frac{f(\alpha_n) - f(0)}{\alpha_n}. \end{aligned}$$

Now let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$\left| \frac{f(x) - f(0)}{x} - f'(0) \right| < \varepsilon$$

if $0 < |x| < \delta$. Then choose N so that $0 < \beta_n < \delta$ and $-\delta < \alpha_n < 0$ for $n > N$. Then for all $n > N$ we have

$$\begin{aligned} |D_n - f'(0)| &\leq \frac{\beta_n}{\beta_n - \alpha_n} \left| \frac{f(\beta_n) - f(0)}{\beta_n} - f'(0) \right| + \\ &\quad + \frac{-\alpha_n}{\beta_n - \alpha_n} \left| \frac{f(\alpha_n) - f(0)}{\alpha_n} - f'(0) \right| \\ &< \frac{\beta_n}{\beta_n - \alpha_n} \varepsilon + \frac{-\alpha_n}{\beta_n - \alpha_n} \varepsilon \\ &= \varepsilon. \end{aligned}$$

(b) If $\frac{\beta_n}{\beta_n - \alpha_n} \leq M$ for all n , and $0 < \alpha_n < \beta_n$, then surely $\frac{\alpha_n}{\beta_n - \alpha_n} < M$ for all n . Hence if $\varepsilon > 0$ is given, choose N so that

$$\left| \frac{f(x) - f(0)}{x} - f'(0) \right| < \frac{\varepsilon}{2M}$$

if $0 < |x| < \delta$. Then choose N so that $0 < \beta_n < \delta$ (hence also $0 < \alpha_n < \delta$) for $n > N$. Then for all $n > N$ we have

$$\begin{aligned} |D_n - f'(0)| &\leq \frac{\beta_n}{\beta_n - \alpha_n} \left| \frac{f(\beta_n) - f(0)}{\beta_n} - f'(0) \right| + \\ &\quad + \frac{\alpha_n}{\beta_n - \alpha_n} \left| \frac{f(\alpha_n) - f(0)}{\alpha_n} - f'(0) \right| \\ &< \frac{\beta_n}{\beta_n - \alpha_n} \frac{\varepsilon}{2M} + \frac{\alpha_n}{\beta_n - \alpha_n} \frac{\varepsilon}{2M} \\ &< \varepsilon. \end{aligned}$$

(c) By the mean-value theorem there exists γ_n between α_n and β_n such that $D_n = f'(\gamma_n)$. Since $\gamma_n \rightarrow 0$ and f' is continuous, it follows that $D_n \rightarrow f'(0)$.

Let $f(x)$ be any function such that $f'(0)$ exists but $\lim_{x \rightarrow 0} f'(x)$ does not exist. We know that $f'(x)$ does not tend to infinity as $x \rightarrow 0$, since if it did, we would have $|f'(x)| > 1 + |f'(0)|$ for all sufficiently small nonzero x , and this contradicts the intermediate-value property of derivatives. Hence there is a sequence $x_n \rightarrow 0$, $x_n \neq 0$, such that $\lim_{n \rightarrow \infty} f'(x_n) = L \neq f'(0)$. Let $\beta_n = x_n$, and let y_n be such that $0 < |y_n - x_n| < \frac{1}{2}|x_n|$ and

$$\left| \frac{f(y_n) - f(x_n)}{y_n - x_n} - f'(x_n) \right| < \frac{|L - f'(0)|}{2n}$$

It is then immediate that

$$\lim_{n \rightarrow \infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} = L \neq f'(0).$$

A suitable example of such a function $f(x)$ is

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

In this case we can get the counterexample in a slightly different form by taking $x_n = \frac{1}{2\pi n}$ and $y_n = \frac{1}{2\pi(n + \frac{1}{4})}$. We then have $f'(0) = 0$ and

$$\frac{f(y_n) - f(x_n)}{y_n - x_n} = \frac{2n}{\pi(n + \frac{1}{4})} \rightarrow \frac{2}{\pi}.$$

Exercise 5.20 Formulate and prove an inequality which follows from Taylor's theorem and which remains valid for vector-valued functions.

Solution. There is a variety of possibilities, of which we choose just one: Suppose $f(x)$ has continuous derivatives up to order n on $[a, b]$. Then there exists $c \in (a, b)$ such that

$$|f(b) - P(b)| \leq \left| \frac{f^n(c)}{n!} \right| (b-a)^n.$$

To prove this assertion true for a vector-valued function \mathbf{f} , we merely observe that it holds for each scalar-valued function $\mathbf{u} \cdot \mathbf{f}$ if \mathbf{u} is any fixed vector of length 1. It is obviously true if $|\mathbf{f}(b) - \mathbf{P}(b)| = 0$, and in all other cases it follows by taking $\mathbf{u} = \frac{1}{|\mathbf{f}(b) - \mathbf{P}(b)|}(\mathbf{f}(b) - \mathbf{P}(b))$.

Exercise 5.21 Let E be a closed subset of R^1 . We saw in Exercise 22, Chap. 4, that there is a real continuous function f on R^1 whose zero set is E . Is it possible, for each closed set E , to find such an f which is differentiable on R^1 , or one which is n times differentiable, or even one which has derivatives of all orders on R^1 ?

Solution. Yes, it is possible. The proof depends on the following lemma:

Let a and b be any real numbers with $a < b$, and let $f(x)$ be defined for all real numbers x by the formulas

$$f(x) = \begin{cases} e^{\frac{1}{(x-a)(x-b)}}, & a < x < b, \\ 0, & x \leq a \text{ or } x \geq b. \end{cases}$$

Then f has derivatives of all orders on R^1 .

It is obvious that f has derivatives of all orders at every point except possibly a and b . To prove that derivatives exist at these points we need two sublemmas: For each nonnegative integer n there exists a polynomial $p_n(z, w)$ such that

$$f^{(n)}(x) = p_n\left(\frac{1}{x-a}, \frac{1}{x-b}\right) e^{\frac{1}{(x-a)(x-b)}}$$

for $a < x < b$.

The proof of this sublemma uses only the partial-fraction decomposition

$$\frac{1}{(x-a)(x-b)} = \frac{1}{b-a} \left[\frac{1}{x-b} - \frac{1}{x-a} \right],$$

together with the chain rule and the fact that the partial derivative of a polynomial is again a polynomial. We omit the details.

The second sublemma is stated as a formula: *For every nonnegative integer n ,*

$$\lim_{x \downarrow a} \frac{e^{\frac{1}{(x-a)(x-b)}}}{(x-a)^n} = 0.$$

Its proof is a consequence of Taylor's formula. To be specific, Taylor's formula with remainder implies the following result:

For each nonnegative integer k and each positive number t

$$e^t > \frac{1}{k!} t^k.$$

This last result follows easily since there is a point $t_k \in (0, t)$ for which

$$e^t = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^{k-1}}{(k-1)!} + \frac{e^{t_k}}{k!} t^k,$$

every term in this last sum is positive, and $e^{t_k} > 1$.

We now apply this result with $k = n$ and $t = \frac{1}{(x-a)(b-x)}$, to obtain

$$\begin{aligned} e^{\frac{1}{(x-a)(x-b)}} &= \frac{1}{e^{\frac{1}{(x-a)(b-x)}}} \\ &< n!(b-x)^n(x-a)^n \end{aligned}$$

for all $n = 0, 1, \dots$. In particular

$$e^{\frac{1}{(x-a)(x-b)}} < n!(b-a)^n(x-a)^n = K_n(x-a)^n.$$

Since the k th derivative of $e^{\frac{1}{(x-a)(x-b)}}$ is a polynomial in $\frac{1}{x-a}$ and $\frac{1}{x-b}$, each derivative also satisfies such an estimate. It follows from this last result that

$$\lim_{x \downarrow a} p\left(\frac{1}{z-b}, \frac{1}{z-a}\right) e^{\frac{1}{(x-a)(x-b)}} = 0$$

for any polynomial $p(z, w)$, and hence that $f^{(n)}(a) = 0$ for all n . The proof that $f^{(n)}(b) = 0$ is similar. We observe that the zero set of $f(x)$ is the complement of the open interval (a, b) .

Identical reasoning shows that the function

$$f(x) = \begin{cases} e^{\frac{1}{a-x}}, & x > a, \\ 0, & x \leq a, \end{cases}$$

has derivatives of all orders, and its zero set is the complement of the semi-infinite open interval $(a, +\infty)$. A similar function can be constructed for a semi-infinite open interval $(-\infty, b)$.

Now let F be any non-empty closed set. The complement of F consists of a countable set of pairwise disjoint finite open intervals (a_k, b_k) , together with possibly one or two semi-infinite open intervals. Define $f(x)$ to be zero on F , let $f(x) = e^{\frac{1}{(x-a_k)(x-b_k)}}$ in each finite open interval complementary to F with endpoints in F , $f(x) = e^{\frac{1}{a-x}}$ for $x > a$ if the complement of F contains a semi-infinite interval $(a, +\infty)$ with endpoint $a \in F$, and $f(x) = e^{\frac{1}{x-b}}$ if the complement of F contains a semi-infinite interval $(-\infty, b)$ with endpoint $b \in F$.

It is now obvious that f is zero precisely on F , and that F has derivatives of all orders at each point of the complement of F and at each interior point of F .

It remains to be shown that f has derivatives of all orders at each boundary point x of F . There are actually 4 cases to consider, but all are handled alike, and we shall settle for just one typical case, in which there is a decreasing sequence of points $x_p \in F$, $x_p \rightarrow x$, and a decreasing sequence of points $y_p \notin F$, $y_p \rightarrow x$, but no increasing sequence of points $z_p \in F$, $z_p \rightarrow x$. This means either $x = b_k$ for some k or $x = b$. Now for each y such that $x < y < x_1$ there is a complementary interval to F , say $(a_l, b_l) \subset (x, x_1)$, with $a_k < y < b_k$. Then for all nonnegative integers k and n we have

$$0 < f^{(k)}(y) < K_{n,k}(y - a_l)^n < K_{n,k}(y - x)^n$$

where $K_{n,k}$ is a positive constant independent of l , hence independent of y . It therefore follows, upon taking $n = 2$ that if $x_1 > y > x$, then

$$\left| \frac{f(y) - f(x)}{(y - x)} \right| \leq K_{2,0}(y - x)$$

(We have just proved this inequality for $y \notin F$, and $f(y) = f(x) = 0$ if $y \in F$.) Hence the right-handed derivative

$$f'_+(x) = \lim_{y \downarrow x} \frac{f(y) - f(x)}{y - x}$$

is zero. That the left-hand derivative is zero follows from the fact that $x = b_k$ or $x = b$. Hence $f'(x) = 0$. We now assume by induction that $f^{(k-1)}(x) = 0$. Then the inequality $f^{(k-1)}(y) \leq K_{2,k-1}(y - x)^2$ shows that

$$f^{(k)}_+(x) = \lim_{y \downarrow x} \frac{f^{(k-1)}(y)}{y - x} = 0.$$

Again, the left-hand k th derivative is zero since $x = b_k$ or $x = b$. It follows easily that $f^{(k)}(x)$ exists and equals zero for all k .

Exercise 5.22 Suppose f is a real function on $(-\infty, \infty)$. Call x a *fixed point* of f if $f(x) = x$.

(a) If f is differentiable and $f'(t) \neq 1$ for every real t , prove that f has at most one fixed point.

(b) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t .

(c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t , prove that a fixed point x of f exists, and that $x = \lim x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, \dots$.

(d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \rightarrow (x_2, x_2) \rightarrow (x_2, x_3) \rightarrow (x_3, x_3) \rightarrow (x_3, x_4) \rightarrow \dots$$

Solution. (a) If a function $f(x)$ has two fixed points x and y , $x \neq y$, the mean-value theorem implies that there exists a point z between x and y such that

$$y - x = f(y) - f(x) = f'(z)(y - x),$$

so that $f'(z) = 1$.

(b) The equation $f(t) = t$ implies that $(1 + e^t)^{-1} = 0$, which is clearly impossible, while $f'(t) = 1 - \frac{e^t}{(1 + e^t)^2}$ always lies in $(0, 1)$.

(c) Since f' is bounded, f is uniformly continuous, and we observe that the sequence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Indeed, if $n > m > N$, we have

$$|x_n - x_m| \leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \dots + |x_{m+1} - x_m|.$$

Now it is easy to show by induction, using the mean-value theorem and the fact that $|f'(x)| \leq A$ for all x , that

$$|x_{n+1} - x_n| \leq A^{n-1} |x_2 - x_1|$$

for $n \geq 1$. We therefore have

$$\begin{aligned} |x_n - x_m| &\leq |x_2 - x_1| (A^{n-2} + A^{n-3} + \dots + A^{m-1}) \\ &< \frac{1}{1-A} A^{m-1} |x_2 - x_1| \\ &\leq \frac{|x_2 - x_1|}{1-A} A^N. \end{aligned}$$

Since $0 \leq A < 1$, it follows that $A^N \rightarrow 0$ as $N \rightarrow \infty$, and so this is a Cauchy sequence. Let its limit be x . We claim that x is a fixed point. Indeed, $x =$

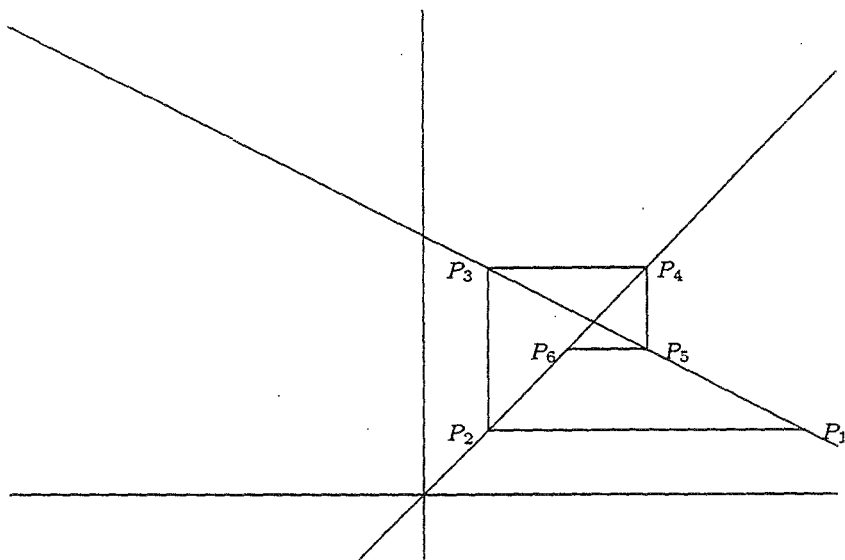


Figure 5.1: Finding a fixed point

$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} x_n) = f(x)$, since f is continuous. There can of course be only one fixed point because of the result proved in (a).

(d) The procedure described can be depicted on the graph of the function f , i.e., the set of points $(x, f(x))$, as follows: Let x_1 be any abscissa; locate the point $(x_1, f(x_1))$ on the graph. Thereafter, for each point (x_n, y_n) located on the graph, let the abscissa of (x_{n+1}, y_{n+1}) be the ordinate of (x_n, y_n) , i.e., $x_{n+1} = y_n$. Thus, from a point (x_n, y_n) on the graph of f we move horizontally to the line $y = x$, then vertically back to the graph of f . It is clear visually that this process leads to the point of intersection of the graph of f with the line $y = x$, as illustrated in Fig. 1 for the case of $f(x) = 2 - \frac{1}{2}x$, where $P_1 = (2, 1)$, $P_2 = (1, 1)$, $P_3 = (1, \frac{3}{2})$, $P_4 = (\frac{3}{2}, \frac{3}{2})$, $P_5 = (\frac{3}{2}, \frac{5}{4})$, and $P_6 = (\frac{5}{4}, \frac{5}{4})$. (The fixed point is $(\frac{4}{3}, \frac{4}{3})$, which is the point of intersection of the graph of f and the line $y = x$.)

Exercise 5.23 The function f defined by

$$f(x) = \frac{x^3 + 1}{3}$$

has three fixed points, say α , β , γ , where

$$-2 < \alpha < -1, \quad 0 < \beta < 1, \quad 1 < \gamma < 2.$$

For arbitrarily chosen x_1 , define $\{x_n\}$ by setting $x_{n+1} = f(x_n)$.

- If $x_1 < \alpha$, prove that $x_n \rightarrow -\infty$ as $n \rightarrow \infty$.
- If $\alpha < x_1 < \gamma$, prove that $x_n \rightarrow \beta$ as $n \rightarrow \infty$.
- If $\gamma < x_1$, prove that $x_n \rightarrow +\infty$ as $n \rightarrow \infty$.

Thus β can be located by this method, but α and γ cannot.

Solution. We shall make use of the auxiliary functions

$$g(x) = f(x) - x = \frac{x^3 + 1}{3} - x$$

and

$$h(x) = \begin{cases} \frac{g(x) - g(\beta)}{x - \beta}, & x \neq \beta, \\ g'(\beta) & x = \beta, \end{cases}$$

i.e., $g(x) = \frac{x^3 + 1}{3} - x$. We observe that the fixed points of f are the zeros of g . Since $g(-2) = -\frac{1}{3} < 0$, $g(-1) = 1 > 0$, $g(0) = \frac{1}{3} > 0$, $g(1) = -\frac{1}{3} < 0$, and $g(2) = 1 > 0$, the intermediate value theorem shows that α , β , and γ are located in the intervals they are asserted to be in.

Since $g(\alpha) = g(\beta) = g(\gamma) = 0$, it follows that $h(\alpha) = h(\gamma) = 0$. Since h is a quadratic function, it has only the two zeros α and γ , and in particular $h(x)$ is negative for $\alpha < x < \gamma$. Now the minimum value of $h(x)$ is attained at $x = -\frac{\beta}{2}$, and this minimum value is c , where $c = \frac{\beta^2}{4} - 1$. Thus $-1 < c < 0$. In particular, for $\alpha < x < \gamma$ there is a number $r \in (0, 1)$ such that

$$f(x) - x = r(\beta - x),$$

i.e.,

$$f(x) - \beta = s(x - \beta),$$

where $s = 1 - r$ is also in the interval $(0, 1)$. This means that $f(x) - \beta$ and $x - \beta$ both have the same sign, but that $|f(x) - \beta| < |x - \beta|$. Thus $f(x)$ is always between β and x . Therefore the sequence $\{x_n\}$ is monotonic and converges to a fixed point in the interval whose endpoints are x_1 and β . Since the only fixed point in this interval is β , the sequence must converge to β .

If $x < \alpha$ (resp. $x > \gamma$), it is easy to see that $f(x) < x$ (resp. $f(x) > x$). Thus the sequence $\{x_n\}$ is monotonically decreasing (resp. increasing), and hence either tends to $-\infty$ (resp. $+\infty$) or converges to a fixed point δ in the interval $(-\infty, x_1)$ (resp. $(x_1, +\infty)$). Since there are no fixed points in this interval, it follows that $x_n \rightarrow -\infty$ (resp. $x_n \rightarrow +\infty$).

Exercise 5.24 The process described in part (c) of Exercise 22 can of course also be applied to functions that map $(0, \infty)$ to $(0, \infty)$.

Fix some $\alpha > 1$, and put

$$f(x) = \frac{1}{2} \left(x + \frac{\alpha}{x} \right), \quad g(x) = \frac{\alpha + x}{1 + x}.$$

Both f and g have $\sqrt{\alpha}$ as their only fixed point in $(0, \infty)$. Try to explain, on the basis of properties of f and g , why the convergence in Exercise 16, Chap. 3, is so much more rapid than it is in Exercise 17. (Compare f' and g' , draw the zig-zag suggested in Exercise 22.)

Do the same when $0 < \alpha < 1$.

Solution. We recall that in Chap. 3 we proved that the first function leads to $|x_n - \sqrt{\alpha}| \leq Ar^{2^n}$ for some $r \in (0, 1)$, while the second leads only to $|x_n - \sqrt{\alpha}| \leq Ar^n$. The exact values of A and r depend on α and x_1 .

The best explanation of the difference between the two methods is that

$$\begin{aligned} f(x) - \sqrt{\alpha} &= \frac{1}{2} \left(1 - \frac{\sqrt{\alpha}}{x} \right) (x - \sqrt{\alpha}), \\ g(x) - \sqrt{\alpha} &= \frac{1 - \sqrt{\alpha}}{1 + x} (x - \sqrt{\alpha}). \end{aligned}$$

The first of these makes it plain that if $x > \sqrt{\alpha}$, the same will be true of $f(x)$, though $f(x)$ will be closer to α than x by a factor that is at most $\frac{1}{2}$ and tends to zero as x tends to $\sqrt{\alpha}$, i.e., the relative improvement in accuracy itself improves as the recursion proceeds. The second equality shows that $g(x) - \alpha$ is on the opposite side of $\sqrt{\alpha}$ from x if $\alpha > 1$, though closer by a factor that is at least the absolute value of $\frac{1 - \sqrt{\alpha}}{1 + x_1}$. Hence the relative improvement in accuracy as the recursion proceeds is limited.

In terms of the zigzag pattern, when we use g , the zigzag keeps circulating around the point of intersection of the graph of g and the line $y = x$ instead of moving steadily toward it in a staircase pattern.

When $0 < \alpha < 1$, the zigzag does stay on one side of the point of intersection of the two curves. However, the relative improvement is still at best a factor of $\frac{1 - \sqrt{\alpha}}{2}$ when x is close to $\sqrt{\alpha}$.

Exercise 5.25 Suppose f is twice differentiable in $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f'(x) \leq M$ for all $x \in [a, b]$. Let ξ be the unique point in (a, b) at which $f(\xi) = 0$.

Complete the details in the following outline of *Newton's method* for computing ξ .

(a) Choose $x_1 \in (\xi, b)$, and define x_n by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Interpret this geometrically, in terms of a tangent to the graph of f .

(b) Prove that $x_{n+1} < x_n$, and that

$$\lim_{n \rightarrow \infty} x_n = \xi.$$

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

(d) If $A = M/2\delta$, deduce that

$$0 \leq x_{n+1} - \xi \leq \frac{1}{A}[A(x_1 - \xi)]^{2^n}.$$

(Compare with Exercises 16 and 18, Chap. 3.)

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

How does $g'(x)$ behave for x near ξ ?

(f) Put $f(x) = x^{1/3}$ on $(-\infty, \infty)$ and try Newton's method. What happens?

Solution. We remark at the outset that x_1 can be found by trying $z_0 = \frac{a+b}{2}$. If $f(z_0) > 0$, take $x_1 = z_0$. Otherwise let $z_{n+1} = (b + z_n)/2$, and let x_1 be the first z_n for which $f(z_n) > 0$. (In a finite number of steps we must reach such a point since $z_n \uparrow b$ and $f(b) > 0$.)

(a) The tangent line to the graph of f at the point x_n has the equation $y - f(x_n) = f'(x_n)(x - x_n)$. Setting $y = 0$ in this equation and solving for x gives $x = x_{n+1}$. Thus the interpretation of Newton's method is that we approximate the point where the graph of f intersects the x -axis by the point at which its tangent line at $(x_n, f(x_n))$ intersects the x -axis.

(b) We can assume by induction that $f(x_n) > 0$, and hence, since $f'(x_n) > 0$, it follows immediately that $x_{n+1} < x_n$. Notice that there exists c between x_n and x_{n+1} such that $f(x_{n+1}) = f(x_n) - f'(c)(x_n - x_{n+1}) > f(x_n) - f'(x_n)(x_n - x_{n+1}) = 0$ since $f'(c) < f'(x_n)$ and $x_n - x_{n+1} > 0$. Thus it follows that $\xi < x_{n+1} < x_n$. Hence $\{x_n\}$ converges to a limit η satisfying $\eta \geq \xi$. Now, however, we have

$$\eta = \eta - \frac{f(\eta)}{f'(\eta)},$$

from which it follows that $f(\eta) = 0$, i.e., $\eta = \xi$.

(c) The required equality can be written as

$$x_n - \xi - \frac{f(x_n)}{f'(x_n)} = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2,$$

while Taylor's theorem can be written as

$$f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2.$$

Since $f(\xi) = 0$, it is clear that these two equations are equivalent.

(d) Since $0 \leq f''(t_n) \leq M$ and $f'(x_n) > \delta$, we have

$$0 \leq x_{n+1} - \xi \leq A(x_n - \xi)^2.$$

In particular

$$0 \leq x_2 - \xi \leq A(x_1 - \xi)^2 = \frac{1}{A}[A(x_1 - \xi)]^2,$$

and then an easy induction gets the general result.

We found this kind of convergence in Exercises 16 and 18 of Chap. 3 with the recursion relation

$$x_{n+1} = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1}.$$

We now recognize this recursion as Newton's method for the function $f(x) = x^p - \alpha$ on the interval $[1, \sqrt[p]{\alpha} + 1]$. Exercise 16 of Chap. 2 was the special case $p = 2$.

(e) Obviously the equation $g(x) = x$ is equivalent to the equation $f(x) = 0$.

Since $g'(x) = \frac{f(x)f''(x)}{[f'(x)]^2}$, we see that $g'(x)$ tends to zero as x tends to ξ , i.e., the graph of $g(x)$ meets the line $y = x$ at a 45° degree angle at the point (ξ, ξ) .

(f) The fixed point of $f(x)$ is $x = 0$. However $f'(x) \rightarrow \infty$ as $x \rightarrow 0$, and $f'(0)$ does not exist. This destroys the convergence of Newton's method. In fact, if $x_n \neq 0$, then $x_{n+1} = -2x_n$, so that x_n oscillates wildly: $\limsup x_n = +\infty$, $\liminf x_n = -\infty$.

Exercise 5.26 Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \leq A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$. *Hint:* Fix $x_n \in [a, b]$, let

$$M_0 = \sup |f(x)|, \quad M_1 = \sup |f'(x)|$$

for $a \leq x \leq x_0$. For any such x ,

$$|f(x)| \leq M_1(x_0 - a) \leq A(x_0 - a)M_n.$$

Hence $M_0 = 0$ if $A(x_0 - a) \leq 1$. That is, $f = 0$ on $[a, x_0]$. Proceed.

Solution. If we anticipate the fundamental result that the function $f(x) = e^x$ satisfies $f'(x) = f(x)$, Exercise 2 above yields the result that $\ln x$ is differentiable and has derivative $\frac{1}{x}$. Hence by the chain rule for any positive differentiable function $f(x)$ the function $g(x) = \ln f(x)$ is differentiable and $g'(x) = \frac{f'(x)}{f(x)}$. (Unfortunately this fundamental result is not proved until Chapter 7, so we shall

just have to wait. However, since certain other functions such as $\sin x$ and $\cos x$ have been introduced without any formal definition, and their derivatives have been assumed known, we might as well continue along this line of reasoning.)

Now suppose there is an interval $(c, d) \subset [a, b]$ such that $f(c) = 0$ but $f(x) \neq 0$ for $c < x < d$. By passing to consideration of $-f(x)$ if necessary, we can assume $f(x) > 0$ for $c < x < d$. The function $g(x) = \ln f(x)$ is then defined for $c < x < d$, and its derivative satisfies

$$|g'(x)| = \left| \frac{f'(x)}{f(x)} \right| \leq A.$$

The mean-value theorem then implies that

$$g(x) \geq g\left(\frac{c+d}{2}\right) - A\left(\frac{d-c}{2}\right)$$

for all $x \in (c, d)$. But this is a contradiction, since $g(x) \rightarrow -\infty$ as $x \rightarrow c$.

This finishes the proof, except that it assumes we know the derivative of e^x . If we don't assume that, we have to fall back on the hint. In that case, let $x_0 = a + \frac{1}{2A}$, and let $M_0 = \sup\{|f(x)| : a \leq x \leq x_0\}$. We then have

$$|f(x)| \leq M_1(x-a) \leq AM_0(x_0-a) = \frac{1}{2}M_0$$

for all $x \in [a, x_0]$. But by definition of M_0 this implies $M_0 \leq \frac{1}{2}M_0$, so that $M_0 \leq 0$, i.e., $M_0 = 0$. We now start over with a replaced by x_0 , $x_1 = x_0 + \frac{1}{2A}$.

In a finite number of steps, we will have $b < x_n + \frac{1}{2A}$, so that $f(x) = 0$ for $a \leq x \leq b$.

Exercise 5.27 Let ϕ be a real function defined on a rectangle R in the plane, given by $a \leq x \leq b$, $\alpha \leq y \leq \beta$. A *solution* of the initial-value problem

$$y' = \phi(x, y), \quad y(a) = c \quad (\alpha \leq c \leq \beta)$$

is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c$, $\alpha \leq f(x) \leq \beta$, and

$$f'(x) = \phi(x, f(x)) \quad (a \leq x \leq b).$$

Prove that such a problem has at most one such solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

whenever $(x, y_1) \in R$ and $(x, y_2) \in R$.

Hint: Apply Exercise 26 to the difference of two solutions. Note that this uniqueness theorem does not hold for the initial-value problem

$$y' = y^{1/2}, \quad y(0) = 0,$$

which has two solutions: $f(x) = 0$ and $f(x) = x^2/4$. Find all other solutions.

Solution. Following the hint, we observe that if $f(x) = f_2(x) - f_1(x)$, then

$$\begin{aligned} |g'(x)| &= |f_2'(x) - f_1'(x)| \\ &= |\phi(x, f_2(x)) - \phi(x, f_1(x))| \\ &\leq A|f_2(x) - f_1(x)| \\ &= A|g(x)|. \end{aligned}$$

By the initial condition $g(a) = f_2(a) - f_1(a) = c - c = 0$. Hence by the preceding exercise $g(x) = 0$ for all $x \in [a, b]$.

As for the equation $y' = \sqrt{y}$, if $f(x)$ is a solution and $f(x) > 0$ on an interval (a, b) , while $f(a) = 0$, we observe that $g(x) = \sqrt{f(x)}$ satisfies $g'(x) = \frac{1}{2}(f(x))^{-1/2}f'(x) = \frac{1}{2}$, so that for some constant c we have $g(x) = \frac{1}{2}(x + c)$. Thus

$$f(x) = (g(x))^2 = \frac{1}{4}(x + c)^2.$$

Since $f(a) = 0$, it follows that $c = -a$, i.e., $f(x) = \frac{(x - a)^2}{4}$. Thus the only possible solutions are

$$f(x) = \begin{cases} 0, & 0 \leq x \leq a, \\ \frac{(x - a)^2}{4}, & a \leq x. \end{cases}$$

Here $a \geq 0$ is arbitrary.

Exercise 5.28 Formulate and prove an analogous uniqueness theorem for systems of differential equations of the form

$$y_j' = \phi_j(x, y_1, \dots, y_k), \quad y_j = c_j \quad (j = 1, \dots, k).$$

Note that this can be rewritten in the form

$$\mathbf{y}' = \boldsymbol{\phi}(x, \mathbf{y}), \quad \mathbf{y}(a) = \mathbf{c},$$

where $\mathbf{y} = (y_1, \dots, y_k)$ ranges over a k -cell, $\boldsymbol{\phi}$ is the mapping of a $(k + 1)$ -cell into the Euclidean k -space whose components are the functions ϕ_1, \dots, ϕ_k , and \mathbf{c} is the vector (c_1, \dots, c_k) . Use Exercise 26 for vector-valued functions.

Solution. The result is the following:

Let $\boldsymbol{\phi}$ be a vector-valued function defined on a $(k + 1)$ -cell $D = [a, b] \times C$ in R^{k+1} whose range is contained in R^k , and suppose that there exists a constant A such that

$$|\boldsymbol{\phi}(x, \mathbf{y}_2) - \boldsymbol{\phi}(x, \mathbf{y}_1)| \leq A|\mathbf{y}_2 - \mathbf{y}_1|$$

for all $\mathbf{y}_1 \in C$, $\mathbf{y}_2 \in C$. Then the initial-value problem

$$\mathbf{y}' = \phi(x, \mathbf{y}) \quad \mathbf{y}(a) = \mathbf{c}$$

has at most one solution $\mathbf{y}: [a, b] \rightarrow C$.

The main tool needed to prove this result is the analogue of Exercise 26 for vector-valued functions, which does hold. Indeed the proof is identical, considering that the original proof depends only on the inequality $|f(d) - f(c)| \leq |f'(r)|(d - c)$ for some $r \in (c, d)$, and this inequality is certainly valid for vector-valued functions. Once that result is obtained, the preceding exercise can be applied verbatim.

Exercise 5.29 Specialize Exercise 28 by considering the system

$$\begin{aligned} y'_j &= y_{j+1} \quad (j = 1, \dots, k-1), \\ y'_k &= f(x) - \sum_{j=1}^k g_j(x)y_j, \end{aligned}$$

where f, g_1, \dots, g_k are continuous real functions on $[a, b]$, and derive a uniqueness theorem for solutions of the equation

$$y^{(k)} + g_k(x)y^{(k-1)} + \dots + g_2(x)y' + g_1(x)y = f(x),$$

subject to initial conditions

$$y(a) = c_1, \quad y'(a) = c_2, \quad \dots, \quad y^{(k-1)}(a) = c_k.$$

Solution. We let $\mathbf{y} = (y_1, y_2, y_3, \dots, y_k) = (y, y', y'', \dots, y^{(k-1)})$ and $\phi(x, \mathbf{y}) = (y_2, y_3, \dots, y_k, f(x) - \sum_{j=1}^k g_j(x)y_j)$. We then observe that if $\mathbf{y}_i = (y_{i1}, \dots, y_{ik})$, then

$$|\phi(x, \mathbf{y}_2) - \phi(x, \mathbf{y}_1)| = \left| \left(y_{22} - y_{12}, y_{23} - y_{13}, \dots, \sum_{j=1}^k g_j(x)(y_{1j} - y_{2j}) \right) \right|.$$

If $M = \sup\{|g_j(x)| : a \leq x \leq b, 1 \leq j \leq k\}$, we then have

$$|\phi(x, \mathbf{y}_2) - \phi(x, \mathbf{y}_1)| \leq (M+1) \sum_{j=1}^k |y_{2j} - y_{1j}| \leq k(M+1)|\mathbf{y}_2 - \mathbf{y}_1|.$$

This provides the hypothesis of the theorem for any $(k+1)$ -cell $[a, b] \times C$ whatsoever in R^{k+1} . Hence there is at most one solution to this initial-value problem.

Chapter 6

The Riemann–Stieltjes Integral

Exercise 6.1 Suppose α increases on $[a, b]$, $a \leq x_0 \leq b$, α is continuous at x_0 , $f(x_0) = 1$, and $f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and that $\int f d\alpha = 0$.

Solution. Let $\varepsilon > 0$, and let δ be such that $|\alpha(x) - \alpha(x_0)| < \varepsilon$ if $|x - x_0| < \delta$. Now consider any partition $a = t_0 < t_1 < \cdots < t_n = b$ with $n \geq 2$ such that $|t_i - t_{i-1}| < \frac{\delta}{2}$. There exists an index i such that $t_{i-1} < x_0 < t_{i+1}$ (there may possibly be 2 such indices). We then have, for any choice of $t_0^*, t_1^*, \dots, t_n^*$,

$$\begin{aligned} \left| \sum_{j=1}^n f(t_j^*) (\alpha(t_j) - \alpha(t_{j-1})) \right| &\leq |f(t_i^*)| |\alpha(t_i) - \alpha(t_{i-1})| + \\ &\quad + |f(t_{i+1}^*)| |\alpha(t_{i+1}) - \alpha(t_i)| \\ &\leq \alpha(t_{i+1}) - \alpha(t_{i-1}) < \varepsilon. \end{aligned}$$

By definition of the Riemann–Stieltjes integral, this means that $f \in \mathcal{R}(\alpha)$ and $\int f d\alpha = 0$.

Exercise 6.2 Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x) dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare this with Exercise 1.)

Solution. Suppose $f(x_0) \neq 0$ for some $x_0 \in [a, b]$. Since $f(x)$ is continuous on $[a, b]$ and $\frac{f(x_0)}{2} > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \frac{f(x_0)}{2}$ for all $x \in [a, b]$ such that $|x - x_0| < \delta$. Let $\eta = \min(\delta, \max(x_0 - a, b - x_0))$, so that $\eta > 0$. Let I be the interval $[x_0 - \eta, x_0]$ if it is contained in $[a, b]$; otherwise let $I = [x_0, x_0 + \eta]$. Whichever is the case, $I \subseteq [a, b]$ and $f(x) =$

$f(x_0) + (f(x) - f(x_0)) \geq f(x_0) - |f(x) - f(x_0)| > \frac{f(x_0)}{2}$ for all $x \in I$. The functions $f_1(x)$ and $f_2(x)$ defined as

$$f_1(x) = \begin{cases} f(x), & x \in I, \\ 0, & x \notin I, \end{cases} \quad f_2(x) = \begin{cases} f(x), & x \notin I, \\ 0, & x \in I, \end{cases}$$

are both nonnegative, bounded, and continuous except possibly at the two endpoints of the interval I . They are therefore both Riemann-integrable. Consideration of Riemann sums shows that

$$\int_a^b f_1(x) dx \geq \eta \frac{\varepsilon}{2},$$

and

$$\int_a^b f_2(x) dx \geq 0,$$

It therefore follows that

$$\int_a^b f(x) dx = \int_a^b f_1(x) dx + \int_a^b f_2(x) dx \geq \eta \frac{\varepsilon}{2} > 0,$$

contradicting the hypothesis that $\int_a^b f(x) dx = 0$.

Exercise 6.3 Define three functions $\beta_1, \beta_2, \beta_3$ as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$ and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1, 1]$.

(a) Prove that $f \in \mathcal{R}(\beta_1)$ if and only if $f(0-) = f(0)$ and that then

$$\int f d\beta_1 = f(0).$$

(b) State and prove a similar result for β_2 .

(c) Prove that $f \in \mathcal{R}(\beta_3)$ if and only if f is continuous at 0.

(d) If f is continuous at 0, prove that

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0).$$

Solution. Let $t_0 < t_1 < \cdots < t_{n-1} < t_n$ be any partition of any interval containing 0. Since the upper Riemann-Stieltjes sums become smaller and the lower ones larger when a point is added to any partition, in deciding whether a function is integrable or not, we may assume that 0 is one of the points of

the partition. Let k be the index such that $t_k = 0$, so that the upper and lower Riemann-Stieltjes sums

$$\sum_{i=1}^n M_i(\beta_j(t_i) - \beta_j(t_{i-1})), \quad j = 1, 2, 3,$$

and

$$\sum_{i=1}^n m_i(\beta_j(t_i) - \beta_j(t_{i-1})), \quad j = 1, 2, 3,$$

are respectively M_k and m_k , M_{k-1} and m_{k-1} , $\frac{M_{k-1} + M_k}{2}$ and $\frac{m_{k-1} + m_k}{2}$.

(a) Since $m_k \leq f(x) \leq M_k$ for $0 \leq x \leq t_{k+1}$ in the first case, the sets of upper and lower sums contain elements arbitrarily near to each other if and only if for each ε there is a partition with $M_k - m_k < \varepsilon$. If such a partition exists, let $\delta = t_{k+1}$. Then we have $|f(x) - f(0)| \leq M_k - m_k < \varepsilon$ for $0 \leq x \leq \delta$, and hence $\lim_{x \rightarrow 0+} f(x) = f(0)$. Conversely, if $\lim_{x \rightarrow 0+} f(x) = f(0)$, then for any ε , let $\delta > 0$ be such that $|f(x) - f(0)| < \varepsilon$ if $0 < x < \delta$, and let P be a partition with $t_k = 0$, $t_{k+1} < \delta$. It is then clear that both upper and lower Riemann sums differ from $f(0)$ by less than ε , i.e., $\int f d\beta_1 = f(0)$.

(b) $f \in \mathcal{R}(\beta_2)$ if and only if $\lim_{x \rightarrow 0-} f(x) = f(0)$ and if this condition holds, then $\int f d\beta_2 = f(0)$. The proof is identical to the proof just given, except that “+” is replaced by “-.”

(c) In the third case, the upper and lower Riemann-Stieltjes sums differ by $\frac{(M_k - m_k) + (M_{k-1} + m_{k-1})}{2}$. If, given ε , there exists a partition containing 0 for which this difference is less than $\frac{\varepsilon}{2}$, let $\delta = \min(t_{k+1}, -t_{k-1})$. Then for $-\delta \leq x \leq \delta$ we certainly have

$$|f(x) - f(0)| \leq \max\left(\frac{M_k - m_k}{2}, \frac{M_{k-1} - m_{k-1}}{2}\right) \leq M_k - m_k + M_{k-1} - m_{k-1} < \varepsilon,$$

so that f is continuous at 0. The same argument shows that in this case

$$\int f d\beta_3 = f(0).$$

(d) This result is contained in (a)–(c).

Exercise 6.4 If $f(x) = 0$ for all irrational x , $f(x) = 1$ for all rational x , prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.

Solution. Every upper Riemann sum equals $b - a$, and every lower Riemann sum equals 0. Hence the set of upper sums and the set of lower sums do not have a common bound.

Exercise 6.5 Suppose f is a bounded real function on $[a, b]$ and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$? Does the answer change if we assume that $f^3 \in \mathcal{R}$?

Solution. The integrability of f^2 does not imply the integrability of f . For example, one could let $f(x) = -1$ if x is irrational and $f(x) = 1$ if x is rational. Then every upper Riemann sum of f is $b - a$ and every lower sum is $a - b$. However, f^2 , being the constant function 1, is integrable.

The integrability of f^3 does imply the integrability of f , by Theorem 6.11 with $\varphi(u) = \sqrt[3]{u}$.

Exercise 6.6 Let P be the Cantor set constructed in Sec. 2.44. Let f be a bounded real function on $[0, 1]$ which is continuous at every point outside P . Prove that $f \in \mathcal{R}$ on $[0, 1]$. [*Hint:* P can be covered by finitely many segments whose total length can be made as small as desired. Proceed as in Theorem 6.10.]

Solution. Let $M = \sup\{|f(x)| : a \leq x \leq b\}$, and let $\varepsilon > 0$ be given. Cover P by a finite collection of open intervals $O = \bigcup_{i=1}^k (a_i, b_i)$ such that $\sum (b_i - a_i) < \frac{\varepsilon}{4M}$. Let $\theta = \inf\{|x - y| : x \in P, y \in [a, b] \setminus O\}$. Since x and y range over disjoint compact sets, θ is a positive number. On the compact set $E = \{x : d(x, P) \geq \frac{1}{2}\theta\}$ the function f is uniformly continuous. Let $\delta > 0$ be such that $|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}$ if $x, y \in E$ and $|x - y| < \delta$. Then consider any partition $\{t_j\}$ of $[a, b]$ with $\max(t_j - t_{j-1}) < \min(\delta, \frac{1}{2}\theta)$. The difference between the upper and lower Riemann sums for this partition can be expressed as two sums:

$$\sum (M_j - m_j)(t_j - t_{j-1}) = \Sigma_1 + \Sigma_2,$$

where Σ_1 contains all the terms for which $[t_{j-1}, t_j]$ is contained in E and Σ_2 all the other terms. It is then obvious that

$$\Sigma_1 < \frac{\varepsilon}{2(b-a)} \sum (t_j - t_{j-1}) \leq \frac{\varepsilon}{2},$$

and, since each interval $[t_{j-1}, t_j]$ that occurs in Σ_2 is contained in O ,

$$\Sigma_2 < 2M \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}.$$

Therefore the upper and lower Riemann sums for any such partition differ by less than ε , and so f is Riemann integrable.

Exercise 6.7 Suppose f is a real function on $[0, 1]$ and $f \in \mathcal{R}$ on $[c, 1]$ for every $c > 0$. Define

$$\int_0^1 f(x) dx = \lim_{c \rightarrow 0^+} \int_c^1 f(x) dx$$

if this limit exists (and is finite).

(a) If $f \in \mathcal{R}$ on $[0, 1]$ show that this definition of the integral agrees with the old one.

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f .

Solution. (a) Suppose $f \in \mathcal{R}$ on $[0, 1]$. Let $\varepsilon > 0$ be given, and let $M = \sup\{|f(x)| : 0 \leq x \leq 1\}$. Let $c \in \left(0, \frac{\varepsilon}{4M}\right]$ be fixed, and consider any partition of $[0, 1]$ containing c for which the upper and lower Riemann sums $\sum M_j(t_j - t_{j-1})$ and $\sum m_j(t_j - t_{j-1})$ of f differ by less than $\frac{\varepsilon}{4}$. Then the partition of $[c, 1]$ formed by the points of this partition that lie in this interval certainly has the property that its upper and lower Riemann sums $\sum' M_j(t_j - t_{j-1})$ and $\sum' m_j(t_j - t_{j-1})$ differ by less than $\frac{\varepsilon}{4}$. Moreover, the terms of the original upper and lower Riemann sums *not* found in the sums for the smaller interval amount to less than $\frac{\varepsilon}{4}$. In short, we have shown that for $c < \frac{\varepsilon}{4M}$ and a suitable partition containing c ,

$$\sum M_j(t_j - t_{j-1}) - \frac{\varepsilon}{4} < \int_0^1 f(x) dx \leq \sum m_j(t_j - t_{j-1}) + \frac{\varepsilon}{4}$$

and

$$\sum' M_j(t_j - t_{j-1}) - \frac{\varepsilon}{4} < \int_c^1 f(x) dx < \sum' m_j(t_j - t_{j-1}) + \frac{\varepsilon}{4}.$$

Moreover, we have also shown that

$$\left| \sum M_j(t_j - t_{j-1}) - \sum' M_j(t_j - t_{j-1}) \right| < \frac{\varepsilon}{4}$$

and

$$\left| \sum m_j(t_j - t_{j-1}) - \sum' m_j(t_j - t_{j-1}) \right| < \frac{\varepsilon}{4}.$$

combining these inequalities, we find that

$$\left| \int_0^1 f(x) dx - \int_c^1 f(x) dx \right| < \varepsilon$$

if $0 < c < \frac{\varepsilon}{4M}$.

(b) Let

$$f(x) = (-1)^n(n+1)$$

for $\frac{1}{n+1} < x \leq \frac{1}{n}$, $n = 1, 2, \dots$. Then if $\frac{1}{N+1} \leq c \leq \frac{1}{N}$ we have

$$\int_c^1 f(x) dx = (-1)^N(N+1)\left(\frac{1}{N} - c\right) + \sum_{k=1}^{N-1} \frac{(-1)^k}{k}.$$

Since $0 \leq \frac{1}{N} - c \leq \frac{1}{N} - \frac{1}{N+1} = \frac{1}{N(N+1)}$, the first term on the right-hand side tends to zero as $c \downarrow 0$, while the sum approaches $\ln 2$. Hence this integral approaches a limit. However,

$$\int_c^1 |f(x)| dx = (N+1) \left(\frac{1}{N} - c \right) + \sum_{k=1}^{N-1} \frac{1}{k},$$

and in this case the first term on the right-hand side tends to zero as $c \downarrow 0$, while the sum tends to infinity.

Exercise 6.8 Suppose $f \in \mathcal{R}$ on $[a, b]$ for every $b > a$, where a is fixed. Define

$$\int_a^\infty f(x) dx = \lim_{x \rightarrow \infty} \int_a^b f(x) dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after f has been replaced by $|f|$, it is said to converge *absolutely*.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^\infty f(x) dx$$

converges if and only if

$$\sum_{n=1}^{\infty} f(n)$$

converges. (This is the so-called “integral test” for convergence of series.)

Solution. Since both the series and the integral are increasing functions of their upper limits, it suffices to show that they are bounded together. Define $f(x) = f(1)$ for $0 \leq x \leq 1$. Then consider a partition of $[0, n]$ consisting of the

$n+1$ points $0, 1, 2, \dots, n$. The upper Riemann sum for this partition is $\sum_{k=0}^{n-1} f(k)$

and the lower Riemann sum is $\sum_{k=1}^n f(k)$. Hence we have

$$\sum_{k=1}^n f(k) \leq \int_0^n f(x) dx = f(0) + \int_1^n f(x) dx \leq \sum_{k=0}^{n-1} f(k) = f(0) + \sum_{k=1}^{n-1} f(k).$$

This shows that

$$-f(0) + \sum_{k=1}^n f(k) \leq \int_1^n f(x) dx \leq \sum_{k=1}^{n-1} f(k),$$

and hence the sum and the integral converge or diverge together.

Exercise 6.9 Show that integration by parts can sometimes be applied to the “improper” integrals defined in Exercises 7 and 8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance show that

$$\int_0^{\infty} \frac{\cos x}{1+x} dx = \int_0^{\infty} \frac{\sin x}{(1+x)^2} dx.$$

Solution. Without striving for ultimate generality we can get the main ideas in the following theorem:

Theorem. Let $f(x)$ and $g(x)$ be continuously differentiable functions defined on $[a, \infty)$ such that $\lim_{b \rightarrow \infty} f(b)g(b)$ exists and the integral $\int_a^{\infty} f(x)g'(x) dx$ converges. Then $\int_a^{\infty} f'(x)g(x) dx$ converges and

$$\int_a^{\infty} f'(x)g(x) dx = \lim_{b \rightarrow \infty} [f(b)g(b) - f(a)g(a)] - \int_a^{\infty} f(x)g'(x) dx.$$

Proof. For each finite value of b larger than a the standard rule for integration by parts gives

$$\int_a^b f'(x)g(x) dx = [f(b)g(b) - f(a)g(a)] - \int_a^b f(x)g'(x) dx.$$

The hypotheses of the theorem guarantee that the limit on the right exists. Therefore, by definition, the integral on the left converges.

Applying this result with $f(x) = \sin x$, $g(x) = \frac{1}{1+x}$, we find, since $f(0)g(0) = 0$ and $\lim_{b \rightarrow \infty} f(b)g(b) = 0$, while $\int_0^{\infty} f(x)g'(x) dx$ converges absolutely, that

$$\int_0^{\infty} \frac{\cos x}{1+x} dx = \int_0^{\infty} \frac{\sin x}{(1+x)^2} dx.$$

Exercise 6.10 Let p and q be positive real number ssuch that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

(b) if $f \in \mathcal{R}(\alpha)$, $g \in \mathcal{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha,$$

then

$$\int_a^b fg d\alpha \leq 1.$$

(c) If f and g are complex functions in $\mathcal{R}(\alpha)$, then

$$\left| \int_a^b fg d\alpha \right| \leq \left\{ \int_a^b |f|^p d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q d\alpha \right\}^{1/q}.$$

This is *Hölder's inequality*. When $p = q = 2$ it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

(d) Show that Hölder's inequality is also true for the "improper" integrals described in Exercises 7 and 8.

Solution. (a) The inequality is obvious if either $u = 0$ or $v = 0$, and equality holds in that case if and only if $u = v = 0$. Hence assume $v > 0$. Keep v fixed. The inequality implies that $p > 1$ and $q > 1$, and hence the function $\varphi(u) = \frac{u^p}{p} + \frac{v^q}{q} - uv$ satisfies

$$\lim_{u \rightarrow +\infty} \varphi(u) = +\infty.$$

We also have $\varphi'(0) = -v < 0$. Hence the function $\varphi(u)$ has a minimum at some point u_0 on $(0, \infty)$ at which $0 = \varphi'(u_0) = u_0^{p-1} - v$, i.e., $u_0 = v^{\frac{1}{p-1}} = v^{q-1}$ and $u_0^p = v^q$. Note that $\varphi(u_0) = \frac{v^q}{p} + \frac{v^q}{q} - v^{q-1}v = v^q - v^q = 0$. Since this point is the only critical point for φ , we have $\varphi(u) > 0$ for all $u \neq u_0$, as required.

(b) Simply integrate the inequality

$$f(x)g(x) \leq \frac{f(x)^p}{p} + \frac{g(x)^q}{q}.$$

(c) The inequality is obviously equality if either of the two integrals on the right-hand side is zero. For the vanishing of, say $\int_a^b |f|^p d\alpha$ implies the vanishing of $\int_a^b M|f| d\alpha$ and hence the vanishing of $\int_a^b |g||f| d\alpha$ if $|g(x)| \leq M$ for all x .

Hence we now assume that $\int_a^b |f|^p d\alpha > 0$ and $\int_a^b |g|^q d\alpha > 0$. In part (b) we replace $f(x)$ by $\frac{|f(x)|}{(\int_a^b |f|^p d\alpha)^{1/p}}$ and $g(x)$ by $\frac{|g(x)|}{(\int_a^b |g|^q d\alpha)^{1/q}}$. We then need only invoke the inequality $\left| \int_a^b h d\alpha \right| \leq \int_a^b |h| d\alpha$.

(d) The inequality holds on each finite interval. If either of the factors on the right-hand side diverges as $b \rightarrow \infty$, the inequality is obvious. If they both converge, it follows that the left-hand side converges absolutely, and to a limit not larger than the limit of the right-hand side.

Exercise 6.11 Let α be a fixed increasing function on $[a, b]$. For $u \in \mathcal{R}(\alpha)$ define

$$\|u\|_2 = \left\{ \int_a^b |u|^2 d\alpha \right\}^{1/2}.$$

Suppose f, g , and $h \in \mathcal{R}(\alpha)$, and prove the triangle inequality

$$\|f - h\|_2 \leq \|f - g\|_2 + \|g - h\|_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Solution. We have

$$\begin{aligned} \|f - h\|_2^2 &= \int_a^b |f - h|^2 d\alpha \\ &= \int_a^b |(f - g) + (g - h)|^2 d\alpha \\ &= \int_a^b |f - g|^2 d\alpha + 2 \int_a^b |f - g| |g - h| d\alpha + \int_a^b |g - h|^2 d\alpha \\ &\leq \|f - g\|_2^2 + 2\|f - g\|_2 \|g - h\|_2 + \|g - h\|_2^2 \\ &= (\|f - g\|_2 + \|g - h\|_2)^2, \end{aligned}$$

from which the desired inequality follows when square roots are taken.

Exercise 6.12 With the notations of Exercise 11, suppose $f \in \mathcal{R}(\alpha)$ and $\varepsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $\|f - g\|_2 < \varepsilon$.

Hint: Let $P = \{x_0, \dots, x_n\}$ be a suitable partition of $[a, b]$, define

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if $x_{i-1} \leq t \leq x_i$.

Solution. Since $g(t)$ is defined on $[x_{i-1}, x_i]$ as the weighted average of the values of $f(x)$ at the endpoints, the weights being proportional to the distances from t to the endpoints, it is clear that $g(t)$ is piecewise linear, hence continuous. For the same reason the maximum value of the function $h = |g - f|$ on the interval $[x_{i-1}, x_i]$ will be at most $M_i - m_i$ where M_i and m_i are the maximum

and minimum values of f on this interval. Let M be the maximum of $|f(x)|$ for $a \leq x \leq b$. If the partition is chosen so that

$$\sum (M_i - m_i)[\alpha(t_i) - \alpha(t_{i-1})] < \frac{\varepsilon^2}{2M},$$

then we will have

$$\sum (M_i - m_i)^2 [\alpha(t_i) - \alpha(t_{i-1})] \leq 2M \sum (M_i - m_i)[\alpha(t_i) - \alpha(t_{i-1})] < \varepsilon^2,$$

and hence the upper Riemann integral for $|g - f|^2$ for this partition will also be less than ε^2 . Therefore $\|g - f\|_2 < \varepsilon$, as required.

Exercise 6.13 Define

$$f(x) = \int_x^{x+1} \sin(t^2) dt.$$

(a) Prove that $|f(x)| < 1/x$ if $x > 0$.

Hint: Put $t^2 = u$ and integrate by parts to show that $f(x)$ is equal to

$$\frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du.$$

Replace $\cos u$ by -1 .

(b) Prove that

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x),$$

where $|r(x)| < c/x$, and c is constant.

(c) Find the upper and lower limits of $xf(x)$ as $x \rightarrow \infty$.

(d) Does $\int_0^\infty \sin(t^2) dt$ converge?

Solution. (a) This inequality is obvious if $0 < x \leq 1$. Hence we assume $x > 1$. Following the hint, we observe that

$$\begin{aligned} f(x) &< \frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} + \frac{1}{2x} - \frac{1}{2(x+1)} \\ &= \frac{1 + \cos(x^2)}{2x} - \frac{1 + \cos[(x+1)^2]}{2(x+1)} \\ &\leq \frac{1 + \cos(x^2)}{2x} \\ &< 1 \end{aligned}$$

A similar argument shows that

$$\begin{aligned}
 f(x) &> \frac{\cos(x^2)}{2x} - \frac{\cos[(x+1)^2]}{2(x+1)} - \frac{1}{2x} + \frac{1}{2(x+1)} \\
 &= \frac{-1 + \cos(x^2)}{2x} - \frac{-1 + \cos[(x+1)^2]}{2(x+1)} \\
 &= \frac{-1 + \cos(x^2)}{2x} + \frac{1 - \cos[(x+1)^2]}{2(x+1)} \\
 &\geq \frac{-1 + \cos(x^2)}{2x} \\
 &\geq \frac{-1}{x}.
 \end{aligned}$$

(b) The expression just written for $f(x)$ shows that

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x),$$

where

$$r(x) = \left(\frac{1}{x+1}\right) \cos[(x+1)^2] - \frac{x}{2} \int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{3/2}} du.$$

If we integrate by parts again, we find that

$$\int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{3/2}} du = \frac{\sin[(x+1)^2]}{(x+1)^3} - \frac{\sin(x^2)}{x^3} + \frac{3}{2} \int_{x^2}^{(x+1)^2} \frac{\sin u}{x^{5/2}} du.$$

We now observe that the absolute value of this last integral is at most

$$\frac{3}{2} \int_{x^2}^{\infty} \frac{1}{u^{5/2}} du = -u^{-3/2} \Big|_{x^2}^{\infty} = x^{-3}.$$

It then follows by collecting the terms that

$$|r(x)| < \frac{3}{x}.$$

(c) Since $r(x) \rightarrow 0$, the upper and lower limits of $xf(x)$ will be the corresponding limits of

$$\frac{\cos(x^2) - \cos[(x+1)^2]}{2} = \sin\left(x^2 + x + \frac{1}{2}\right) \sin\left(x + \frac{1}{2}\right).$$

We can write this last expression as $\sin s \sin\left(s^2 + \frac{1}{4}\right)$, where $s = x + \frac{1}{2}$. We claim that the upper limit of this expression is 1 and the lower limit is -1 . Indeed, let $\varepsilon > 0$ be given. Choose n to be any positive integer larger than $\frac{2-\varepsilon}{8\varepsilon}$.

Then the interval $\left(\frac{1}{4} + \left((2n + \frac{1}{2})\pi - \varepsilon\right)^2, \frac{1}{4} + \left((2n + \frac{1}{2})\pi + \varepsilon\right)^2\right)$ is longer than 2π , and hence there exists a point $t \in \left((2n + \frac{1}{2})\pi - \varepsilon, (2n + \frac{1}{2})\pi + \varepsilon\right)$

at which $\sin\left(t^2 + \frac{1}{4}\right) = 1$ and also a point u in the same interval at which $\sin\left(u^2 + \frac{1}{4}\right) = -1$. But then $tf(t) > 1 - \varepsilon$ and $uf(u) < -1 + \varepsilon$. It follows that the upper limit is 1 and the lower limit is -1 . (This argument actually shows that the limit points of $xf(x)$ fill up the entire interval $[-1, 1]$.)

(d) The integral does converge. We observe that for integers N we have

$$\begin{aligned} \int_0^N \sin(t^2) dt &= \sum_{k=0}^N f(k) \\ &= f(0) + \sum_{k=1}^N \frac{r(k)}{k} + \sum_{k=1}^N \frac{\cos(k^2) - \cos[(k+1)^2]}{k} \\ &= f(0) + \sum_{k=1}^N \frac{r(k)}{k} + \left[\frac{\cos 1}{2} - \frac{\cos[(N+1)^2]}{N} \right] + \sum_{k=2}^N \frac{\cos(k^2)}{k(k-1)}. \end{aligned}$$

The first sum on the right converges since $|r(k)| < \frac{3}{k}$, and the rest obviously converges. Hence we will be finished if we show that

$$\lim_{x \rightarrow \infty} \int_{[x]}^x \sin(t^2) dt = 0,$$

where $[x]$ is the integer such that $[x] \leq x < [x] + 1$. But this is easily done using integration by parts. The integral equals

$$\frac{\cos([x]^2)}{2[x]} - \frac{\cos(x^2)}{x^2} - \int_{[x]^2}^{x^2} \frac{\cos u}{4u^{3/2}} du,$$

and this expression obviously tends to zero as $x \rightarrow \infty$.

Exercise 6.14 Deal similarly with

$$f(x) = \int_x^{x+1} \sin(e^t) dt.$$

Show that

$$e^x |f(x)| < 2$$

and that

$$e^x f(x) = \cos(e^x) - e^{-1} \cos(e^{x+1}) + r(x),$$

where $|r(x)| < Ce^{-x}$ for some constant C .

Solution. The arguments are completely analogous to the preceding problem. The substitution $u = e^t$ changes $f(x)$ into

$$f(x) = \int_{e^x}^{e^{x+1}} \frac{\sin u}{u} du,$$

and then integration by parts yields

$$f(x) = \frac{\cos(e^x)}{e^x} - \frac{\cos(e^{x+1})}{e^{x+1}} - \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du$$

from which it then follows that

$$-\frac{1 - \cos(e^x)}{e^x} \leq f(x) \leq \frac{1 + \cos(e^x)}{e^x}.$$

We have the equality

$$e^x f(x) = \cos(e^x) - e^{-1} \cos(e^{x+1}) - e^x \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du,$$

and one more integration by parts shows that

$$\left| e^x \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du \right| < \frac{3}{e^x}.$$

In this case $f(x)$ decreases so rapidly that there is no difficulty at all proving the convergence of the integral.

Exercise 6.15 Suppose f is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f^2(x) dx = 1.$$

Prove that

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x) dx \geq \frac{1}{4}.$$

Solution. To prove the first assertion we merely integrate by parts, taking $u = x$, $dv = f(x)f'(x) dx$, so that $du = dx$ and $v = \frac{1}{2}f^2(x)$. Since v vanishes at both endpoints, the result is

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2} \int_a^b f^2(x) dx = -\frac{1}{2}.$$

The second inequality is an immediate consequence of the Schwarz inequality applied to the two functions $xf(x)$ and $f'(x)$.

Exercise 6.16 For $1 < s < \infty$, define

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(This is Riemann's zeta function, of great importance in the study of the distribution of prime numbers.) Prove that

$$(a) \quad \zeta(s) = s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx$$

and that

$$(b) \quad \zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{x - [x]}{x^{s+1}} dx,$$

where $[x]$ denotes the greatest integer $\leq x$.

Prove that the integral in (b) converges for all $x > 0$.

Hint: To prove (a) compute the difference between the integral over $[1, N]$ and the N th partial sum of the series that defines $\zeta(s)$.

Solution. (a) Ignoring the author's advice, we note that

$$\begin{aligned} s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx &= s \sum_{n=1}^{\infty} n \int_n^{n+1} \frac{1}{x^{s+1}} dx \\ &= \sum_{n=1}^{\infty} n \left[\frac{1}{n^s} - \frac{1}{(n+1)^s} \right] \\ &= 1 \left[\frac{1}{1^s} - \frac{1}{2^s} \right] + 2 \left[\frac{1}{2^s} - \frac{1}{3^s} \right] + \cdots \\ &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= \zeta(s). \end{aligned}$$

(b) This result is a trivial consequence of (a) and the identity

$$\frac{s}{s-1} = \int_1^{\infty} \frac{x}{x^{s+1}} dx.$$

Exercise 6.17 Suppose α increases monotonically on $[a, b]$, g is continuous, and $g(x) = G'(x)$ for $a \leq x \leq b$. Prove that

$$\int_a^b \alpha(x) g(x) dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G d\alpha.$$

Hint: Take g real, without loss of generality. Given $P = \{x_0, x_1, \dots, x_n\}$, choose $t_i \in (x_{i-1}, x_i)$ so that $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$. Show that

$$\sum_{i=1}^n \alpha(x_i) g(t_i) \Delta x_i = G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G(x_{i-1}) \Delta \alpha_i.$$

Solution. The identity just given is a trivial consequence of Abel's method of rearranging the sums:

$$\begin{aligned}\sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i &= \sum_{i=1}^n \alpha(x_i)(G(x_i) - G(x_{i-1})) \\ &= G(x_n)\alpha(x_n) - G(x_0)\alpha(x_0) - \sum_{i=1}^G (x_{i-1})(\alpha(x_i) - \alpha_{i-1})).\end{aligned}$$

Now the fact that $G(x)$ is continuous and α is nondecreasing means that the right-hand side can be made arbitrarily close to

$$G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G d\alpha,$$

whenever the partition is sufficiently fine. It does not follow immediately that the function $\alpha(x)g(x)$ is integrable on $[a, b]$. However, since α is nondecreasing, its only discontinuities are jumps, and for any given $\varepsilon > 0$ there can be only a finite number of jumps larger than ε . These can be enclosed in a finite number of open intervals of arbitrarily small length. We can then argue, as in Exercise 6 above, that *any* partition that is sufficiently fine will have upper and lower Riemann sums that differ by less than ε . Hence $\alpha(x)g(x)$ is integrable, and its integral is given by the stated relation.

Exercise 6.18 Let $\gamma_1, \gamma_2, \gamma_3$ be curves in the complex plane defined on $[0, 2\pi]$ by

$$\gamma_1(t) = e^{it}, \quad \gamma_2(t) = e^{2it}, \quad \gamma_3(t) = e^{2\pi it \sin(1/t)}.$$

Show that these curves have the same range, that γ_1 and γ_2 are rectifiable, that the length of γ_1 is 2π , that the length of γ_2 is 4π , and that γ_3 is not rectifiable.

Solution. Since e^{it} has period 2π it is obvious that γ_1 and γ_2 have the same range, namely the set of all complex numbers of absolute value 1. To show that this is also the range of γ_3 , we need to show that the mapping $t \mapsto 2\pi t \sin(1/t)$, $0 \leq t \leq 2\pi$, covers an interval of length 2π , i.e., that the mapping $t \mapsto t \sin(1/t)$, $0 \leq t \leq 2\pi$ covers an interval of length 1. (We naturally take the value to be zero when $t = 0$.) Since this range is connected, it suffices to find two points a and b in the range with $a - b > 1$. We choose those points to be $a = \frac{3}{\pi}$ (the image of $t = \frac{6}{\pi}$) and $b = \frac{-2}{3\pi}$, (the image of $t = \frac{2}{3\pi}$). We have $a - b = \frac{11}{3\pi} > 1$.

The rectification of γ_1 and γ_2 is straightforward:

$$\begin{aligned}l(\gamma_1) &= \int_0^{2\pi} |\gamma_1'(t)| dt = 2\pi, \\ l(\gamma_2) &= \int_0^{2\pi} |\gamma_2'(t)| dt = \int_0^{2\pi} 2 dt = 4\pi.\end{aligned}$$

To show that γ_3 is not rectifiable, we observe that its length would be

$$\int_0^{2\pi} \left| \sin(1/t) - \frac{1}{t} \cos(1/t) \right| dt \geq \int_0^{2\pi} \left| \frac{\cos(1/t)}{t} \right| dt - 2\pi.$$

By making the substitution $u = \frac{1}{t}$ in this last integral we get

$$\int_{\frac{1}{2\pi}}^{\infty} \left| \frac{\cos u}{u} \right| du.$$

But we already know that this integral diverges, since

$$\sum_{n=1}^{\infty} \int_{2n\pi}^{(2n+\frac{1}{2})\pi} \frac{\cos u}{u} du \geq \sum_{n=1}^{\infty} \frac{1}{(2n+\frac{1}{2})\pi} = \infty.$$

Exercise 6.19 Let γ_1 be a curve in R^k defined on $[a, b]$; let ϕ be a continuous 1-1 mapping of $[c, d]$ onto $[a, b]$ such that $\phi(c) = a$, and define $\gamma_2(x) = \gamma_1(\phi(x))$. Prove that γ_2 is an arc, a closed curve, or a rectifiable curve if and only if the same is true of γ_1 . Prove that γ_1 and γ_2 have the same length.

Solution. We know that ϕ has a continuous 1-1 inverse φ , and that the composition of one-to-one functions is one-to-one. Hence, since $\gamma_1(x) = \gamma_2(\varphi(x))$, we see that γ_1 and γ_2 are both arcs (one-to-one) if either is. Since necessarily $\phi(d) = b$, we see that $\gamma_1(a) = \gamma_1(b)$ if and only if $\gamma_2(c) = \gamma_2(d)$. Hence both are closed curves if either is. Finally, since ϕ and φ establish a one-to-one correspondence between partitions $\{s_i\}$ of $[a, b]$ and $\{t_i\}$ of $[c, d]$ such that $\sum |\gamma_1(s_i) - \gamma_1(s_{i-1})| = \sum |\gamma_2(t_i) - \gamma_2(t_{i-1})|$, it follows that the two curves have the same length.

Chapter 7

Sequences and Series of Functions

Exercise 7.1 Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.

Solution. Let $\{f_n(x)\}_{n=1}^{\infty}$ be a uniformly convergent sequence of bounded functions, say $|f_n(x)| \leq M_n$ for all x and all n . Since the sequence converges uniformly, it is a uniformly Cauchy sequence. Hence there exists N such that $|f_m(x) - f_n(x)| < 1$ for all $m, n \geq N$. In particular if $m \geq N$, we have $|f_m(x)| \leq |f_N(x)| + |f_m(x) - f_N(x)| \leq M_N + 1$, and therefore if $M = 1 + \max(M_1, \dots, M_N)$ we have $|f_n(x)| \leq M$ for all n and x .

Exercise 7.2 If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E , prove that $\{f_n + g_n\}$ converges uniformly on E . If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E .

Solution. Let f and g denote the limits of the two sequences. Let $\varepsilon > 0$. There exist N_1 and N_2 such that $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$ for all x if $n > N_1$ and $|g_n(x) - g(x)| < \frac{\varepsilon}{2}$ for all x if $n > N_2$. Let $N = \max(N_1, N_2)$. Then for $n > N$ we have, for all x ,

$$|(f_n + g_n)(x) - (f + g)(x)| \leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \varepsilon.$$

Hence $\{f_n + g_n\}$ converges uniformly.

Suppose now that each of the functions f_n and g_n is bounded. By the previous problem, both sequences are uniformly bounded. Hence there exists M such that $|f_n(x)| \leq M$ and $|g_n(x)| \leq M$ for all n and all x . It follows that $|g(x)| \leq M$ also. Then, given $\varepsilon > 0$, choose N_1 and N_2 such that $|f_n(x) - f(x)| \leq \frac{\varepsilon}{2M}$ for all x and all $n > N_1$ and $|g_n(x) - g(x)| < \frac{\varepsilon}{2M}$ for all x and $n > N_2$.

Again let $N = \max(N_1, N_2)$. We then have, for all x and all $n > N$,

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)g_n(x) - f_n(x)g(x)| + \\ &\quad + |f_n(x)g(x) - f(x)g(x)| \\ &\leq M|g_n(x) - g(x)| + M|f_n(x) - f(x)| \\ &< M\frac{\varepsilon}{2M} + M\frac{\varepsilon}{2M} \\ &= \varepsilon. \end{aligned}$$

Exercise 7.3 Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E , but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).

Solution. Let $f_n(x) = x$ for all x and all n , and let $g_n(x) = \frac{1}{n}$ for all x and all n . Then $f_n(x)$ converges uniformly to x , and $g_n(x)$ converges uniformly to 0. Therefore $f_n(x)g_n(x)$ converges to 0, but not uniformly. In fact for every n there is an x , namely $x = n$, such that $f_n(x)g_n(x) = 1$. Hence, no matter how large n is taken, the inequality $|f_n(x)g_n(x)| < 1$ will never hold for all x .

Exercise 7.4 Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

Solution. The series converges for all x except 0 and $x = \frac{-1}{n^2}$, $n = 1, 2, \dots$. For $x = 0$ all the terms of the series are defined, but the terms do not tend to 0. For $x = \frac{-1}{n^2}$ the n th term is not defined. For all other values of x the series converges. By Theorem 7.10 (the Weierstrass M -test) the series converges uniformly on the interval $[\delta, \infty)$ if $\delta > 0$, since on that interval

$$\frac{1}{1+n^2x} \leq \frac{1}{n^2\delta}.$$

Likewise, the series converges uniformly on $(-\infty, -\delta]$ except at the points $x = -\frac{1}{n^2}$, since for $n \geq \sqrt{\frac{2}{\delta}}$ we have

$$\left| \frac{1}{1+n^2x} \right| \leq \frac{1}{n^2} \cdot \frac{1}{\delta - \frac{1}{n^2}} \leq \frac{2}{\delta n^2}.$$

The series does not converge uniformly on any interval having 0 as an endpoint. This is easy to see in the case when 0 is the left-hand endpoint. For each

of the terms of the series is a bounded function on $[0, \infty)$. If the series converged uniformly, the limit would be bounded by Problem 1 above. But we have

$$f\left(\frac{1}{m^2}\right) \geq \sum_{n=1}^m \frac{1}{1 + \frac{n^2}{m^2}} \geq \frac{m}{2}.$$

Likewise the series cannot be a uniformly Cauchy series (i.e., the sequence of partial sums cannot be a uniformly Cauchy sequence) on any interval $(-\delta, 0)$, since, no matter how large n is taken, there is a point x in this interval, namely $x = -\frac{1}{2n^2}$, at which the n th term has the value 2. Hence, if S_n denotes the sum of the first n terms, then $|S_n(x) - S_{n-1}(x)| = 2$.

The uniform convergence shows that the limiting function $f(x)$ is continuous wherever it is defined on $(-\infty, \delta] \cup [\delta, +\infty)$. Since δ is arbitrary, $f(x)$ is continuous wherever it is defined. The argument given above shows that $f(x)$ is not bounded.

Exercise 7.5 Let

$$f_n(x) = \begin{cases} 0 & (x < \frac{1}{n+1}), \\ \sin^2 \frac{\pi}{x} & (\frac{1}{n+1} \leq x \leq \frac{1}{n}), \\ 0 & (\frac{1}{n} < x). \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x does not imply uniform convergence.

Solution. The limit of $f_n(x)$ is zero. If $x \leq 0$ or $x \geq 1$, then $f_n(x) = 0$ for all n , and so this assertion is obvious. If $0 < x < 1$, then $f_n(x) = 0$ for all $n \geq \frac{1}{x}$, and so once again the assertion is obvious.

The convergence is not uniform, since, no matter how large n is taken, there is a point x , namely $x = \frac{1}{2n + \frac{1}{2}}$, for which $f_n(x) = 1$.

The series $\sum f_n(x)$ converges to 0 for $x \leq 0$ and $x \geq 1$, and to $\sin^2 \frac{\pi}{x}$ for $0 < x < 1$. Since the terms are nonnegative, the series obviously converges absolutely. Since the sum is not continuous at 0, the series does not converge uniformly on any interval containing 0.

Exercise 7.6 Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x .

Solution. The series is the sum of two series:

$$x^2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

The first of these converges both uniformly and absolutely on any bounded interval $[a, b]$ by the M -test (with $M_n = \frac{M^2}{n^2}$, where $M = \max(|a|, |b|)$). The second is independent of x and converges, hence it converges uniformly in x . By Exercise 2 above, the sum of the two series converges uniformly.

The series does not converge absolutely since the absolute value of each term is at least $\frac{1}{n}$ for any x .

Exercise 7.7 For $n = 1, 2, 3, \dots$, x real, put

$$f_n(x) = \frac{x}{1 + nx^2}.$$

Show that $\{f_n\}$ converges uniformly to a function f , and that the equation

$$f'(x) = \lim_{n \rightarrow \infty} f'_n(x)$$

is correct if $x \neq 0$, but false if $x = 0$.

Solution. The Schwarz inequality, which implies that $|f_n(x)| \leq \frac{|x|}{2\sqrt{n|x|}} = \frac{1}{2\sqrt{n}}$ for $x \neq 0$, shows that $f_n(x)$ tends uniformly to 0. Now $f'_n(x) = \frac{1-nx^2}{(1+nx^2)^2}$, which tends to 0 if $x \neq 0$, though $f'_n(0) = 1$ for all n .

Exercise 7.8 If

$$I(x) = \begin{cases} 0 & (x \leq 0), \\ 1 & (x > 0), \end{cases}$$

if $\{x_n\}$ is a sequence of distinct points of (a, b) , and if $\sum |c_n|$ converges, prove that the series

$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \quad (a \leq x \leq b)$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

Solution. The uniform convergence is a consequence of the M -test with $M_n = |c_n|$. Hence f is continuous wherever each of the individual terms is continuous, in particular, at least for $x \neq x_n$.

Exercise 7.9 Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function f on a set E . Prove that

$$\lim_{n \rightarrow \infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \rightarrow x$, and $x \in E$. Is the converse of this true?

Solution. Let $\varepsilon > 0$. Choose N_1 so large that $|f_m(x) - f(x)| < \frac{\varepsilon}{2}$ for all $m > N_1$. Then, since f is continuous at x , choose $\delta > 0$ so small that $|f(y) - f(x)| < \frac{\varepsilon}{2}$ if $|y - x| < \delta$. Finally, choose N_2 so large that $|x_n - x| < \delta$ if $n > N_2$. Then if $n > \max(N_1, N_2)$ we have

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \varepsilon.$$

The converse is not true in general. For example let $f_n(x)$ be given by $f_n(x) = \sin^2 \pi x$ for $n \leq |x| \leq n+1$ and $f_n(x) = 0$ for $|x| \leq n$ or $|x| \geq n+1$. Thus $f_n(x)$ tends to zero, since $f_n(x) = 0$ if $n \geq |x|$, but $f_n(x)$ does not converge uniformly, since $f_n(n + \frac{1}{2}) = 1$. Then for any convergent sequence, say $x_n \rightarrow x$, let $N \geq \max(|x|, |x_1|, |x_2|, \dots, |x_n|, \dots)$. We then have $f_n(x_n) = 0$ for all $n \geq N$, and so $f_n(x_n) \rightarrow f(x)$.

This condition does guarantee uniform convergence on any *compact* set, however. For if $\{f_n(x)\}$ is *not* a uniformly Cauchy sequence, then for some $\varepsilon_0 > 0$ there is a sequence of integers $n_1 < n_2 < \dots$ and a sequence of points x_1, x_2, \dots such that

$$|f_{n_{2k-1}}(x_k) - f_{n_{2k}}(x_k)| \geq \varepsilon_0$$

for $k = 1, 2, \dots$. Since K is compact, some subsequence of $\{x_k\}$ converges, say $x_{k_r} \rightarrow x$ as $r \rightarrow \infty$. Now define $y_n = x$ for all $n \neq n_{2k_r}, n \neq n_{2k_r-1}$, and let $y_{n_{2k_r-1}} = y_{n_{2k_r}} = x_{k_r}$, so that $y_n \rightarrow x$. Then the sequence $\{z_n\} = \{f_n(y_n)\}$ is not a Cauchy sequence, since $|z_{n_{2k_r}} - z_{n_{2k_r-1}}| \geq \varepsilon_0$.

Exercise 7.10 Let $f(x) = x - n$, where n is the unique integer such that $n \leq x < n+1$. Prove that

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2}$$

is discontinuous at a dense set of points.

Solution. We shall prove that $f(x)$ is discontinuous at every rational number. Since $f(x)$ has period 1, it suffices to prove this for $0 \leq x < 1$. To that end, let $x = \frac{p}{q}$ where p and q are relatively prime integers, $0 \leq p < q$. We “stratify” the sum that defines $f(x)$ by grouping all the indices n that are congruent modulo q , i.e., we let $n = kq + r$, where $1 \leq r \leq q$:

$$f(x) = \sum_{k=0}^{\infty} \sum_{r=1}^q \frac{((kq+r)x)}{(kq+r)^2}.$$

Reversing the order of summation, we find

$$f(x) = f_1(x) + f_2(x) + \cdots + f_{q-1}(x) + f_q(x),$$

where

$$f_r(x) = \sum_{k=0}^{\infty} \frac{((kq+r)x)}{(kq+r)^2}.$$

Now it is easy to see that $f_1(x), \dots, f_{q-1}(x)$ are continuous at $x = \frac{p}{q}$. For if $1 \leq r < q$, then $((kq+r)x)$ is continuous at that point, since (x) is continuous at the point $x = (kq+r)\frac{p}{q} = kp + \frac{rp}{q}$. (This point is not an integer, since p and q are relatively prime.) Since the series defining $f_r(x)$ converges uniformly, its limit is continuous at each point where all of the terms are continuous. In particular $f_r(x)$ is continuous at $x = \frac{p}{q}$, for $1 \leq r < q$.

We shall now show that $f_q(x)$ is discontinuous at $x = \frac{p}{q}$. It will then follow that $f(x)$ is discontinuous at that point. Observe that

$$f_q(x) = \frac{1}{q^2} \sum_{k=0}^{\infty} \frac{((k+1)qx)}{(k+1)^2} = \frac{1}{q^2} \sum_{k=1}^{\infty} \frac{(kqx)}{k^2},$$

so that

$$f_q\left(\frac{p}{q}\right) = \frac{1}{q^2} \sum_{k=1}^{\infty} \frac{(kp)}{k^2} = 0.$$

We shall prove that $\lim_{x \rightarrow \frac{p}{q}+} f_q(x) > 0$, and this will show that $f_q(x)$ is discontinuous at $x = \frac{p}{q}$. Since all the terms of the series for f_q are nonnegative, it suffices to show that the limit of the first term is positive. To that end, let $\delta = \frac{1}{2q}$. If $\frac{p}{q} - \delta < x < \frac{p}{q}$, then $p - \frac{1}{2} < qx < p$, and hence $(qx) > \frac{1}{2}$, from which it follows that $f_q(x) \geq \frac{1}{2q^2}$. Therefore the lower left-hand limit of $f_q(x)$ at $x = \frac{p}{q}$ is at least $\frac{1}{2q^2}$.

Since, by the M -test with $M_n = \frac{1}{n^2}$, this series converges uniformly and each of its terms is Riemann-integrable, it follows from Theorem 7.16 that the sum of the series is Riemann-integrable.

Exercise 7.11 Suppose $\{f_n\}, \{g_n\}$ are defined on E and

- (a) $\sum f_n$ has uniformly bounded partial sums;
- (b) $g_n \rightarrow 0$ uniformly on E ;

(c) $g_1(x) \geq g_2(x) \geq g_3(x) \geq \cdots$ for every $x \in E$.

Prove that $\sum f_n g_n$ converges uniformly on E . *Hint:* Compare with Theorem 3.42.

Solution. Following the hint, we let $S_N(x) = \sum_{n=1}^N f_n(x)g_n(x)$ and $F_N(x) = \sum_{n=1}^N f_n(x)$ ($F_0(x) = 0$), so that $|F_N(x)| \leq B$ for all x . Then if $N > M$, we have

$$\begin{aligned} |S_N(x) - S_M(x)| &= \left| \sum_{n=M+1}^N [F_n(x) - F_{n-1}(x)]g_n(x) \right| \\ &= \left| F_N(x)g_N(x) - F_M(x)g_{M+1}(x) + \right. \\ &\quad \left. + \sum_{n=M+1}^{N-1} F_n(x)[g_n(x) - g_{n+1}(x)] \right| \\ &\leq B \left\{ |g_N(x)| + |g_{M+1}(x)| + \sum_{n=M+1}^{N-1} [g_n(x) - g_{n+1}(x)] \right\} \\ &= B[|g_N(x)| + |g_{M+1}(x)| + g_{M+1}(x) - g_N(x)], \end{aligned}$$

and this last expression can be made uniformly small by choosing M sufficiently large by hypothesis (b). Hypothesis (c) was used in moving the summation sign outside the absolute value.

Exercise 7.12 Suppose g and f_n ($n = 1, 2, 3, \dots$) are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \leq g$, $f_n \rightarrow f$ uniformly on every compact subset of $(0, \infty)$, and

$$\int_0^\infty g(x) dx < \infty.$$

Prove that

$$\lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx = \int_0^\infty f(x) dx.$$

(See Exercises 7 and 8 of Chap. 6 for the relevant definitions.)

This is a rather weak form of Lebesgue's dominated convergence theorem (Theorem 11.32). Even in the context of the Riemann integral, uniform convergence can be replaced by pointwise convergence if it is assumed that $f \in \mathcal{R}$. (See the articles by F. Cunningham in *Math. Mag.*, vol. 40, 1967, pp. 179–186, and by H. Kestelman in *Amer. Math. Monthly*, vol. 77, 1970, pp. 182–187.)

Solution. We shall prove that $\int_0^\infty f_n(x) dx$ converges for each n , that the limit $\lim_{n \rightarrow \infty} \int_0^\infty f_n(x) dx$ exists, that $\int_0^\infty f(x) dx$ converges and that these last two quantities are equal.

Since we obviously have $|f(x)| \leq g(x)$ also, it follows that for any interval $[r, s] \subset (0, \infty)$ we have

$$\begin{aligned} \left| \int_r^s f_n(x) dx \right| &\leq \int_r^s g(x) dx, \\ \left| \int_r^s f(x) dx \right| &\leq \int_r^s g(x) dx, \\ \left| \int_r^s f_n(x) - f(x) dx \right| &\leq 2 \int_r^s g(x) dx. \end{aligned}$$

Now let $\varepsilon > 0$. Choose a and b with $0 < a < b < \infty$ so that if $0 < c < a < b < d < \infty$, then

$$\left| \int_c^d g(x) dx - \int_0^\infty g(x) dx \right| < \frac{\varepsilon}{2}.$$

It follows in particular that if $d > e > b$ we have

$$\begin{aligned} \int_e^d g(x) dx &= \int_{\frac{a}{2}}^d g(x) dx - \int_{\frac{a}{2}}^e g(x) dx \\ &\leq \left| \int_0^\infty g(x) dx - \int_{\frac{a}{2}}^d g(x) dx \right| \\ &\quad + \left| \int_0^\infty g(x) dx - \int_{\frac{a}{2}}^e g(x) dx \right| \\ &< \varepsilon. \end{aligned}$$

Then for any $d > e > b > r$ and any n we certainly have

$$\left| \int_r^d f_n(x) dx - \int_r^e f_n(x) dx \right| = \left| \int_d^e f_n(x) dx \right| \leq \int_d^e g(x) dx < \varepsilon.$$

Thus by the Cauchy criterion $\lim_{d \rightarrow \infty} \int_r^d f_n(x) dx$ exists. A similar argument shows that all the improper integrals in question converge. Moreover the argument shows that

$$\left| \int_c^d \varphi(x) dx - \int_0^\infty \varphi(x) dx \right| < \varepsilon$$

when $0 < c < a < b < d$, whether $\varphi(x) = f_n(x)$, $\varphi(x) = f(x)$, or $\varphi(x) = g(x)$.

We now merely observe that

$$\begin{aligned} \left| \int_0^\infty f_n(x) dx - \int_0^\infty f(x) dx \right| &\leq \left| \int_0^\infty f_n(x) dx - \int_c^d f_n(x) dx \right| + \\ &\quad + \left| \int_c^d [f_n(x) - f(x)] dx \right| + \\ &\quad + \left| \int_c^d f(x) dx - \int_0^\infty f(x) dx \right|. \end{aligned}$$

Given $\varepsilon > 0$ we can choose c and d so that the first and last terms on the right are less than $\frac{\varepsilon}{3}$ (for all n , in the case of the first term). Then, since $f_n(x) \rightarrow f(x)$ uniformly on c, d , we can choose n_0 so large that the middle term is less than $\frac{\varepsilon}{3}$ if $n > n_0$.

Exercise 7.13 Assume that $\{f_n\}$ is a sequence of monotonically increasing functions on R^1 with $0 \leq f_n(x) \leq 1$ for all x and all n .

(a) Prove that there is a function f and a sequence $\{n_k\}$ such that

$$f(x) = \lim_{k \rightarrow \infty} f_{n_k}(x)$$

for every $x \in R^1$. (The existence of such a pointwise convergent subsequence is usually called *Helly's selection theorem*.)

(b) If, moreover f is continuous, prove that $f_{n_k} \rightarrow f$ uniformly on R^1 .

Hint: (i) Some subsequence $\{f_{n_i}\}$ converges at all rational points r , say to $f(r)$. (ii) Define $f(x)$ for any $x \in R^1$ to be $\sup f(r)$, the sup being taken over all $r \leq x$. (iii) Show that $f_{n_i}(x) \rightarrow f(x)$ at every x at which f is continuous. (This where monotonicity is strongly used.) (iv) A subsequence of $\{f_{n_i}\}$ converges at every point of discontinuity of f since there are at most countably many such points. This proves (a). To prove (b), modify your proof of (iii) appropriately.

Solution. (a) Following the hint, we enumerate the rational numbers (or any countable dense set) as $\{r_n\}$ and use the well-known diagonal procedure to get first a subsequence that converges at r_1 , then a further subsequence that converges at r_2 , etc. The sequence formed by taking the n th term of the n th subsequence is itself a subsequence and converges at each r_n . (Note that we have not used the fact that $0 \leq f_n(x) \leq 1$ for all x and n , only the much weaker fact that for each x there is an $M(x)$ such that $|f_n(x)| \leq M(x)$ for all x and n .) Let the function $f(x)$ be defined as $f(r_k) = \lim_{n_i} f_{n_i}(r_k)$ and $f(x) = \sup\{f(r_k) : r_k \leq x\}$ for all other x . The second definition could be taken as the general one if we wished, since it is consistent with the definition already given at the points $x = r_k$.

Since each of the functions is nondecreasing, it is clear that the function $f(x)$ is nondecreasing. By its definition it is continuous from the left. Suppose $f(x)$ is continuous at x_0 . Let $\varepsilon > 0$ be given. Choose rational numbers r and s with $r \leq x_0 \leq s$, $f(x_0) - \frac{\varepsilon}{4} \leq f(r) \leq f(x_0) \leq f(s) \leq f(x_0) + \frac{\varepsilon}{4}$. Then choose i_0 so large that $|f_{n_i}(t) - f(t)| < \frac{\varepsilon}{4}$ for all $i > i_0$, $t = r$ or $t = s$. We then have

$$f(x_0) - \frac{\varepsilon}{2} \leq f(r) - \frac{\varepsilon}{4} < f_{n_i}(r) \leq f_{n_i}(x_0) \leq f_{n_i}(s) < f(s) + \frac{\varepsilon}{4} \leq f(x_0) + \frac{\varepsilon}{2}.$$

Hence $|f(x_0) - f_{n_i}(x_0)| < \varepsilon$ if $i > i_0$, which proves the convergence at points of continuity. One more application of the diagonal procedure now allows us to assure that some subsequence converges at every point (since the set of discontinuities of a nondecreasing function is countable). We can then modify the definition of $f(x)$ at these points.

The claim that the convergence is uniform if the limit is continuous is not true. Let

$$f_n(x) = \begin{cases} \frac{1}{3} + \frac{x}{1+3|x|}, & \text{if } x \leq n, \\ 1, & \text{if } x > n. \end{cases}$$

It is clear that $\lim_{n \rightarrow \infty} f_n(x) = \frac{1}{3} + \frac{x}{1+3|x|}$ for each x , yet $f_n(y) - f(y) \geq \frac{1}{3}$ if $y > n$, so that the convergence is not uniform. Here the functions $f_n(x)$ are not continuous, but they could easily be made so without violating the conditions of the problem.

To get uniform convergence we must assume in addition that $f(x) \rightarrow 1$ as $x \rightarrow \infty$ and $f(x) \rightarrow 0$ as $x \rightarrow -\infty$. Let us grant these relations and assume that $f(x)$ is continuous at all points x . To simplify the notation we shall write f_k instead of f_{n_k} . Given $\varepsilon > 0$, choose an interval $[a, b]$ such that $f(x) < \frac{\varepsilon}{2}$ if $x \leq a$ and $f(x) > 1 - \frac{\varepsilon}{2}$ if $x > b$. Then, since $f(x)$ is uniformly continuous on $[a, b]$, let $a = t_0 < t_1 < \dots < t_n = b$ be such that $f(t_i) - f(t_{i-1}) < \frac{\varepsilon}{5}$. Choose k so large that $|f_l(t_i) - f(t_i)| < \frac{\varepsilon}{5}$ for all $i = 1, \dots, n$ and all $l > k$. Then for all $y \geq b = t_n$ we have

$$1 \geq f_l(y) \geq f_l(t_k) > 1 - \frac{4\varepsilon}{5}$$

and

$$1 \geq f(y) > 1 - \frac{\varepsilon}{2} > 1 - \frac{4\varepsilon}{5}.$$

Hence certainly

$$|f_l(y) - f(y)| \leq 1 - \left(1 - \frac{4\varepsilon}{5}\right) < \varepsilon$$

for all $l > k$ and all $y \geq b$.

A similar argument shows that f_l converges uniformly to f on $(-\infty, a]$. The argument that f_l converges uniformly to f on $[t_{i-1}, t_i]$ is identical to that given above.

Exercise 7.14 Let f be a continuous real function on R^1 with the following properties: $0 \leq f(t) \leq 1$, $f(t+2) = f(t)$ for every t , and

$$f(t) = \begin{cases} 0 & (0 \leq t \leq \frac{1}{3}) \\ 1 & (\frac{2}{3} \leq t \leq 1). \end{cases}$$

Put $\Phi(t) = (x(t), y(t))$, where

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t), \quad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t).$$

Prove that Φ is *continuous* and that Φ maps $I = [0, 1]$ onto the unit square $I^2 \subset R^2$. In fact, show that Φ maps the Cantor set onto I^2 .

Hint: Each $(x_0, y_0) \in I^2$ has the form

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}, \quad y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n},$$

where each a_i is 0 or 1. If

$$t_0 = \sum_{i=1}^{\infty} 3^{-i-1} (2a_i),$$

show that $f(3^k t_0) = a_k$, and hence that $x(t_0) = x_0$, $y(t_0) = y_0$.

(This simple example of a so-called "space-filling curve" is due to I. J. Schoenberg, *Bull. A.M.S.*, vol. 44, 1938, p. 519.)

Solution. We note that $3^k t_0$ is the sum of the even integer $2(3^{k-2} a_1 + \cdots + 3a_{k-2} + a_{k-1})$ and a fractional part $\sum_{i=k}^{\infty} \frac{2a_i}{3^{i-k+1}}$. This fractional part lies in $[\frac{2}{3}, 1]$ if $a_k = 1$, while if $a_k = 0$ it is at least 0 and at most $\frac{2}{9}$. Thus it lies in the interval $[0, \frac{1}{3}]$ if $a_k = 0$. In either case $f(3^k t_0) = a_k$, as claimed. We therefore have

$$x(t_0) = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1} = x_0, \quad y(t_0) = \sum_{n=1}^{\infty} 2^{-n} a_{2n} = y_0,$$

as asserted.

Exercise 7.15 Suppose f is a real continuous function on R^1 , $f_n(t) = f(nt)$ for $n = 1, 2, 3, \dots$, and $\{f_n\}$ is equicontinuous on $[0, 1]$. What conclusion can you draw about f ?

Solution. The function $f(t)$ must be constant on $[0, \infty)$. For if $f(x) \neq f(y)$ and $0 \leq x < y < \infty$, say $|f(x) - f(y)| = \varepsilon > 0$, it follows that $|f_n(\frac{x}{n}) - f_n(\frac{y}{n})| = \varepsilon$ for all n . Since $\frac{x-y}{n} \rightarrow 0$, it follows that the family $\{f_n\}$ cannot be equicontinuous on $[0, 1]$, or, indeed, on any neighborhood of 0.

Exercise 7.16 Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K , and $\{f_n\}$ converges pointwise on K . Prove that $\{f_n\}$ converges uniformly on K .

Solution. Let $\varepsilon > 0$. Choose $\delta > 0$ such that $|f_n(x) - f_n(y)| < \frac{\varepsilon}{3}$ for all $n \neq m$ if $x, y \in K$ and $|x - y| < \delta$. Choose a finite number of points x_1, \dots, x_N such that for every $x \in K$ there exists j with $|x - x_j| < \delta$. (Such a finite set exists; otherwise we could inductively select a sequence $\{x_n\}$ such that $|x_m - x_n| \geq \delta$

for all n , and this sequence would have no Cauchy subsequence, contradicting the compactness of K .) Then choose n_0 so large that $|f_m(x_j) - f_n(x_j)| < \frac{\varepsilon}{3}$ for all $m, n > n_0$ and all $j = 1, 2, \dots, N$. Then for any point $x \in K$, fix j so that $|x - x_j| < \delta$. If $m, n > n_0$ we have

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f_m(x_j)| + |f_m(x_j) - f_n(x_j)| + |f_n(x_j) - f_n(x)|.$$

The first and last terms are smaller than $\frac{\varepsilon}{3}$ because $|x - x_j| < \delta$; the middle term is smaller than $\frac{\varepsilon}{3}$ since $m, n > n_0$. Thus the sequence is a uniformly Cauchy sequence.

Exercise 7.17 Define the notions of uniform convergence and equicontinuity for mappings into any metric space. Show that Theorems 7.9 and 7.12 are valid for mappings into any metric space, that Theorems 7.8 and 7.11 are valid for mappings into any complete metric space, and that Theorems 7.10, 7.16, 7.17, 7.24, and 7.25 hold for vector-valued functions, that is, for mappings into any R^n .

Solution. Let X and Y be any metric spaces. The sequence $\{f_n\}$, where $f_n : X \rightarrow Y$, converges uniformly to $f : X \rightarrow Y$ if for every $\varepsilon > 0$ there exists N such that $d_Y(f_n(x), f(x)) < \varepsilon$ for all $x \in X$ and all $n > N$. A family of functions \mathcal{F} is equicontinuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(x_1), f(x_2)) < \varepsilon$ for all $f \in \mathcal{F}$ whenever $d_X(x_1, x_2) < \delta$.

An immediate consequence of this definition is that $\{f_n\}$ converges uniformly to f if and only if $M_n \rightarrow 0$, where $M_n = \sup_{x \in X} d_Y(f_n(x), f(x))$ (Theorem 7.9).

The same $\frac{\varepsilon}{3}$ argument that proves Theorem 7.12 shows that the uniform limit of a sequence of continuous functions is continuous.

The Cauchy convergence criterion accepts the additional word *uniformly* without any change, provided Y is complete. Suppose for every $\varepsilon > 0$ there exists N such that $d_Y(f_m(x), f_n(x)) < \varepsilon$ for all $m, n > N$ and all x . Then, in particular, for each $x \in X$, the sequence $\{f_n(x)\}$ is a Cauchy sequence in Y . Since Y is complete, this sequence converges to a value that we shall call $f(x)$. We now claim that $\{f_n\}$ converges uniformly to f . Indeed, given $\varepsilon > 0$ choose N so that $d_Y(f_m(x), f_n(x)) < \frac{\varepsilon}{2}$ if $m, n > N$. Since a metric is a continuous function, it follows that $d_Y(f(x), f_n(x)) \leq \frac{\varepsilon}{2} < \varepsilon$ if $n > N$, that is $\{f_n\}$ converges uniformly to f . This is Theorem 7.8.

Suppose now $\{f_n\}$ converges uniformly to f , Y is complete, $x_0 \in X$, and $\lim_{x \rightarrow x_0} f_n(x) = A_n$ for $n = 1, 2, \dots$. Then $\{A_n\}$ converges, and $\lim_{x \rightarrow x_0} f(x) = \lim_{n \rightarrow \infty} A_n$. (This is Theorem 7.11.) The proof is as follows. Given $\varepsilon > 0$ choose N so that $d_Y(f(x), f_n(x)) < \frac{\varepsilon}{3}$ for all x if $n \geq N$. Let $n > N$ be fixed. Choose $\delta > 0$ (depending on n and ε in general) such that $d_Y(f_n(x), A_n) < \frac{\varepsilon}{3}$ if $0 < d_X(x, x_0) < \delta$. (The fact that $\lim_{x \rightarrow x_0} f_n(x) = A_n$ implies that there must exist x satisfying these inequalities, i.e., that x_0 is an accumulation point of X .) We then have, for $m, n > N$,

$$d_Y(A_m, A_n) \leq d_Y(A_m, f_m(x)) + d_Y(f_m(x), f_n(x)) + d_Y(f_n(x), f(x))$$

The middle term is less than $\frac{\varepsilon}{3}$ for all $m, n > N$ and all $x \in X$. If m and n are then fixed integers larger than N , the first and last terms can be made smaller than $\frac{\varepsilon}{3}$ by choosing x sufficiently close to x_0 . Hence we have $d_Y(A_m, A_n) < \varepsilon$ if $m, n > N$. Since Y is complete, the sequence $\{A_n\}$ converges, say to A . Now observe that

$$d_Y(f(x), A) \leq d_Y(f(x), f_n(x)) + d_Y(f_n(x), A_n) + d_Y(A_n, A).$$

If N is chosen sufficiently large, the first and last terms on the right-hand side will be less than $\frac{\varepsilon}{3}$ (for all x , in the case of the first term). For a fixed n satisfying these conditions, if $\delta > 0$ is sufficiently small, the second term will be less than $\frac{\varepsilon}{3}$ whenever $0 < d_X(x, x_0) < \delta$, and hence $d_Y(f(x), A) < \varepsilon$ if $0 < d_X(x, x_0) < \delta$.

The proof of the stated theorems for vector-valued functions is a consequence of the obvious facts that a vector-valued function \mathbf{f} is integrable, differentiable or continuous if and only if each of its components has the corresponding property, and that a series of vector-valued functions $\{\mathbf{f}_n\}$ is Cauchy, bounded, convergent, uniformly convergent, majorized by a convergent sequence, equicontinuous, etc., if and only if each component has those properties. A typical proof proceeds as follows (Theorem 7.25). Suppose $\{\mathbf{f}_n\}$ is a bounded equicontinuous sequence of vector-valued functions on a compact set K . Let $\|\mathbf{f}_n(x)\| \leq M$ for all $x \in K$ and all n , and given $\varepsilon > 0$ choose $\delta > 0$ such that $\|\mathbf{f}_n(x) - \mathbf{f}_n(y)\| < \varepsilon$ whenever $d(x, y) < \delta$. Then for each component f_n^i of \mathbf{f}_n we have $|f_n^i(x)| \leq \|\mathbf{f}_n\| \leq M$ and $|f_n^i(x) - f_n^i(y)| \leq \|\mathbf{f}_n(x) - \mathbf{f}_n(y)\| < \varepsilon$ whenever $d(x, y) < \delta$. Hence each sequence of components $\{f_n^i\}_n$, $i = 1, \dots, k$, is bounded and equicontinuous. Therefore for each i there is a subsequence $\{n_r\}$ such that $f_{n_r}^i$ converges uniformly. By refining to subsubsequences, we can obtain a single subsequence $\{n_r\}$ such that $\{f_{n_r}^i\}$ converges uniformly for all i , say to $f^i(x)$. Then, given $\varepsilon > 0$, choose r_0 so large that $|f_{n_r}^i(x) - f^i(x)| < \frac{\varepsilon}{k}$ for $i = 1, 2, \dots, k$ and $r > r_0$. It then follows that $\|\mathbf{f}_{n_r}(x) - \mathbf{f}(x)\| < \varepsilon$ if $r > r_0$. The proofs of the other results all follow this model argument.

Exercise 7.18 Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann integrable on $[a, b]$, and put

$$F_n(x) = \int_a^x f_n(t) dt \quad (a \leq x \leq b).$$

Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

Solution. Let M be such that $|f_n(x)| \leq M$ for all n and x . Then clearly $|F_n(x)| \leq M(b - a)$ for all n , so that $\{F_n\}$ is uniformly bounded. Also, given $\varepsilon > 0$, let $\delta = \frac{\varepsilon}{M}$. Then if $x < y$ and $|x - y| < \delta$, we have

$$|F_n(y) - F_n(x)| = \left| \int_x^y f_n(t) dt \right| < M|x - y| < \varepsilon.$$

Hence $\{F_n\}$ is also uniformly equicontinuous. Therefore by Ascoli's Theorem (Theorem 7.25), there exists a uniformly convergent subsequence of $\{F_n\}$.

Exercise 7.19 Let K be a compact metric space, let S be a subset of $\mathcal{C}(K)$. Prove that S is compact (with respect to the metric defined in Section 7.14) if and only if S is uniformly closed, pointwise bounded, and equicontinuous. (If S is not equicontinuous, then S contains a sequence which has no equicontinuous subsequence, hence has no sequence that converges uniformly on K .)

Solution. First suppose S is compact. Then we know that S has to be closed and bounded (in this metric *bounded* means the same thing as *uniformly bounded*). If S is not equicontinuous, then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exist x, y and $g \in S$ such that $d(x, y) < \delta$ and $|g(x) - g(y)| \geq \varepsilon$. Let $x_n, y_n \in K$ and $g_n \in S$ be such that $d(x_n, y_n) < \frac{1}{n}$ and $|g_n(x_n) - g_n(y_n)| \geq \varepsilon$. Then no subsequence of $\{g_n\}$ can be equicontinuous, since $|g_{n_k}(x_{n_k}) - g_{n_k}(y_{n_k})| \geq \varepsilon$. Hence by Theorem 7.24 no subsequence of $\{g_n\}$ can converge in $\mathcal{C}(K)$, and so S cannot be compact. We conclude, then, that if S is compact, then S is closed, bounded, and equicontinuous.

Conversely, if S is closed, pointwise bounded, and equicontinuous, then every sequence $\{g_n\}$ contains a subsequence that converges uniformly, hence converges in the metric of $\mathcal{C}(K)$ (by Ascoli's theorem). Since S is closed, the limit belongs to S , and so S is compact by Exercise 26 of Chapter 2.

Exercise 7.20 If f is continuous on $[0, 1]$ and if

$$\int_0^1 f(x)x^n dx = 0 \quad (n = 0, 1, 2, \dots),$$

prove that $f(x) = 0$ on $[0, 1]$. *Hint:* The integral of the product of f with any polynomial is zero. Use the Weierstrass theorem to show that $\int_0^1 f^2(x) dx = 0$.

Solution. There exists a sequence of polynomials $p_n(x)$ such that $p_n(x)$ converges uniformly to $f(x)$. Since f is bounded, $\{p_n\}$ is uniformly bounded, and hence $p_n f$ converges uniformly to f^2 . Then by Theorem 7.16

$$\int_0^1 f^2(x) dx = \lim_{n \rightarrow \infty} \int_0^1 p_n(x)f(x) dx = 0.$$

But we know already (Exercise 2 of Chapter 6) that this implies $f^2(x) \equiv 0$.

Exercise 7.21 Let K be the unit circle in the complex plane (i.e., the set of all z with $|z| = 1$), and let \mathcal{A} be the algebra of all functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^N c_n e^{in\theta} \quad (\theta \text{ real}).$$

The \mathcal{A} separates points on K , and \mathcal{A} vanishes at no point of K , but nevertheless there are continuous functions on K which are not in the uniform closure of \mathcal{A} .

Hint: For every $f \in \mathcal{A}$

$$\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} d\theta = 0,$$

and this is also true for every f in the closure of \mathcal{A} .

Solution. The function $f(z) = z \in \mathcal{A}$ separates points on K and never vanishes. The equality given in the hint is a straightforward computation. It implies that the continuous function $\frac{1}{z}$, which is $e^{-i\theta}$, is not in the uniform closure of \mathcal{A} , since

$$\int_0^{2\pi} e^{-i\theta} e^{i\theta} d\theta = 2\pi.$$

Exercise 7.22 Assume $f \in \mathcal{R}(\alpha)$ on $[a, b]$, and prove that there are polynomials P_n such that

$$\lim_{n \rightarrow \infty} \int_a^b |f - P_n|^2 d\alpha = 0.$$

(Compare with Exercise 12, Chap. 6.)

Solution. The parenthetical remark refers to the proof that there is a sequence of *continuous* functions $\{f_n\}$ such that

$$\lim_{n \rightarrow \infty} \int_a^b |f - f_n|^2 d\alpha = 0.$$

All that is now needed is to note that one can find polynomials P_n such that $|f_n(x) - P_n(x)| < \frac{1}{n}$ for all $x \in [a, b]$ and all n .

Exercise 7.23 Put $P_n = 0$, and define, for $n = 0, 1, 2, \dots$,

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}$$

Prove that

$$\lim_{n \rightarrow \infty} P_n(x) = |x|,$$

uniformly on $[-1, 1]$.

(This makes it possible to prove the Stone-Weierstrass theorem without first proving Theorem 7.26.)

Hint: Use the identity

$$|x| - P_{n+1}(x) = [|x| - P_n(x)] \left[1 - \frac{|x| + P_n(x)}{2} \right]$$

to prove that $0 \leq P_n(x) \leq P_{n+1}(x) \leq |x|$ if $|x| \leq 1$, and that

$$|x| - P_n(x) \leq |x| \left(1 - \frac{|x|}{2}\right)^n < \frac{2}{n+1}$$

if $|x| \leq 1$.

Solution. The identity given in the hint is a trivial consequence of the identity $x^2 - P_n^2(x) = [|x| - P_n(x)][|x| + P_n(x)]$. Then, granting that $0 \leq P_n(x) \leq |x|$, we conclude that $0 \leq 1 - \frac{|x| + P_n(x)}{2} < 1$ for $|x| \leq 1$, and hence that $0 \leq |x| - P_{n+1}(x) \leq |x| - P_n(x)$, which gives all of the desired inequalities. An immediate corollary of the same identity (obtained by replacing $P_n(x)$ by 0 in the second factor on the right-hand side) is

$$|x| - P_{n+1}(x) \leq [|x| - P_n(x)] \left(1 - \frac{|x|}{2}\right),$$

and this inequality makes it possible to obtain the inequality

$$|x| - P_n(x) \leq |x| \left(1 - \frac{|x|}{2}\right)^n$$

by induction on n . Finally, by symmetry, the maximum of $|x|(1 - \frac{|x|}{2})^n$ on $[-1, 1]$ is its maximum on $[0, 1]$, and this can be found by simple calculus to occur at $x = \frac{2}{n+1}$. Since this function is always less than $|x|$, the final inequality now follows.

Exercise 7.24 Let X be a metric space, with metric d . Fix a point $a \in X$. Assign to each $p \in X$ the function f_p defined by

$$f_p(x) = d(x, p) - d(x, a) \quad (x \in X).$$

Prove that $|f_p(x)| \leq d(a, p)$ for all $x \in X$, and therefore $f_p \in \mathcal{C}(X)$.

Prove that

$$\|f_p - f_q\| = d(p, q)$$

for all $p, q \in X$.

If $\Phi(p) = f_p$, it follows that Φ is an *isometry* (a distance-preserving mapping) of X onto $\Phi(X) \subset \mathcal{C}(X)$.

Let Y be the closure of $\Phi(X)$ in $\mathcal{C}(X)$. Show that Y is complete.

Conclusion: X is isometric to a dense subset of a complete metric space Y . (Exercise 24, Chap. 3 contains a different proof of this.)

Solution. The inequality $|f_p(x)| \leq d(a, p)$ is well-known, i.e., the fact that

$$|d(x, p) - d(x, a)| \leq d(a, p)$$

and follows from the triangle inequality by merely transposing a term. (The left-hand side is either $d(x, p) - d(x, a)$ or $d(x, a) - d(x, p)$. Whichever is the case, if the subtracted term is moved to the other side, we have the ordinary triangle inequality.)

As for the isometry, we certainly have, for all x ,

$$|f_q(x) - f_p(x)| = |d(x, q) - d(x, p)| \leq d(p, q)$$

and equality holds here if $x = q$ or $x = p$. Hence the supremum over all x is exactly $d(p, q)$.

As for the closure Y of $\Phi(X)$ being complete, it is a closed subset of a complete metric space, hence necessarily complete. By definition of closure, $\Phi(X)$ is dense in Y .

Exercise 7.25 Suppose ϕ is a continuous bounded real function in the strip defined by $0 \leq x \leq 1$, $-\infty < y < \infty$. Prove that the initial-value problem

$$y' = \phi(x, y), \quad y(0) = c$$

has a solution. (Note that the hypotheses of this existence theorem are less stringent than those of the corresponding uniqueness theorem; see Exercise 27, Chap. 5.)

Hint: Fix n . For $i = 0, \dots, n$; put $x_i = i/n$. Let f_n be a continuous function on $[0, 1]$ such that $f_n(0) = c$,

$$f'_n(t) = \phi(x_i, f_n(x_i)) \quad \text{if } x_i < t < x_{i+1},$$

and put

$$\Delta_n(t) = f'_n(t) - \phi(t, f_n(t)),$$

except at the points x_i , where $\Delta_n(t) = 0$. Then

$$f_n(x) = c + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)] dt.$$

Choose $M < \infty$ so that $|\phi| \leq M$. Verify the following assertions.

(a) $|f'_n| \leq M$, $|\Delta_n| \leq 2M$, $\Delta_n \in \mathcal{R}$, and $|f_n| \leq |c| + M = M_1$, say, on $[0, 1]$ for all n .

(b) $\{f_n\}$ is equicontinuous on $[0, 1]$, since $|f'_n| \leq M$.

(c) Some $\{f_{n_k}\}$ converges to some f , uniformly on $[0, 1]$.

(d) Since ϕ is uniformly continuous on the rectangle $0 \leq x \leq 1$, $|y| \leq M_1$,

$$\phi(t, f_{n_k}(t)) \rightarrow \phi(t, f(t))$$

uniformly on $[0, 1]$.

(e) $\Delta_n(t) \rightarrow 0$ uniformly on $[0, 1]$, since

$$\Delta_n(t) = \phi(x_i, f_n(x_i)) - \phi(t, f_n(t))$$

in (x_i, x_{i+1}) .

(f) Hence

$$f(x) = c + \int_0^x \phi(t, f(t)) dt.$$

This f is a solution of the given problem.

Solution. It will save trouble if we assume that ϕ is a bounded continuous mapping from $[0, 1] \times R^k$ into R^k and that c is a vector in R^k . That way we can do Exercise 26 simultaneously with this one. Since we are defining the functions $f_n(t)$ to be piecewise-linear, there is no difficulty in doing this with vector-valued functions. We simply define $f_n(t) = c + t\phi(0, c)$ for $0 \leq t \leq x_1$, and then, by induction on i ,

$$f_n(t) = f_n(x_i) + (t - x_i)\phi(x_i, f_n(x_i))$$

for $x_i < t \leq x_{i+1}$.

Then, if $\Delta_n(t)$ is defined as indicated, we have $f'_n(t) = \Delta_n(t) + \phi(t, f_n(t))$ except at a finite set of points, and therefore

$$f_n(x) = f_n(0) + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)] dt.$$

(a) The assertions $|f'_n| \leq M$ and $|\Delta_n| \leq 2M$ are immediate consequences of the definitions of these two functions and the fact that $|\phi(x, y)| \leq M$ for all x and y (here in general $y \in R^k$). Since $\Delta_n(t)$ is bounded and continuous except at x_i , it is Riemann-integrable. The inequality $|f_n| \leq |c| + M = M_1$ is then immediate.

(b) $|f_n(x) - f_n(y)| \leq \int_x^y |f'_n(t)| dt \leq M|x - y|$.

(c) This is Ascoli's Theorem (Theorem 7.25).

(d) Given $\varepsilon > 0$ let $\delta > 0$ be such that $|\phi(t, y) - \phi(t, z)| < \varepsilon$ if $|y - z| < \delta$, for all $t \in [0, 1]$, and $y, z \in R^k$. Then if $|f_{n_k}(t) - f(t)| < \delta$ for all t (which is the case if k is large), we have $|\phi(t, f_{n_k}(t)) - \phi(t, f(t))| < \varepsilon$ for all t .

(e) For each t and n let $i(n)$ be chosen so that $t \in [x_{i(n)}, x_{i(n)+1}]$, so that $|t - x_{i(n)}| \leq \frac{1}{n}$. Since $f_{n_k}(t)$ converges uniformly to $f(t)$ and $x_{i(n)} \rightarrow t$, it follows that $\phi(x_{i(n)}, f_n(x_{i(n)})) - \phi(t, f_n(t)) \rightarrow 0$.

(f) We now invoke Theorem 7.16 to get

$$f(x) = c + \int_0^x \phi(t, f(t)) dt.$$

Clearly $f(0) = c$, and since the right-hand side has a continuous derivative, so does the left-hand side, and $f'(x) = \phi(x, f(x))$.

Exercise 7.26 Prove an analogous existence theorem for the initial-value problem

$$\mathbf{y}' = \Phi(x, \mathbf{y}), \quad \mathbf{y}(0) = \mathbf{c},$$

where now $\mathbf{c} \in R^k$, $\mathbf{y} \in R^k$, and Φ is a continuous bounded mapping of the part of R^{k+1} defined by $0 \leq x \leq 1$, $\mathbf{y} \in R^k$ into R^k . (Compare Exercise 28, Chap. 5.) *Hint:* Use the vector-valued version of Theorem 7.25.

Solution. Since we were foresightful enough to make all the necessary notes in the solution of the previous problem, there is nothing to be done. Observe that an k -th order initial-value problem

$$y^{(k)} = \phi(x, y, y', y'', \dots, y^{(k-1)})$$

with $y(0) = c_0$, $y'(0) = c_1, \dots$, $y^{(k-1)}(0) = c_{k-1}$ falls under this theorem if we let

$$\Phi(x, y_1, y_2, \dots, y_k) = (y_2, y_3, \dots, y_k, \phi(x, y_1, \dots, y_{k-1})),$$

$\mathbf{y}(0) = (c_0, \dots, c_{k-1})$ Any solution of this problem provides a solution of the k -th order equation (namely y_1 if $\mathbf{y} = (y_1, \dots, y_k)$).

Chapter 8

Some Special Functions

Exercise 8.1 Define

$$f(x) = \begin{cases} e^{-1/x^2} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$ and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \dots$.

Solution. We have $\lim_{x \rightarrow 0} x^k e^{-1/x^2} = 0$ for all $k = 0, \pm 1, \pm 2, \dots$ by L'Hospital's rule. It is easily shown by induction that there is a polynomial p_n such that $f^{(n)}(x) = p_n(\frac{1}{x})e^{-1/x^2}$ for $x \neq 0$. Assuming (by induction) that $f^{(n)}(0) = 0$, we then have $f^{(n+1)}(0) = \lim_{x \rightarrow 0} q_n(\frac{1}{x})e^{-1/x^2} = 0$, where $q_n(x) = xp_n(x)$.

Exercise 8.2 Let a_{ij} be the number in the i th row and j th column of the array

$$\begin{array}{cccccc} -1 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & -1 & 0 & 0 & \dots \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \dots \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{array}$$

so that

$$a_{ij} = \begin{cases} 0 & (i < j), \\ -1 & (i = j), \\ 2^{j-i} & (i > j). \end{cases}$$

Prove that

$$\sum_i \sum_j a_{ij} = -2, \quad \sum_j \sum_i a_{ij} = 0.$$

Solution. This is a routine computation:

$$\begin{aligned}\sum_i \sum_j a_{ij} &= \sum_{i=1}^{\infty} \left[-1 + \sum_{j=1}^{i-1} 2^{j-i} \right] \\ &= \sum_{i=1}^{\infty} [-1 + (1 - 2^{1-i})] \\ &= \sum_{i=1}^{\infty} -2^{1-i} = -2,\end{aligned}$$

while

$$\begin{aligned}\sum_j \sum_i a_{ij} &= \sum_{j=1}^{\infty} \left[-1 + \sum_{i=j+1}^{\infty} 2^{j-i} \right] \\ &= \sum_{j=1}^{\infty} [-1 + 1] \\ &= 0.\end{aligned}$$

Exercise 8.3 Prove that

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$$

if $a_{ij} \geq 0$ for all i and j (the case $+\infty = +\infty$ may occur).

Solution. In fact the only case that we need to consider is the case when one of the two sums is infinite. If either sum is finite, we merely invoke Theorem 8.3, which explicitly states that the two sums are equal. Hence if either sum is infinite, then both are.

Exercise 8.4 Prove the following limit relations:

$$(a) \lim_{x \rightarrow 0} \frac{b^x - 1}{x} = \log b \quad (b > 0).$$

$$(b) \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1.$$

$$(c) \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e.$$

$$(d) \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

Solution. (a) Consider the function $f(x) = b^x = e^{x \log b}$. The limit we are considering is $f'(0)$. By the chain rule

$$f'(x) = e^{x \log b} \log b.$$

Now take $x = 0$.

(b) Let $y = \log(1+x)$, so that $x = e^y - 1$. It is easy to justify the relation

$$\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = \lim_{y \rightarrow 0} \frac{y}{e^y - 1} = \frac{1}{\lim_{y \rightarrow 0} \frac{e^y - 1}{y}} = 1,$$

since $\lim_{y \rightarrow 0} \frac{e^y - 1}{y} = E'(0)$.

(c) Consider the function $(1+x)^{1/x} = e^{\frac{\log(1+x)}{x}}$. By part (b) $\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} = e^1 = e$.

(d) As above, we have $\left(1 + \frac{x}{n}\right)^n = \left[\left(1 + \frac{x}{n}\right)^{1/(x/n)}\right]^x$, and by part (c) the limit of the expression inside the brackets is e .

Exercise 8.5 Find the following limits

(a) $\lim_{x \rightarrow 0} \frac{e - (1+x)^{1/x}}{x}.$

(b) $\lim_{n \rightarrow \infty} \frac{n}{\log n} [n^{1/n} - 1].$

(c) $\lim_{x \rightarrow 0} \frac{\tan x - x}{x(1 - \cos x)}.$

(d) $\lim_{x \rightarrow 0} \frac{x - \sin x}{\tan x - x}.$

Solution. (a) This limit is $f'(0)$, where $f(x) = (1+x)^{1/x}$ (by part (c) of the previous problem). Now for $x \neq 0$, we have

$$f'(x) = (1+x)^{1/x} \left[\frac{(1+x) \log(1+x) - x}{x^2(x+1)} \right].$$

Since we know that the limit of the first factor is e , we need only consider the limit inside the brackets. Since

$$(1+x) \log(1+x) = \left(x - \frac{x^2}{2} + \cdots\right) + x \left(x - \frac{x^2}{2} + \cdots\right),$$

we can cancel x^2 from the numerator and denominator of the expression in brackets, and we see that the limit of this expression is $\frac{1}{2}$. Hence the limit of $f'(x)$ as $x \rightarrow 0$ exists and equals $\frac{e}{2}$. It then follows from the mean-value theorem that $f'(0)$ equals this limit (see the corollary to Theorem 5.12).

(b) Write this expression as

$$\frac{e^{\frac{\log n}{n}} - 1}{\frac{\log n}{n}}.$$

Since $\frac{\log n}{n}$ tends to 0 as $n \rightarrow \infty$, this fraction tends to the derivative of e^x at 0, i.e., it tends to 1.

(c) Write this expression as

$$\frac{\sin x - x \cos x}{x \cos x(1 - \cos x)}.$$

We can then use either Maclaurin series or L'Hospital's rule to prove that the limit is $\frac{2}{3}$.

(d) Write this expression as

$$\frac{(x - \sin x) \cos x}{\sin x - x \cos x}$$

and again either by Maclaurin series or L'Hospital's rule the limit is $\frac{1}{2}$.

Exercise 8.6 Suppose $f(x)f(y) = f(x+y)$ for all real x and y .

(a) Assuming that f is differentiable and not zero, prove that

$$f(x) = e^{cx}$$

where c is a constant.

(b) Prove the same thing, assuming only that f is continuous.

Solution. (a) Since f is not 0, it follows that $f(0) = 1$ (take $x = y = 0$ in the basic relation that defines f). It then follows that $f'(x) = f(x)f'(0)$, and hence that the function $g(x) = e^{-xf'(0)}f(x)$ satisfies $g'(x) = 0$ for all x . Therefore $g(x) = g(0) = f(0) = 1$ for all x , i.e., $f(x) = e^{cx}$, where $c = f'(0)$.

(b) The relation $f(x)f(y) = f(x+y)$ shows that either $f(x)$ is always zero, or it is never zero. In the latter case, since f is continuous, it cannot change sign, and therefore (since $f(0) = 1$) it is always positive. Let $g(x) = \log[f(x)]$. Then $g(x+y) = g(x) + g(y)$, and g is continuous. It suffices then to show that $g(x) = cx$ for some constant $c = g(1)$. To this end, we note that the additive property of g implies that $g(0) = 0$, $g(-x) = -g(x)$, and (by an easy induction) $g(nx) = ng(x)$ for all integers $n = 0, \pm 1, \pm 2, \dots$. Consider the set of x such that $g(x) = g(1)x$. Obviously 0 and 1 belong to this set. If a belongs to this set, so does na for any n , since $g(na) = ng(a) = ng(1)a = g(1)(na)$. Finally, if a belongs to this set, so does $\frac{a}{n}$, $n = 1, 2, \dots$, since $g(a) = g(n\frac{a}{n}) = ng(\frac{a}{n})$. That is, $g(\frac{a}{n}) = \frac{1}{n}g(a) = \frac{1}{n}g(1)a = g(1)\frac{a}{n}$. It now follows that r belongs to this set for all rational numbers r , that is, the two continuous functions $g(x)$ and $g(1)x$ have the same values at all rational numbers r . Since the rational numbers are dense, and the set of points at which two continuous functions are equal is a closed set, it follows that $g(x) = g(1)x$ for all x .

Exercise 8.7 If $0 < x < \frac{\pi}{2}$, prove that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1.$$

Solution. To show the left-hand inequality, consider the function $f(x) = \sin x - \frac{2x}{\pi}$ on the interval $0 \leq x \leq \frac{\pi}{2}$. We have $f(0) = f(\frac{\pi}{2}) = 0$. Since $f''(x) = -\sin x \leq 0$, the function $f'(x)$ is strictly decreasing on this interval. Therefore it has at most one zero on this interval; by Rolle's theorem, it has exactly one zero. Since $f''(x) < 0$ at that point, the function $f(x)$ has a maximum at that point. Therefore $f(x) > 0$ for $0 < x < \frac{\pi}{2}$.

The proof of the right-hand inequality is similar, but easier. The function $g(x) = x - \sin x$ has derivative $1 - \cos x$, which is nonnegative. Therefore $g(x)$ is strictly increasing, and so $g(x) > g(0) = 0$ for all $x > 0$ (the restriction $x < \frac{\pi}{2}$ is superfluous in this case).

Exercise 8.8 For $n = 0, 1, 2, \dots$, and x real, prove that

$$|\sin nx| \leq n|\sin x|.$$

Note that this inequality may be false for other values of n . For instance,

$$|\sin \frac{1}{2}\pi| > \frac{1}{2}|\sin \pi|.$$

Solution. The inequality is obvious if $n = 0$ or $n = 1$. Then by induction we have

$$\begin{aligned} |\sin nx| &= |\sin((n-1)x + x)| \\ &= |\sin((n-1)x) \cos x + \cos((n-1)x) \sin x| \\ &\leq |\sin((n-1)x)| + |\sin x| \\ &\leq (n-1)|\sin x| + |\sin x| = n|\sin x|. \end{aligned}$$

A stronger remark can be made: If c is *not* an integer, then $|\sin c\pi| > |c||\sin \pi|$. Hence this inequality fails for $x = \pi$ unless c is an integer.

Exercise 8.9 (a) Put $s_N = 1 + (\frac{1}{2}) + \dots + (1/N)$. Prove that

$$\lim_{N \rightarrow \infty} (s_N - \log N)$$

exists. (The limit, often denoted by γ , is called Euler's constant. Its numerical value is $0.5772\dots$. It is not known whether γ is rational or not.)

(b) Roughly how large must m be so that $N = 10^m$ satisfies $s_N > 100$?

Solution. (a) We observe that $\log(N+1) - \log N = \int_N^{N+1} \frac{1}{t} dt$, so that $(s_{N+1} - \log(N+1)) - (s_N - \log N) = \frac{1}{N+1} - \int_N^{N+1} \frac{1}{t} dt < 0$. Thus the sequence is a decreasing sequence. On the other hand, it consists of positive numbers, since

$\log N = \int_1^N \frac{1}{t} dt < 1 + \frac{1}{2} + \cdots + \frac{1}{N-1} < s_N$. It follows that the sequence must converge to a nonnegative number γ .

(b) The answer here depends on how “rough” an estimate is desired. We observe that $s_{10^{N+1}} - s_{10^N}$ lies between $9 \cdot 10^N (\frac{1}{10^{N+1}})$ and $9 \cdot 10^N (\frac{1}{10^N})$, i.e., between 0.9 and 9. Hence by an easy induction $0.9N < s_{10^N} < 9N$. Thus $m = 112$ will certainly work, and m must be at least 12.

Exercise 8.10 Prove that $\sum 1/p$ diverges; the sum extends over all primes.

(This shows that the primes form a fairly substantial subset of the positive integers.)

Hint: Given N , let p_1, \dots, p_k be those primes that divide at least one integer $\leq N$. Then

$$\begin{aligned} \sum_{n=1}^N \frac{1}{n} &\leq \prod_{j=1}^k \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots\right) \\ &= \prod_{j=1}^k \left(1 - \frac{1}{p_j}\right)^{-1} \\ &\leq \exp \sum_{j=1}^k \frac{2}{p_j}. \end{aligned}$$

The last inequality holds because

$$(1-x)^{-1} \leq e^{2x}$$

if $0 \leq x \leq \frac{1}{2}$.

(There are many proofs of this result. See, for instance, the article by I. Niven in *Amer. Math. Monthly*, vol. 78, 1971, pp. 272–273, and the one by R. Bellman in *Amer. Math. Monthly*, vol. 50, 1943, pp. 318–319.)

Solution. We observe that the primes p_1, \dots, p_k form the set of all primes not greater than N . Each of them is at least 2, and therefore each integer from 1 to N is a unique product of the form $p_1^{e_1} \cdots p_k^{e_k}$ for nonnegative integers e_j , $0 \leq e_j \leq \log_2 N$. For simplicity let m be the greatest integer in $\log_2 N$. Then certainly

$$\begin{aligned} \sum_{n=1}^N \frac{1}{n} &\leq \sum_{e_1, \dots, e_k=0}^m \frac{1}{p_1^{e_1} \cdots p_k^{e_k}} \\ &= \prod_{j=1}^k \left(1 + \frac{1}{p_j} + \cdots + \frac{1}{p_j^m}\right) \end{aligned}$$

$$\begin{aligned}
&= \prod_{j=1}^k \left(\frac{1 - p_j^{-m-1}}{1 - \frac{1}{p_j}} \right) \\
&\leq \prod_{j=1}^k \left(\frac{1}{1 - \frac{1}{p_j}} \right) \\
&= \prod_{j=1}^k \left(1 - \frac{1}{p_j} \right)^{-1} \\
&\leq \exp \left(\sum_{j=1}^k \frac{2}{p_j} \right).
\end{aligned}$$

To establish the inequality $(1-x)^{-1} \leq e^{2x}$ on $[0, \frac{1}{2}]$, we simply observe that the function $f(x) = (1-x)e^{2x}$ has derivative $(1-2x)e^{2x}$, which is positive on this interval. Hence the smallest value this function has on the interval is its value at $x = 0$, which is 1.

We have now established the inequality

$$\sum_{j=1}^k \frac{1}{p_j} \geq \frac{1}{2} \log \left(\sum_{n=1}^N \frac{1}{n} \right)$$

for any integer N less than p_{k+1} . Since the right-hand side of this inequality tends to ∞ , so does the left.

Exercise 8.11 Suppose $f \in \mathcal{R}$ on $[0, A]$ for all $A < \infty$, and $f(x) \rightarrow 1$ as $x \rightarrow +\infty$. Prove that

$$\lim_{t \rightarrow 0} \int_0^{\infty} e^{-tx} f(x) dx = 1 \quad (t > 0).$$

Solution. We first observe that the improper integral converges absolutely for all $t > 0$, since

$$\int_R^S e^{-tx} |f(x)| dx \leq \frac{M}{t} (e^{-Rt} - e^{-St}) \rightarrow 0,$$

where $M = \sup_{x \geq R} |f(x)|$, as $R, S \rightarrow \infty$.

We also note that

$$t \int_0^{\infty} e^{-tx} f(x) dx = \int_0^{\infty} e^{-u} f\left(\frac{u}{t}\right) du,$$

and this last improper integral also converges for all $t > 0$. Hence we have

$$\begin{aligned}
\left| t \int_0^{\infty} e^{-tx} f(x) dx - 1 \right| &= \left| \int_0^{\infty} e^{-u} f\left(\frac{u}{t}\right) du - 1 \right| \\
&\leq \int_0^{\infty} e^{-u} \left| f\left(\frac{u}{t}\right) - 1 \right| du.
\end{aligned}$$

Since $f(x)$ has a limit at infinity and $f(x)$ is Riemann-integrable on $[0, 1]$, it follows that $f(x) \leq K$ for some constant K and all x . Thus for any $\eta > 0$ we have

$$\begin{aligned} \left| t \int_0^\infty e^{-tx} f(x) dx - 1 \right| &\leq (K+1) \int_0^\eta e^{-u} du + \int_\eta^\infty \left| f\left(\frac{u}{t}\right) - 1 \right| du \\ &\leq \eta(K+1) + M_{\eta,t}, \end{aligned}$$

where $M_{\eta,t} = \sup_{z \geq \frac{\eta}{t}} |f(z) - 1|$. Hence, given $\varepsilon > 0$ we take $\eta = \frac{\varepsilon}{2(K+1)}$. We then choose $X > 0$ so large that $|f(z) - 1| < \frac{\varepsilon}{2}$ if $z > X$, and we let $\delta = \frac{\eta}{X}$. It then follows that $M_{\eta,t} < \frac{\varepsilon}{2}$ if $0 < t < \delta$.

Exercise 8.12 Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \leq \delta$, $f(x) = 0$ if $\delta < |x| < \pi$, and $f(x + 2\pi) = f(x)$ for all x .

(a) Compute the Fourier coefficients of f .

(b) Conclude that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \quad (0 < \delta < \pi).$$

(c) Deduce from Parseval's theorem that

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}.$$

(d) Let $\delta \rightarrow 0$, and prove that

$$\int_0^\infty \left(\frac{\sin x}{x} \right)^2 dx = \frac{\pi}{2}.$$

(e) Put $\delta = \pi/2$ in (c). What do you get?

Solution. (a) Since $f(x)$ is an even real-valued function, it makes sense to use the real form of the Fourier series, since symmetry shows that $b_n = 0$ for all n . Then $a_0 = \frac{2\delta}{\pi}$, and for $n \geq 1$ we have $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^\delta \cos nx dx = \frac{2 \sin n\delta}{\pi n}$.

(b) Since $f(x)$ satisfies the Lipschitz condition of Theorem 8.14 at $x = 0$, it follows that the series actually converges to $f(0)$ at that point, i.e.,

so that

$$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}.$$

(c) Parseval's theorem now implies that

$$\frac{2\delta}{\pi} = \frac{1}{\pi} \int_{-\delta}^{\delta} |f(x)|^2 dx = \frac{1}{2} \left(\frac{2\delta}{\pi} \right)^2 + \sum_{n=1}^{\infty} \frac{4 \sin^2(n\delta)}{\pi^2 n^2}.$$

Now multiplying both sides by $\frac{\pi^2}{4\delta}$ gives the required result.

(d) Let R be any fixed number, N any positive integer, and let $\delta_N = \frac{R}{N}$. As $N \rightarrow \infty$ we have $\sum_{n=1}^N \frac{\sin^2(n\delta_N)}{n^2 \delta_N} \rightarrow \int_0^R \left(\frac{\sin x}{x} \right)^2 dx$, since the left-hand side of this equality is a Riemann sum for this integral. Note that

$$\sum_{n=N+1}^{\infty} \frac{\sin^2(n\delta_N)}{n^2 \delta_N} < \frac{1}{N\delta_N} = \frac{1}{R}.$$

(The inequality results from the fact that $\sum_{n=k}^{\infty} \frac{1}{n^2} < \int_{k-1}^{\infty} \frac{1}{t^2} dt = \frac{1}{k-1}$.) Given ε , choose $R > \frac{\varepsilon}{3}$ such that

$$\left| \int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx - \int_0^S \left(\frac{\sin x}{x} \right)^2 dx \right| < \frac{\varepsilon}{3}$$

if $S > R$. Then choose $N_0 > \frac{3}{\varepsilon}$ so large that

$$\left| \sum_{n=1}^N \frac{\sin^2(n\delta_N)}{n^2 \delta_N} - \int_0^R \left(\frac{\sin x}{x} \right)^2 dx \right| < \frac{\varepsilon}{3}$$

whenever $N > N_0$. Then for $N > N_0$, $\delta_N = \frac{R}{N}$ we have

$$\left| \sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2 \delta} - \int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx \right| < \varepsilon.$$

Consequently

$$\int_0^{\infty} \left(\frac{\sin x}{x} \right)^2 dx = \lim_{N \rightarrow \infty} \frac{\pi - \delta_N}{2} = \frac{\pi}{2}.$$

(e) Taking $\delta = \pi/2$ yields

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}.$$

Exercise 8.13 Put $f(x) = x$ if $0 \leq x < 2\pi$, and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Solution. By computation we see that $a_n = 0$ for $n > 0$, and $a_0 = 2\pi$. Computation shows that $b_n = \frac{-2}{n}$. Hence Parseval's relation gives

$$\frac{8\pi^2}{3} = \frac{1}{2}(2\pi)^2 + 4 \sum_{n=1}^{\infty} \frac{1}{n^2},$$

so that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

There is another way of deriving this result. Since

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2},$$

denoting this last sum by X , we find that

$$X - \frac{1}{4}X = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8}$$

and hence, by part (e) of the previous problem

$$X = \frac{4}{3} \frac{\pi^2}{8} = \frac{\pi^2}{6}.$$

Exercise 8.14 If $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$, prove that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(A recent article by E. L. Stark contains many references to series of the form $\sum n^{-s}$, where s is a positive integer. See *Math. Mag.*, vol. 47, 1974, pp. 197–202.)

Solution. Since $f(x)$ is an even function, $b_n = 0$ for all n . The a_n 's are computed in a straightforward manner:

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} (\pi - x)^2 dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{3} \pi^2;$$

and

$$a_n = \frac{2}{\pi} \int_0^\pi (\pi - x)^2 \cos nx \, dx = (-1)^n \frac{2}{\pi} \int_0^\pi x^2 \cos nx \, dx,$$

so that, eventually, we find $a_n = \frac{4}{n^2}$.

This gives the stated Fourier series, and since $f(x)$ satisfies the Lipschitz condition of Theorem 8.14, the series converges to $f(x)$ at every point. Taking $x = 0$ gives the first of the two desired equalities:

$$\pi^2 = f(0) = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Parseval's theorem yields

$$\frac{2\pi^4}{5} = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 \, dx = \frac{2\pi^4}{9} + 16 \sum_{n=1}^{\infty} \frac{1}{n^4},$$

which easily transforms to the desired relation.

Exercise 8.15 With D_n as defined in (77), put

$$K_n(x) = \frac{1}{N+1} \sum_{n=0}^N D_n(x).$$

Prove that

$$K_N(x) = \frac{1}{N+1} \frac{1 - \cos(N+1)x}{1 - \cos x}$$

and that

$$(a) \, K_n \geq 0,$$

$$(b) \, \frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x) \, dx = 1.$$

$$(c) \, K_n(x) \leq \frac{1}{N+1} \frac{2}{1 - \cos \delta} \text{ if } 0 < \delta \leq |x| \leq \pi.$$

If $s_N = s_N(f; x)$ is the N th partial sum of the Fourier series of f , consider the arithmetic means

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}.$$

Prove that

$$\sigma_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t) \, dt$$

and hence prove Fejér's theorem:

If f is continuous, with period 2π , then $\sigma_N(f; x) \rightarrow f(x)$ uniformly on $[-\pi, \pi]$.

Hint: Use properties (a), (b), (c) to proceed as in Theorem 7.26.

Solution. Using the formula $1 - \cos \theta = 2 \sin^2 \frac{1}{2} \theta$, and the formula $D_n(x) = \frac{\sin(n + \frac{1}{2})x}{\sin \frac{1}{2}x}$, we deduce that

$$(1 - \cos x)K_N(x) = \frac{1}{N+1} \sum_{n=0}^N \sin \frac{1}{2}x \sin(n + \frac{1}{2})x.$$

Now, however, $\sin \alpha \sin \beta = \frac{1}{2} \cos(\alpha - \beta) - \cos(\alpha + \beta)$, so that

$$\begin{aligned} (1 - \cos x)K_N(x) &= \frac{1}{N+1} \sum_{n=0}^N (\cos(nx) - \cos((n+1)x)) = \\ &= \frac{1}{N+1} (1 - \cos(N+1)x). \end{aligned}$$

The formula is now established. Notice that it could also be written

$$K_N(x) = \frac{1}{N+1} \left[\frac{\sin(\frac{N+1}{2}x)}{\sin \frac{1}{2}x} \right]^2.$$

(a) The nonnegativity of $K_N(x)$ is an immediate consequence of either of the formulas just written.

(b) It was established in the text that $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(x) dx = 1$, and so the same result for $K_N(x)$, which is an average of the $D_n(x)$, must follow by routine computation.

(c) This inequality is an immediate consequence of the facts that $\cos(N+1)x \geq -1$ and that $\cos x$ is decreasing on $[0, \pi]$.

The formula for $\sigma_N(f; x)$ is an immediate consequence of the definition of $\sigma_N(f; x)$ and the corresponding formula for $s_n(f; x)$.

Now let $M = \sup |f(x)|$, the supremum being taken over all x . By (a) and (b) we have

$$\begin{aligned} |\sigma_N(x) - f(x)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x-t) - f(x)] K_N(t) dt \right| \\ &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-t) - f(x)| K_N(t) dt \\ &\leq \frac{1}{2\pi} \int_{-\delta}^{\delta} |f(x-t) - f(x)| K_N(t) dt + \\ &\quad + \frac{1}{\pi} (\pi - \delta) \frac{1}{N+1} \frac{2}{1 - \cos \delta} 2M \\ &\leq \sup_{|t| \leq \delta} |f(x-t) - f(x)| + \frac{Q_\delta}{N+1} \end{aligned}$$

where $Q_\delta = \frac{4M(\pi-\delta)}{\pi(N+1)(1-\cos\delta)}$.

Given $\varepsilon > 0$, we first choose $\delta > 0$ so small that $\sup_{|t| \leq \delta} |f(x-t) - f(x)| < \frac{\varepsilon}{2}$ for all x . With this δ fixed, we then have $|\sigma_N(f; x) - f(x)| < \varepsilon$ for all $N > \frac{2Q_\delta}{\varepsilon}$ and all x .

Exercise 8.16 Prove a pointwise version of Fejér's Theorem:

If $f \in \mathcal{R}$ and $f(x+)$, $f(x-)$ exist for some x , then

$$\lim_{N \rightarrow \infty} \sigma_N(f; x) = \frac{1}{2}[f(x+) + f(x-)].$$

Solution. We need only a slight modification of the argument just given, namely the formula

$$\begin{aligned} \sigma_N(f; x) - \frac{1}{2}[f(x+) + f(x-)] &= \\ &= \frac{1}{2\pi} \int_0^\pi [f(x-t) - f(x-)] K_N(t) dt + \frac{1}{2\pi} \int_{-\pi}^0 [f(x-t) - f(x+)] K_N(t) dt. \end{aligned}$$

Each of these two integrals can be broken up into an integral over a half-neighborhood of 0 and an integral outside that neighborhood. The first of the integrals can be made small if the neighborhood is taken small enough (independently of N). With that neighborhood fixed, the second integral in each case can be made small if N is large enough using the same inequalities just stated.

Exercise 8.17 Assume f is bounded and monotonic on $[-\pi, \pi]$ with Fourier coefficients c_n , as given by (62).

- (a) Use Exercise 17 of Chap. 6 to prove that $\{nc_n\}$ is a bounded sequence.
 (b) Combine (a) with Exercise 16 and with Exercise 14(e) of Chap. 3, to conclude that

$$\lim_{n \rightarrow \infty} s_N(f; x) = \frac{1}{2}[f(x+) + f(x-)]$$

for every x .

- (c) Assume only that $f \in \mathcal{R}$ on $[-\pi, \pi]$ and that f is monotonic in some segment $(\alpha, \beta) \subset [-\pi, \pi]$. Prove that the conclusion of (b) holds for every $x \in (\alpha, \beta)$.

(This is an application of the localization theorem.)

Solution. (a) by Exercise 17 of Chap. 6 we have

$$\frac{1}{2\pi} \int_{-\pi}^\pi f(x) e^{-inx} dx = \frac{-1}{2\pi n} \int_{-\pi}^\pi e^{-inx} df(x),$$

from which it follows that

$$|nc_n| \leq \frac{1}{2\pi} [f(\pi) - f(-\pi)].$$

(b) Since $f(x+)$ and $f(x-)$ exist at every point, it follows from the previous exercise that $\sigma_N(f; x) \rightarrow \frac{1}{2}[f(x+) + f(x-)]$. Then Exercise 14(e) of Chap. 3 assures us that $s_n(f; x)$ has the same limit.

(c) Let $g(x) = f(x)$ for $\alpha \leq x \leq \beta$, $g(x) = f(\alpha)$ for $0 \leq x \leq \alpha$ and $g(x) = f(\beta)$ for $\beta \leq x \leq 2\pi$. Then $s_N(g; x) \rightarrow \frac{1}{2}[g(x+) + g(x-)]$ for all x by part (b). Since $s_N(g; x) - s_N(f; x) \rightarrow 0$ for $\alpha < x < \beta$ by the Corollary to Theorem 8.14, it follows that $s_N(f; x) \rightarrow \frac{1}{2}[g(x+) + g(x-)] = \frac{1}{2}[f(x+) + f(x-)]$ for these values of x .

Exercise 8.18 Define

$$\begin{aligned} f(x) &= x^3 - \sin^2 x \tan x \\ g(x) &= 2x^2 - \sin^2 x - x \tan x. \end{aligned}$$

Find out, for each of these two functions, whether it is positive or negative for all $x \in (0, \pi/2)$, or whether it changes sign. Prove your answer.

Solution. Both functions tend to $-\infty$ as $x \rightarrow \frac{\pi}{2}$. Hence the only question is whether they ever become positive. We note that the derivative of the first function is $3x^2 - \sin^2 x - \tan^2 x$. By writing $\sin^2 x$ as $\frac{1}{2} - \frac{1}{2} \cos 2x$ and making repeated use of the formula $\frac{d}{dx} \tan^k x = k \tan^{k-1} x + k \tan^{k+1} x$, we find that the first six derivatives of this function vanish at 0, and that the sixth derivative is

$$-32 \sin 2x - 272 \tan x - 1232 \tan^3 x - 1104 \tan^5 x - 144 \tan^7 x,$$

which is negative on $(0, \frac{\pi}{2})$. Hence all of the first six derivatives are negative on this interval, and therefore the function itself is negative.

The same technique applies to the second function. All of its first five derivatives vanish at $x = 0$ and the fifth is

$$\begin{aligned} -[16 \sin 2x + 16x + 80 \tan x + 136x \tan^2 x + \\ + 200 \tan^3 x + 240x \tan^4 x + 120 \tan^5 x + 120x \tan^6 x], \end{aligned}$$

which is negative on $(0, \frac{\pi}{2})$. Hence this function is always negative on that interval.

Exercise 8.19 Suppose f is a continuous function on R^1 , $f(x + 2\pi) = f(x)$, and α/π is irrational. Prove that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x + n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

for every x . *Hint:* Do it first for $f(x) = e^{ikx}$.

Solution. Following the hint, we observe that both sides of the desired equality equal 1 trivially when $k = 0$. In any other case the right-hand side is zero, and the left-hand side is

$$\lim_{N \rightarrow \infty} e^{ikx} \frac{1 - e^{i(n+1)k\alpha}}{N(1 - e^{i\alpha})},$$

which tends to zero as $N \rightarrow \infty$.

Since both sides are linear functions of f , it now follows that the relation holds for all trigonometric polynomials. Finally, since both sides are bounded by the supremum of f , given ε , we can approximate f uniformly within ε by a trigonometric polynomial. It then follows that all the means on the left, for N sufficiently large, are within 2ε of the integral on the right. Since ε is arbitrary, it follows that the limit on the left equals the integral on the right.

Exercise 8.20 The following simple computation yields a good approximation to Stirling's formula.

For $m = 1, 2, 3, \dots$, define

$$f(x) = (m+1-x) \log m - (x-m) \log(m+1)$$

if $m \leq x \leq m+1$, and define

$$g(x) = \frac{x}{m} - 1 + \log m$$

if $m - \frac{1}{2} \leq x < m + \frac{1}{2}$. Draw the graphs of f and g . Note that $f(x) \leq \log x \leq g(x)$ if $x \geq 1$ and that

$$\int_1^n f(x) dx = \log(n!) - \frac{1}{2} \log n > -\frac{1}{8} + \int_1^n g(x) dx.$$

Integrate $\log x$ over $[1, n]$. Conclude that

$$\frac{7}{8} < \log(n!) - \left(n + \frac{1}{2}\right) \log n + n < 1$$

for $n = 2, 3, 4, \dots$ (Note: $\log \sqrt{2\pi} \sim 0.918\dots$.) Thus

$$e^{7/8} < \frac{n!}{(n/e)^n \sqrt{n}} < e.$$

Solution. We first draw the graphs of f and g in the range $x = 1$ to $x = 10$. We note that f is merely the broken set of chords joining the points on the graph of $\log x$ at integer values of x , and g is made up of segments of the tangents at these points (g is not continuous). Because the downward side of the graph of $\log x$ is convex, $f(x) \leq \log x \leq g(x)$ for all x . The estimate for the integral of f is straightforward: The integral is the sum of the areas of one triangle and $n-2$ trapezoids with base 1 and parallel sides $\log k$ and $\log(k+1)$ ($k = 2, \dots, n-1$).

We find it equal to $\frac{1}{2} \log 1 + \log 2 + \log 3 + \cdots + \log(n-1) + \frac{1}{2} \log n = \log(n!) - \frac{1}{2} \log n$, as asserted. Meanwhile the integral of $g(x)$ is also a sum of trapezoids and two triangles and equals $\frac{1}{8} + \log(n!) - \frac{1}{2} \log n - \frac{1}{8n}$. Hence we have

$$\log(n!) - \frac{1}{2} \log n < \int_1^n \log x \, dx < \frac{1}{8} + \log(n!) - \frac{1}{2} \log n - \frac{1}{8n} < \frac{1}{8} + \log(n!) - \frac{1}{2} \log n.$$

Now straightforward computation reveals that

$$\int_1^n \log x \, dx = (n \log n - n) - (1 \log 1 - 1) = n \log n - n + 1.$$

The desired inequalities are now deduced by taking exponentials of the three expressions.

Exercise 8.21 Let

$$L_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)| \, dt \quad (n = 1, 2, 3, \dots).$$

Prove that there exists a constant $C > 0$ such that

$$L_n > C \log n \quad (n = 1, 2, 3, \dots),$$

or, more precisely, that the sequence

$$\left\{ L_n - \frac{4}{\pi^2} \log n \right\}$$

is bounded.

Solution. We observe that

$$\begin{aligned} L_n &= \frac{1}{\pi} \int_0^{\frac{2\pi}{2n+1}} \frac{\sin(n + \frac{1}{2})t}{\sin \frac{1}{2}t} \, dt + \\ &+ \sum_{k=1}^{n-1} \frac{1}{\pi} \int_{\frac{2\pi k}{2n+1}}^{\frac{2\pi(k+1)}{2n+1}} \frac{(-1)^k \sin(n + \frac{1}{2})t}{\sin \frac{1}{2}t} \, dt + \frac{1}{\pi} \int_{\frac{2n\pi}{2n+1}}^{\pi} \frac{(-1)^n \sin(n + \frac{1}{2})t}{\sin \frac{1}{2}t} \, dt \end{aligned}$$

The substitution $u = (n + \frac{1}{2})t$ changes the first and last terms into the sum

$$\frac{1}{\pi} \int_0^{\pi} \frac{\sin u}{(n + \frac{1}{2}) \sin(\frac{u}{2n+1})} \, du + \int_{n\pi}^{(n+\frac{1}{2})\pi} \frac{(-1)^n \sin u}{(n + \frac{1}{2}) \sin(\frac{u}{2n+1})} \, du.$$

The first of these terms tends to $\frac{1}{2\pi} \int_0^{\pi} \sin u \, du = \frac{1}{\pi}$ as $n \rightarrow \infty$. The second tends to 0 (for $u \in [n\pi, (n + \frac{1}{2})\pi]$ we have $\sin(\frac{u}{2n+1}) \geq \sin \frac{n\pi}{2n+1}$, which tends to 1 as $n \rightarrow \infty$).

Thus we find that

$$\frac{1}{\pi} + \varepsilon_n + \sum_{k=1}^{n-1} \frac{1}{\pi \sin(\frac{\pi(k+1)}{2n+1})} \left| \int_{\frac{2\pi k}{2n+1}}^{\frac{2\pi(k+1)}{2n+1}} \sin(n + \frac{1}{2})t \, dt \right| < L_n,$$

where $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$.

If we take out the first two terms of the sum instead of just the first, we find similarly that

$$\begin{aligned} L_n &= \frac{1}{\pi} \int_0^{\frac{4\pi}{2n+1}} \frac{|\sin(n + \frac{1}{2})t|}{\sin \frac{1}{2}t} \, dt + \\ &+ \sum_{k=2}^{n-1} \frac{1}{\pi} \int_{\frac{2\pi k}{2n+1}}^{\frac{2\pi(k+1)}{2n+1}} \frac{(-1)^k \sin(n + \frac{1}{2})t}{\sin \frac{1}{2}t} \, dt + \frac{1}{\pi} \int_{\frac{2n\pi}{2n+1}}^{\pi} \frac{(-1)^n \sin(n + \frac{1}{2})t}{\sin \frac{1}{2}t} \, dt. \end{aligned}$$

Again the substitution $u = (n + \frac{1}{2})t$ changes the first and last terms into the sum

$$\frac{1}{\pi} \int_0^{2\pi} \frac{\sin u}{(n + \frac{1}{2}) \sin(\frac{u}{2n+1})} \, du + \int_{n\pi}^{(n+\frac{1}{2})\pi} \frac{(-1)^n \sin u}{(n + \frac{1}{2}) \sin(\frac{u}{2n+1})} \, du.$$

The first of these terms tends to $\frac{1}{2\pi} \int_0^{2\pi} |\sin u| \, du = \frac{2}{\pi}$ as $n \rightarrow \infty$, and once again the second tends to zero.

Thus we find that

$$L_n < \frac{2}{\pi} + \eta_n + \sum_{k=2}^{n-1} \frac{1}{\pi \sin(\frac{\pi k}{2n+1})} \left| \int_{\frac{2\pi k}{2n+1}}^{\frac{2\pi(k+1)}{2n+1}} \sin(n + \frac{1}{2})t \, dt \right|,$$

where $\eta_n \rightarrow 0$ as $n \rightarrow \infty$.

Once again, in each of the integrals under the sigma in the last two inequalities we make the substitution $u = (n + \frac{1}{2})t$. When we do so, we have

$$\frac{1}{\pi} + \varepsilon + \sum_{k=1}^{n-1} \frac{2}{(n + \frac{1}{2})\pi \sin(\frac{\pi(k+1)}{2n+1})} < L_n < \frac{2}{\pi} + \eta_n + \sum_{k=2}^{n-1} \frac{2}{(n + \frac{1}{2})\pi \sin(\frac{\pi k}{2n+1})},$$

where $\varepsilon_n \rightarrow 0$ and $\eta_n \rightarrow 0$. It therefore follows that

$$\frac{1}{\pi} + \varepsilon_n < L_n - \sum_{k=1}^{n-1} \frac{2}{\pi} \frac{1}{(n + \frac{1}{2}) \sin(\frac{\pi(k+1)}{2n+1})} < \frac{2}{\pi} + \eta_n - \frac{1}{(n + \frac{1}{2}) \sin(\frac{\pi n}{2n+1})}.$$

The extremes in these inequalities are both bounded. Hence we will be done if we can show that

$$\frac{2}{\pi} \log n - \sum_{k=1}^{n-1} \frac{1}{(n + \frac{1}{2}) \sin(\frac{\pi(k+1)}{2n+1})}$$

remains bounded. To do this, we use the fact that there is a constant K such that

$$\left| \frac{1}{\sin x} - \frac{1}{x} \right| \leq Kx$$

for $0 < x \leq \frac{\pi}{2}$. This fact in turn is a consequence of the fact that, by L'Hospital's rule,

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^2 \sin x} = -\frac{1}{2}.$$

We thus have

$$\sum_{k=1}^{n-1} \frac{1}{(n + \frac{1}{2}) \sin(\frac{\pi(k+1)}{2n+1})} = E_n + \sum_{k=1}^{n-1} \frac{1}{(n + \frac{1}{2})(\frac{\pi(k+1)}{2n+1})},$$

where

$$\begin{aligned} |E_n| &\leq K \frac{1}{n + \frac{1}{2}} \sum_{k=1}^{n-1} \frac{\pi(k+1)}{2n+1} = \\ &= \frac{2K}{\pi(2n+1)^2} \sum_{k=1}^n k+1 = \frac{2K}{\pi(2n+1)^2} \left[\frac{(n+1)(n+2)}{2} - 1 \right]. \end{aligned}$$

Since the right-hand side tends to $\frac{K}{4\pi}$ as $n \rightarrow \infty$, we see that E_n remains bounded as $n \rightarrow \infty$. We will be finished if we can show that

$$\log n - \sum_{k=1}^{n-1} \frac{1}{k+1}$$

remains bounded. But this was done in Exercise 9 above.

Exercise 8.22 If α is real and $-1 < x < 1$, prove Newton's binomial theorem

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n.$$

Hint: Denote the right side by $f(x)$. Prove that the series converges. Prove that

$$(1+x)f'(x) = \alpha f(x).$$

and solve this differential equation.

Show also that

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} x^n$$

if $-1 < x < 1$ and $\alpha > 0$.

Solution. Following the hint, we use the ratio test to establish that the radius of convergence of the power series that defines $f(x)$ is 1. This amounts merely to the statement that

$$\lim_{n \rightarrow \infty} \left| \frac{\alpha - n}{n + 1} \right| = 1.$$

The differential equation then results from termwise operations on the series and the fact that

$$\frac{\alpha(\alpha - 1) \cdots (\alpha - n)}{n!} + \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{(n - 1)!} = \alpha \frac{\alpha(\alpha - 1) \cdots (\alpha - n + 1)}{n!}.$$

Then, given that $f(0) = 1 \neq 0$, it follows that for x near 0 we have

$$\frac{f'(x)}{f(x)} = \frac{\alpha}{1 + x}.$$

so that $\log f(x)$ and $\log(1 + x)^\alpha$ have the same derivative, and hence differ by a constant, which turns out to be zero, since both equal 1 at $x = 0$. It thus follows that $f(x) = (1 + x)^\alpha$.

To prove the other relation, we merely observe that

$$(-\alpha)(-\alpha - 1) \cdots (-\alpha - n + 1) = (-1)^n \alpha(\alpha + 1) \cdots (\alpha + n - 1) = (-1)^n \frac{\Gamma(n + \alpha)}{\Gamma(\alpha)}.$$

Exercise 8.23 Let γ be a continuously differentiable *closed* curve in the complex plane with parameter interval $[a, b]$, and assume that $\gamma(t) \neq 0$ for every $t \in [a, b]$. Define the *index* of γ to be

$$\text{Ind}(\gamma) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt.$$

Prove that $\text{Ind}(\gamma)$ is always an integer.

Hint: There exists φ on $[a, b]$ with $\varphi' = \gamma'/\gamma$, $\varphi(a) = 0$. Hence $\gamma \exp(-\varphi)$ is constant. Since $\gamma(a) = \gamma(b)$, it follows that $\exp(\varphi(a)) = \exp(\varphi(b)) = 1$. Note that $\varphi(b) = 2\pi i \text{Ind}(\gamma)$.

Compute $\text{Ind}(\gamma)$ when $\gamma(t) = e^{int}$, $a = 0$, $b = 2\pi$.

Explain why $\text{Ind}(\gamma)$ is often called the *winding number* of γ around 0.

Solution. Again, following the hint leaves very little to do. We define

$$\varphi(x) = \int_a^x \frac{\gamma'(t)}{\gamma(t)} dt,$$

so that we automatically have $\varphi'(t) = \frac{\gamma'(t)}{\gamma(t)}$. The fact that $\gamma \exp(-\varphi)$ is constant is now a consequence of the chain rule. It then follows immediately that $\exp(\varphi(b)) = 1$, so that $\varphi(b) = 2\pi in$ for some integer n .

Routine computation shows that $\text{Ind}(\gamma) = n$ if $\gamma(t) = e^{int}$, $0 \leq t \leq 2\pi$. Since this curve winds counterclockwise about 0 a total of n times, the name *winding number* is appropriate.

Exercise 8.24 Let γ be as in Exercise 23, and assume in addition that the range of γ does not intersect the negative real axis. Prove that $\text{Ind}(\gamma) = 0$. *Hint:* For $0 \leq c < \infty$, $\text{Ind}(\gamma + c)$ is a continuous integer-valued function of c . Also, $\text{Ind}(\gamma + c) \rightarrow 0$ as $c \rightarrow \infty$.

Solution. Following the hint, we observe that

$$f(c) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t) + c} dt,$$

is a continuous function of c on $[0, \infty)$, since

$$|f(c_1) - f(c_2)| = \frac{1}{2\pi} \left| \int_a^b \frac{\gamma'(t)(c_1 - c_2)}{(\gamma(t) + c_1)(\gamma(t) + c_2)} dt \right| \leq K|c_1 - c_2|,$$

where $K = \frac{1}{2\pi r^2} \int_a^b |\gamma'(t)| dt$ and r is the supremum of the integrand for $c_1, c_2 \geq 0$ and $0 \leq t \leq 2\pi$. (This supremum is finite, since the integrand tends to zero as either c_1 or c_2 tends to infinity.) Furthermore

$$|f(c)| \leq \frac{1}{2\pi c} \int_a^b \frac{|\gamma'(t)|}{|1 + \frac{\gamma(t)}{c}|} dt,$$

and this last expression tends to 0 as $c \rightarrow \infty$. It follows, since f assumes only integer values, that $f(c) \equiv 0$. In particular $f(0) = \text{Ind}(\gamma) = 0$.

Exercise 8.25 Suppose γ_1 and γ_2 are curves as in Exercise 23, and

$$|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)| \quad (a \leq t \leq b)$$

Prove that $\text{Ind}(\gamma_1) = \text{Ind}(\gamma_2)$.

Hint: Put $\gamma = \gamma_2/\gamma_1$. Then $|1 - \gamma| < 1$. Hence $\text{Ind}(\gamma) = 0$ by Exercise 24. Also,

$$\frac{\gamma'}{\gamma} = \frac{\gamma_2'}{\gamma_2} - \frac{\gamma_1'}{\gamma_1}.$$

Solution. The hint leaves almost nothing to be done. The inequality established for γ shows that the real part of γ is always positive, so that the hypotheses of Exercise 24 are satisfied. The relation for $\frac{\gamma'}{\gamma}$ is a routine computation, and shows in general that $\text{Ind}(\gamma\delta) = \text{Ind}(\gamma) + \text{Ind}(\delta)$.

Exercise 8.26 Let γ be a *closed* curve in the complex plane (not necessarily differentiable) with parameter interval $[0, 2\pi]$, such that $\gamma(t) \neq 0$ for every $t \in [0, 2\pi]$.

Choose $\delta > 0$ such that $|\gamma(t)| > \delta$ for all $t \in [0, 2\pi]$. If P_1 and P_2 are trigonometric polynomials such that $|P_i(t) - \gamma(t)| < \delta/4$ for all $t \in [0, 2\pi]$, (their existence is assured by Theorem 8.15), prove that

$$\text{Ind}(P_1) = \text{Ind}(P_2)$$

by applying Exercise 25.

Define this common value to be $\text{Ind}(\gamma)$.

Prove that the statements of Exercises 24 and 25 hold without any differentiability assumptions.

Solution. Since $|P_1(t) - P_2(t)| < \frac{\delta}{2} < |P_1(t)|$, (because $|P_1(t)| \geq |f(t)| - |f(t) - P_1(t)| > \frac{3\delta}{4}$), the equality of the indices follows from Exercise 25, as stated.

Exercise 24 remains valid, since if $\gamma(t)$ does not intersect the negative real axis, there is a positive number $\delta > 0$ such that $|\gamma(t) - x| \geq \delta$ for all $x \leq 0$. Then if $|P_j(t) - \gamma(t)| < \delta$ for all $t \in [0, 2\pi]$, it follows that $P_j(t)$ also does not intersect the negative real axis, hence has winding number 0.

Exercise 25 remains valid, since if $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)|$ for all t , we can let $\delta = \min_t |\gamma_1(t)| - |\gamma_1(t) - \gamma_2(t)|$. Then if $|P_i(t) - \gamma_i(t)| < \delta/4$ for all t , it follows that $|P_1(t) - P_2(t)| \leq |\gamma_1(t) - \gamma_2(t)| + (\delta/2) < |\gamma_1(t)| - (\delta/4) \leq |P_1(t)|$, and so $\text{Ind}(P_1) = \text{Ind}(P_2)$, by Exercise 25.

Exercise 8.27 Let f be a continuous complex function defined in the complex plane. Suppose there is a positive integer n and a complex number $c \neq 0$ such that

$$\lim_{|z| \rightarrow \infty} z^{-n} \gamma(z) = c.$$

Prove that $f(z) = 0$ for at least one complex number z .

Note that this is a generalization of Theorem 8.8.

Hint: Assume $f(z) \neq 0$ for all z , define

$$\gamma_r(t) = f(re^{it})$$

for $0 \leq r < \infty$, $0 \leq t \leq 2\pi$, and prove the following statements about the curve γ .

(a) $\text{Ind}(\gamma_0) = 0$.

(b) $\text{Ind}(\gamma_r) = n$ for all sufficiently large r .

(c) $\text{Ind}(\gamma_r)$ is a continuous function of r on $[0, \infty)$.

[In (b) and (c), use the last part of Exercise 26.]

Show that (a), (b), and (c) are contradictory, since $n > 0$.

Solution. (a) Since $\gamma_0(t) = f(0)$ for all t , we have $\gamma'_0(t) = 0$ for all t , and hence by definition $\text{Ind}(\gamma_0) = 0$.

(b) Choose R so large that $|z^{-n}f(z) - c| < \frac{|c|}{2}$ whenever $|z| > R$. Then for all r we have $\text{Ind}(\gamma_r) = \text{Ind}(\gamma_{r_1}) + \text{Ind}(\gamma_{r_2})$, where $\gamma_{r_1}(t) = r^n e^{int}$ and

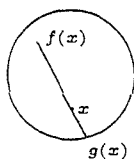


Figure 8.1: The Brouwer fixed-point theorem

$\gamma_{r2}(t) = r^{-n} e^{-int} f(re^{it})$. By Exercise 25 we have $\text{Ind}(\gamma_{r2}) = 0$ for $r > R$, and by direct computation we have $\text{Ind}(\gamma_{r1}) = n$ for all r .

(c) Fix $r_0 > 0$, and let $\varepsilon = \min_{0 \leq t \leq 2\pi} |f(r_0 e^{it})|$. Then choose $\delta \in (0, r_0)$ such that $|f(r_0 e^{it}) - f(re^{it})| < \varepsilon$ if $|r - r_0| < \delta$. Then by Exercise 25 we again have $\text{Ind}(\gamma_r) = \text{Ind}(\gamma_{r_0})$ for $|r - r_0| < \delta$. Hence $\text{Ind}(\gamma_r)$ is a locally constant function of r . By the connectivity of $[0, \infty)$, it follows that it is globally constant, which contradicts (a) and (b).

Exercise 8.28 Let \overline{D} be the closed unit disc in the complex plane. (Thus $z \in \overline{D}$ if and only if $|z| \leq 1$.) Let g be a continuous mapping of \overline{D} into the unit circle T . (Thus $|g(z)| = 1$ for every $z \in \overline{D}$.)

Prove that $g(z) = -z$ for at least one $z \in T$.

Hint: For $0 \leq r \leq 1$, $0 \leq t \leq 2\pi$, put

$$\gamma_r(t) = g(re^{it}),$$

and put $\psi(t) = e^{-it} \gamma_1(t)$. If $g(z) \neq -z$ for every $z \in T$, then $\psi(t) \neq -1$ for every $t \in [0, 2\pi]$. Hence $\text{Ind}(\psi) = 0$, by Exercises 24 and 25. It follows that $\text{Ind}(\gamma_1) = 1$. But $\text{Ind}(\gamma_0) = 0$. Derive a contradiction, as in Exercise 27.

Solution. The hint tells us that $\psi(t)$ does not meet the negative real axis, hence has index 0, by Exercise 24. Hence by Exercise 25, γ_1 has index 1. Again, since $\gamma_0 = g(0) \neq 0$ (since $g(0) \neq -0 = 0$), it follows that $\text{Ind}(\gamma_0) = 0$. But, as before, since $|g(z)| = 1$ for all z , it follows that $\text{Ind}(\gamma_r)$ is locally constant and hence by the connectivity of $[0, 1]$, globally constant. Thus, once again, we have a contradiction.

Exercise 8.29 Prove that every continuous mapping f of \overline{D} into \overline{D} has a fixed point in \overline{D} .

(This is the 2-dimensional case of Brouwer's fixed-point theorem.)

Hint: Assume $f(z) \neq z$ for every $z \in \overline{D}$. Associate to each $z \in \overline{D}$ the point $g(z) \in T$ which lies on the ray that starts at $f(z)$ and passes through z . Then g maps \overline{D} into T , $g(z) = z$ if $z \in T$, and g is continuous, because

$$g(z) = z - s(z)[f(z) - z],$$

where $s(z)$ is the unique nonnegative root of a certain quadratic equation whose coefficients are continuous functions of f and z . Apply Exercise 28.

Solution. The number $s = s(z)$ is a nonnegative real number because of the geometry of the situation (see figure). The quadratic equation in question is given by the relation $|g(z)|^2 = 1$, i.e.,

$$|f(z) - z|^2 s^2 + 2(|z|^2 - \operatorname{Re}(\bar{z}f(z)))s + |z|^2 - 1 = 0.$$

It is well-known that a quadratic equation $az^2 + bz + c = 0$ has one and only one nonnegative root if a , b , and c are real and $ac < 0$. We can write explicitly

$$s(z) = \frac{|z|^2 - \operatorname{Re}(\bar{z}f(z)) + \sqrt{(|z|^2 - \operatorname{Re}(\bar{z}f(z)))^2 + |f(z) - z|^2(1 - |z|^2)}}{|f(z) - z|^2},$$

which makes it clear that $s(z)$ is a continuous function of z . Hence $g(z)$ is continuous.

We now know that there must be a value at which $g(z) = -z$. But this is impossible, since $|g(z)| = 1$ for all z and $g(z) = z$ if $|z| = 1$.

Exercise 8.30 Use Stirling's formula to prove that

$$\lim_{x \rightarrow \infty} \frac{\Gamma(x+c)}{x^c \Gamma(x)} = 1$$

for every real constant c .

Solution. We need Stirling's formula in the form

$$\lim_{z \rightarrow \infty} \frac{\Gamma(z)}{\left(\frac{z-1}{e}\right)^{z-1} \sqrt{2\pi(z-1)}} = 1.$$

Applying this result with $z = x + c$ and $z = x$, we get

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{\Gamma(x+c)}{x^c \Gamma(x)} &= \\ &= \lim_{x \rightarrow \infty} f(x) \cdot \frac{\Gamma(x+c)}{\left(\frac{x+c-1}{e}\right)^{x+c-1} \sqrt{2\pi(x+c-1)}} \cdot \frac{\left(\frac{x-1}{e}\right)^{x-1} \sqrt{2\pi(x-1)}}{\Gamma(x)}, \end{aligned}$$

where

$$f(x) = \frac{1}{x^c} \cdot \frac{\left(\frac{x+c-1}{e}\right)^{x+c-1}}{\left(\frac{x-1}{e}\right)^{x-1}} \cdot \sqrt{\frac{x+c-1}{x-1}} = \frac{\left(1 + \frac{c-1}{x}\right)^x \left(1 + \frac{c-1}{x}\right)^{c-1}}{e^c \left(1 - \frac{1}{x}\right)^x \left(1 - \frac{1}{x}\right)^{-1}} \cdot \sqrt{\frac{x+c-1}{x-1}}.$$

Since $x^x \rightarrow 1$ as $x \rightarrow \infty$, it now follows that $\lim_{x \rightarrow \infty} f(x) = 1$, which, combined with Stirling's formula, gives the desired result.

Exercise 8.31 In the proof of Theorem 7.26 it was shown that

$$\int_{-1}^1 (1-x^2)^n dx \geq \frac{4}{3\sqrt{\pi}}.$$

for $n = 1, 2, 3, \dots$. Use Theorem 8.20 and Exercise 30 to show the more precise result

$$\lim_{n \rightarrow \infty} \sqrt{n} \int_{-1}^1 (1-x^2)^n dx = \sqrt{\pi}.$$

Solution. Let $u = x^2$ in the integral, so that $dx = \frac{1}{2}u^{-\frac{1}{2}} du$. We then have

$$\sqrt{n} \int_{-1}^1 (1-x^2)^n dx = \sqrt{n} \int_0^1 (1-u)^n u^{-\frac{1}{2}} du = \frac{\sqrt{n} \Gamma(n + \frac{1}{2}) \Gamma(\frac{1}{2})}{\Gamma(n+1)},$$

and taking $c = \frac{1}{2}$ in Exercise 30, we find that this last expression tends to $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Chapter 9

Functions of Several Variables

Exercise 9.1 If S is a nonempty subset of a vector space X , prove (as asserted in Sec. 9.1) that the span of S is a vector space.

Solution. We need only verify that the span of S is closed under the two vector space operations. All the other properties of a vector space hold in the span of S , since it is contained in a vector space in which they hold.

To that end, let \mathbf{x} and \mathbf{y} be elements of the span of S , and let c be any real number. By definition there are elements $\mathbf{x}_1, \dots, \mathbf{x}_m, \mathbf{y}_1, \dots, \mathbf{y}_n$, and scalars $c_1, \dots, c_m, d_1, \dots, d_n$ such that $\mathbf{x} = c_1\mathbf{x}_1 + \dots + c_m\mathbf{x}_m$ and $\mathbf{y} = d_1\mathbf{y}_1 + \dots + d_n\mathbf{y}_n$. We then have

$$\mathbf{x} + \mathbf{y} = c_1\mathbf{x}_1 + \dots + c_m\mathbf{x}_m + d_1\mathbf{y}_1 + \dots + d_n\mathbf{y}_n,$$

which is a finite linear combination of elements of S , hence belongs to the span of S . Likewise, by the distributive law,

$$c\mathbf{x} = c(c_1\mathbf{x}_1 + \dots + c_m\mathbf{x}_m) = (cc_1)\mathbf{x}_1 + \dots + (cc_m)\mathbf{x}_m,$$

which belongs to the span of S .

Exercise 9.2 Prove (as asserted in Sec. 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible.

Solution. Let $A : X \rightarrow Y$ and $B : Y \rightarrow Z$ be linear transformations, and let \mathbf{x} and \mathbf{y} be any elements of X and c any scalar. Then $BA : X \rightarrow Z$ satisfies

$$\begin{aligned} BA(\mathbf{x} + \mathbf{y}) &= B(A(\mathbf{x} + \mathbf{y})) \\ &= B(A(\mathbf{x}) + A(\mathbf{y})) \\ &= B(A(\mathbf{x})) + B(A(\mathbf{y})) \\ &= BA(\mathbf{x}) + BA(\mathbf{y}). \end{aligned}$$

Similarly,

$$\begin{aligned}
 BA(cx) &= B(A(cx)) \\
 &= B(cA(x)) \\
 &= cB(A(x)) \\
 &= cBA(x).
 \end{aligned}$$

If A is a one-to-one mapping of X onto Y , and z and w are any elements of Y , let $x = A^{-1}(z)$ and $y = A^{-1}(w)$. Then by definition $A(x) = z$ and $A(y) = w$. It therefore follows from the linearity of A that $A(x + y) = z + w$. Again, by definition, this means that $A^{-1}(z + w) = x + y = A^{-1}(z) + A^{-1}(w)$, so that A^{-1} preserves vector addition. Similarly, $A(cx) = cA(x) = cz$, so that $A^{-1}(cz) = cx = cA^{-1}(z)$, and hence A^{-1} also preserves scalar multiplication.

Exercise 9.3 Assume $A \in L(X, Y)$ and $Ax = 0$ only when $x = 0$. Prove that A is then 1-1.

Solution. Suppose $A(x) = A(y)$. It then follows that $A(x - y) = A(x) - A(y) = 0$. Hence by assumption $x - y = 0$, and so $x = y$; therefore A is one-to-one.

Exercise 9.4 Prove (as asserted in Sec. 9.30) that null spaces and ranges of linear transformations are vector spaces.

Solution. Let N be the null space of the linear transformation $A : X \rightarrow Y$, let x and y be elements of N , and let c be any scalar. By definition $A(x) = 0 = A(y)$, and $A(x + y) = A(x) + A(y) = 0 + 0 = 0$, so that, by definition, $x + y \in N$. Likewise $A(cx) = cA(x) = c0 = 0$, and so $cx \in N$. Therefore N is a subspace of X .

Let R be the range of A , let z and w be any elements of R , and let c be any scalar. By definition, there exist vectors $x \in X$ and $y \in X$ such that $z = A(x)$ and $w = A(y)$. Then $A(x + y) = A(x) + A(y) = z + w$, and hence $z + w \in R$. Likewise $A(cx) = cA(x) = cz$, so that $cz \in R$. Therefore R is a subspace of Y .

Exercise 9.5 Prove that to every $A \in L(R^n, R^1)$ corresponds a unique $y \in R^n$ such that $Ax = x \cdot y$. Prove also that $\|A\| = |y|$.

Hint: Under certain conditions, equality holds in the Schwarz inequality.

Solution. Let e_1, \dots, e_n be the standard basis of R^n , and let $y = A(e_1)e_1 + \dots + A(e_n)e_n$. Then for any $x = c_1e_1 + \dots + c_n e_n$ we have

$$\begin{aligned}
 A(x) &= c_1A(e_1) + \dots + c_nA(e_n) \\
 &= y \cdot x.
 \end{aligned}$$

There can be at most one such \mathbf{y} , since if $A(\mathbf{x}) = \mathbf{z} \cdot \mathbf{x}$, then $|\mathbf{y} - \mathbf{z}|^2 = \mathbf{y} \cdot \mathbf{y} - \mathbf{y} \cdot \mathbf{z} - \mathbf{z} \cdot \mathbf{y} + \mathbf{z} \cdot \mathbf{z} = A(\mathbf{y}) - A(\mathbf{y}) - A(\mathbf{z}) + A(\mathbf{z}) = 0$.

By the Schwarz inequality we have

$$|A(\mathbf{x})| = |\mathbf{y} \cdot \mathbf{x}| \leq |\mathbf{y}| |\mathbf{x}|$$

for all \mathbf{x} , so that $\|A\| \leq |\mathbf{y}|$. On the other hand $A(\mathbf{y}) = \mathbf{y} \cdot \mathbf{y} = |\mathbf{y}|^2$, so that $\|A\| \geq |\mathbf{y}|$.

Exercise 9.6 If $f(0,0) = 0$ and

$$f(x, y) = \frac{xy}{x^2 + y^2} \quad \text{if } (x, y) \neq (0, 0),$$

prove that $(D_1 f)(x, y)$ and $(D_2 f)(x, y)$ exist at every point of R^2 , although f is not continuous at $(0, 0)$.

Solution. At any point (x, y) except $(0, 0)$ the differentiability of $f(x, y)$ follows from the rules for differentiation and the principles of Chapter 5. At $(0, 0)$ it is a routine computation to verify that both partial derivatives equal zero:

$$(D_1 f)(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = 0.$$

However, $f(x, y)$ is not continuous at $(0, 0)$, since $f(x, x) = \frac{1}{2}$ for all $x \neq 0$, and hence $\lim_{x \rightarrow 0} f(x, x) = \frac{1}{2} \neq f(0, 0)$.

Exercise 9.7 Suppose that f is a real-valued function defined in an open set $E \subset R^n$, and that the partial derivatives $D_1 f, \dots, D_n f$ are bounded in E . Prove that f is continuous in E .

Hint: Proceed as in the proof of Theorem 9.21.

Solution. Let $\varepsilon > 0$ be given, and let $\mathbf{x}^0 = (x_1^0, \dots, x_n^0)$ be any point of E . First choose $\delta_0 > 0$ so that $\mathbf{y} \in E$ if $|\mathbf{y} - \mathbf{x}^0| < 2\delta_0$. Then, if $M = \max_{\mathbf{x} \in E} ((D_1 f)(\mathbf{x}), \dots, (D_n f)(\mathbf{x}))$, choose $\delta = \min \left(\delta_0, \frac{\varepsilon}{(n+1)M} \right)$. It then follows that if $|\mathbf{y} - \mathbf{x}^0| < \delta$, we have

$$\begin{aligned} |f(\mathbf{y}) - f(\mathbf{x}^0)| &= |f(y_1, \dots, y_n) - f(x_1^0, \dots, x_n^0)| \\ &\leq |f(y_1, y_2, \dots, y_n) - f(x_1^0, y_2, \dots, y_n)| + \\ &\quad + |f(x_1^0, y_2, \dots, y_n) - f(x_1^0, x_2^0, \dots, y_n)| + \dots \\ &\quad \dots + |f(x_1^0, x_2^0, \dots, x_{n-1}^0, y_n) - f(x_1^0, x_2^0, \dots, x_{n-1}^0, x_n^0)|, \end{aligned}$$

where the ellipsis indicates terms of the form

$$|f(x_1^0, x_2^0, \dots, x_{k-1}^0, y_k, y_{k+1}, \dots, y_n) - f(x_1^0, x_2^0, \dots, x_{k-1}^0, x_k^0, y_{k+1}, \dots, y_n)|.$$

By the mean-value theorem there is a number c_k between x_k^0 and y_k such that this last difference equals

$$|(D_k f)(x_1^0, x_2^0, \dots, x_{k-1}^0, c_k, y_{k+1}, \dots, y_n)(y_k - x_k^0)|,$$

which is at most $M\delta$. Since by definition $M\delta$ is at most $\frac{\varepsilon}{n+1}$, and there are only n such terms, it follows that $|f(\mathbf{x}^0) - f(\mathbf{y})| < \varepsilon$. Thus f is continuous.

Remark: We have actually shown that $f(\mathbf{x})$ satisfies a Lipschitz condition on any convex subset of E , i.e., that $|f(\mathbf{x}) - f(\mathbf{y})| \leq nM|\mathbf{x} - \mathbf{y}|$ on each convex subset.

Exercise 9.8 Suppose that f is a differentiable real function in an open set $E \subset R^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = 0$.

Solution. Let \mathbf{y} be any element of R^n , and consider the real-valued function $\varphi(t) = f(\mathbf{x} + t\mathbf{y})$, defined near $t = 0$. This function is differentiable (by Theorem 9.15 $\varphi(t) = f'(\mathbf{x} + t\mathbf{y})(\mathbf{y})$). Since $\varphi(t)$ has a maximum at $t = 0$, it follows that $\varphi'(0) = 0$, i.e., that $f'(\mathbf{x})(\mathbf{y}) = 0$. Since \mathbf{y} is arbitrary, it follows by definition of the zero linear transformation that $f'(\mathbf{x})$ is the zero linear transformation.

Exercise 9.9 If \mathbf{f} is a differentiable mapping of a *connected* open set $E \subset R^n$ into R^m , and if $\mathbf{f}'(\mathbf{x}) = \mathbf{0}$ for every $\mathbf{x} \in E$, prove that \mathbf{f} is constant in E .

Solution. The mean-value argument given in Exercise 7 above, applied to each component of \mathbf{f} , shows that \mathbf{f} is *locally* constant (the partial derivatives are all zero). Hence, if \mathbf{x}^0 is any point of E , the set of \mathbf{x} such that $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}^0)$ is an open set. Since this set is also closed in E , and E is connected, it follows that it must be all of E .

Exercise 9.10 If f is a real function defined in a convex open set $E \subset R^n$, such that $(D_1 f)(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $f(\mathbf{x})$ depends only on x_2, \dots, x_n .

Show that the convexity of E can be replaced by a weaker condition, but that some condition is required. For example, if $n = 2$ and E is shaped like a horseshoe, the statement may be false.

Solution. We need to show that $f(x_1^0, x_2, \dots, x_n) = f(x_1^1, x_2, \dots, x_n)$ whenever $\mathbf{x}^0 = (x_1^0, x_2, \dots, x_n)$ and $\mathbf{x}^1 = (x_1^1, x_2, \dots, x_n)$ both belong to E . Since E is convex, the line segment joining \mathbf{x}^0 and \mathbf{x}^1 is contained in E . The mean-value theorem applies on this line segment, showing that $f(\mathbf{x}^0) - f(\mathbf{x}^1) = (x_1^0 - x_1^1)(D_1 f)(\mathbf{x})$ for some point \mathbf{x} on this interval. Hence the result now follows from the hypothesis.

Note that convexity is needed only on each line segment through E parallel to the x_1 -axis. Thus if the intersection of E with each line parallel to the x_1 -axis is an interval and $(D_1f)(\mathbf{x}) = 0$ for all $\mathbf{x} \in E$, then f is independent of x_1 .

If we define $f(x, y)$ on all of R^2 except the nonnegative portion of the y -axis by specifying

$$f(x, y) = \begin{cases} 0 & \text{if } y < 0 \text{ or } x < 0, \\ y^2 & \text{if } y \geq 0 \text{ and } x > 0, \end{cases}$$

then $f(x, y)$ is continuously differentiable on its domain, $(D_1f)(x, y) = 0$ everywhere on that domain, yet $f(-1, 1) = 0 \neq 1 = f(1, 1)$, so that f is not independent of x .

Exercise 9.11 If f and g are differentiable real functions in R^n , prove that

$$\nabla(fg) = f\nabla g + g\nabla f$$

and that $\nabla(1/f) = -f^{-2}\nabla f$ wherever $f \neq 0$.

Solution. This is a routine computation applied to the i th component of the various quantities.

Exercise 9.12 Fix two real numbers a and b , $0 < a < b$. Define a mapping $\mathbf{f} = (f_1, f_2, f_3)$ of R^2 into R^3 by

$$\begin{aligned} f_1(s, t) &= (b + a \cos s) \cos t \\ f_2(s, t) &= (b + a \cos s) \sin t \\ f_3(s, t) &= a \sin s \end{aligned}$$

Describe the range K of \mathbf{f} . (It is a certain compact subset of R^3 .)

(a) Show that there are exactly 4 points $\mathbf{p} \in K$ such that

$$(\nabla f_1)(\mathbf{f}^{-1}(\mathbf{p})) = 0.$$

Find these points.

(b) Determine the set of all $\mathbf{q} \in K$ such that

$$(\nabla f_3)(\mathbf{f}^{-1}(\mathbf{q})) = 0.$$

(c) Show that one of the points \mathbf{p} found in part (a) corresponds to a local maximum of f_1 , one corresponds to a local minimum, and that the other two are neither (they are so-called "saddle points").

Which of the points \mathbf{q} found in part (b) correspond to maxima or minima?

(d) Let λ be an irrational number, and define $\mathbf{g}(t) = \mathbf{f}(t, \lambda t)$. Prove that \mathbf{g} is a 1-1 mapping of R^1 onto a dense subset of K . Prove that

$$|\mathbf{g}'(t)|^2 = a^2 + \lambda^2(b + a \cos t)^2.$$

Solution. The range K is a torus obtained by moving a circle of radius a with center on a circle of radius b , always keeping the planes of the two circles perpendicular and each plane passing through the center of the other circle. This can be seen by observing that in cylindrical coordinates the parametric equations say $r = b + a \cos s$, $z = a \sin s$, i.e., $(r - b)^2 + z^2 = a^2$, which, together with the equation $\theta = \text{const}$, gives the equation of a circle with center at $(b, 0)$ and radius a in the half-plane $\theta = \text{const}$.

(a) The equation $(\nabla f_1)(s, t) = \mathbf{0}$ says $-a \sin s \cos t = 0$ and $-(b + a \cos s) \sin t = 0$. This second equation requires $t = k\pi$, and since these functions have period 2π in both s and t , we may as well assume $t = 0$ or $t = \pi$. In that case the first equation implies $s = 0$ or $s = \pi$. Hence the only points \mathbf{p} satisfying this equation are the images of the points $(0, 0)$, $(0, \pi)$, $(\pi, 0)$, and (π, π) , i.e., the points $(b + a, 0, 0)$, $(b - a, 0, 0)$, $(-b + a, 0, 0)$, and $(-b - a, 0, 0)$.

(b) The equation $(\nabla f_3)(s, t) = \mathbf{0}$ says only that $a \cos s = 0$, i.e., $s = \frac{\pi}{2}$ or $s = \frac{3\pi}{2}$. The image of these two conditions consists of the two circles of radius b about the z -axis in the planes $z = \pm a$.

(c) The point $(a + b, 0, 0)$ is the maximum possible value of $f_1(s, t)$, and occurs only when $\cos s = 1$ and $\cos t = 1$. Likewise the point $(-a - b, 0, 0)$ is the minimum possible value, and occurs only when $\cos s = -1$ and $\cos t = -1$. The other two points, which occur when $s = 0, t = \pi$ and when $s = \pi, t = 0$, lie near points of both larger and smaller values of $f_1(s, t)$. For example, when $s = 0$, the point $t = \pi$ is a minimum for the function $\varphi(t) = f_1(0, t) = b \cos t$; but when $t = \pi$, the point $s = 0$ is a maximum of $\psi(s) = f_1(s, \pi) = -(b + a \cos s)$. Hence the point $(0, \pi)$ is neither a maximum nor a minimum for $f_1(s, t)$.

The points with $z = +a$ are obviously absolute maxima of $f_3(s, t)$, while those with $z = -a$ are the absolute minima.

(d) Suppose $\mathbf{g}(t_1) = \mathbf{g}(t_2)$. Then because $a \sin t_1 = a \sin t_2$, and

$$\sqrt{(f_1(t_1, \lambda t_1))^2 + (f_2(t_1, \lambda t_1))^2} = \sqrt{(f_1(t_2, \lambda t_2))^2 + (f_2(t_2, \lambda t_2))^2}$$

(that is, $b + a \cos t_1 = b + a \cos t_2$), we have $\sin t_1 = \sin t_2$ and $\cos t_1 = \cos t_2$. Therefore $\sin(t_1 - t_2) = 0$, which means $t_2 = t_1 + k\pi$ for some integer k . Because $\sin t_1 = \sin t_2$, it follows that k is an even integer, say $k = 2m$. It then follows, since $f_i(t_1, \lambda t_1) = f_i(t_2, \lambda t_2)$, $\lambda = 1, 2$, that $\cos \lambda t_1 = \cos \lambda t_2$ and $\sin \lambda t_1 = \sin \lambda t_2$. This in turn implies that $\lambda t_2 = \lambda t_1 + 2r\pi$ for some integer r . Combining these two results, we find that $m\lambda = r$. Since λ is irrational, this means that $m = 0 = r$, i.e., $t_2 = t_1$. Thus $\mathbf{g}(t)$ is one-to-one.

To show that the range is dense in K , we need only show that the numbers $2\pi n\lambda$, $n = 0 \pm 1, \pm 2, \dots$, are dense "modulo 2π ," meaning that for any real number θ and any $\varepsilon > 0$ there are integers m and n such that $|2\pi n\lambda - 2\pi m - \theta| < \varepsilon$. A proposition easily seen to be equivalent is that for any $\eta > 0$ and any real number c there exist integers m and n such that $|n\lambda - m - c| < \eta$. (This statement is obvious ($m = n = 0$) if $c = 0$.) To prove that, fix an integer r larger than $\frac{1}{\eta}$, and consider the numbers $0, \lambda - [\lambda], 2\lambda - [2\lambda], \dots, r\lambda - [r\lambda]$. There are $r + 1$ such numbers, all lying in the interval $[0, 1)$. Hence two of them must

be closer than $\frac{1}{r}$ to each other, say $0 < s\lambda - [s\lambda] - t\lambda + [t\lambda] < \frac{1}{r}$. In particular, the number $(s - t)\lambda$ lies within $\frac{1}{r}$ of an integer (namely $[s\lambda] - [t\lambda]$). Thus we have, say $(s - t)\lambda = k + \delta$, where $0 < \delta < \frac{1}{r}$. Let p be the unique integer such that $p\delta \leq c < (p + 1)\delta$. We then have $p(s - t)\lambda = pk + p\delta$, and hence, taking $n = p(s - t)$ and $m = pk$, we find $|n\lambda - m - c| = |p\delta - c| < \delta < \frac{1}{r} < \eta$.

This being established, consider any point in K , say the point $\mathbf{p} = (b + a \cos s_0) \cos t_0, (b + a \cos s_0) \sin t_0, a \sin s_0$, and let $\varepsilon > 0$ be given. According to what was just established, there are integers m, n such that $|2\pi m\lambda - 2\pi n - (t_0 - s_0\lambda)| < \frac{\varepsilon}{3a+3b}$. It then follows that

$$\begin{aligned} |\cos((s_0 + 2\pi m)\lambda) - \cos t_0| &= |\cos((s_0 + 2\pi m)\lambda - 2\pi n) - \cos t_0| \\ &\leq \frac{\varepsilon}{3a + 3b}, \end{aligned}$$

where we have used the inequality $|\cos u - \cos v| \leq |u - v|$, with $u = (s_0 + 2\pi m)\lambda - 2\pi n$ and $v = t_0$. A similar inequality applies with \sin in place of \cos . It then follows that $|\mathbf{g}(s_0 + 2\pi m) - \mathbf{p}| \leq \frac{2\varepsilon}{3} < \varepsilon$. Therefore the range of \mathbf{g} is dense in K .

The equation

$$|\mathbf{g}'(t)|^2 = a^2 + \lambda^2(b + a \cos t)^2$$

is a routine, though tedious, computation.

Exercise 9.13 Suppose \mathbf{f} is a differentiable mapping of R^1 into R^3 such that $|\mathbf{f}(t)| = 1$ for every t . Prove that $\mathbf{f}'(t) \cdot \mathbf{f}(t) = 0$.

Interpret this result geometrically.

Solution. This result is obtained by merely differentiating the relation $\mathbf{f}(t) \cdot \mathbf{f}(t) = 1$. Geometrically it asserts that the velocity vector of a point moving over a sphere is tangent to the sphere (perpendicular to the radius vector from the center of the sphere to the point).

Exercise 9.14 Define $f(0, 0) = 0$ and

$$f(x, y) = \frac{x^3}{x^2 + y^2} \quad \text{if } (x, y) \neq (0, 0).$$

(a) Prove that $D_1 f$ and $D_2 f$ are bounded functions in R^2 . (Hence f is continuous.)

(b) Let \mathbf{u} be any unit vector in R^2 . Show that the directional derivative $(D_{\mathbf{u}} f)(0, 0)$ exists, and that its absolute value is at most 1.

(c) Let γ be a differentiable mapping of R^1 into R^2 (in other words, γ is a differentiable curve in R^2), with $\gamma(0) = (0, 0)$ and $|\gamma'(0)| > 0$. Put $g(t) = f(\gamma(t))$ and prove that g is differentiable for every $t \in R^1$.

If $\gamma \in C'$, prove that $g \in C'$.

(d) In spite of this, prove that f is not differentiable at $(0, 0)$.

Hint: Formula (40) fails.

Solution. (a) For $(x, y) \neq (0, 0)$ we have

$$D_1 f(x, y) = \frac{x^2(x^2 + 3y^2)}{(x^2 + y^2)^2}, \quad D_2 f(x, y) = -\frac{2x^3 y}{(x^2 + y^2)^2}.$$

It follows that

$$0 \leq D_1 f(x, y) \leq \frac{3x^2}{x^2 + y^2} \leq 3$$

and

$$|D_2 f(x, y)| \leq \frac{x^2}{x^2 + y^2} \leq 1.$$

Also $D_1 f(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{x - 0}{x} = 1$, and $D_2 f(0, 0) = \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{0}{y} = 0$. Hence, as asserted, $f(x, y)$ is continuous.

(b) Let $\mathbf{u} = (\cos \theta, \sin \theta)$. Then $D_{\mathbf{u}} f(0, 0) = \lim_{t \rightarrow 0} \frac{f(t \cos \theta, t \sin \theta) - f(0, 0)}{t} = \cos^3 \theta$.

(c) Suppose $u(t)$ and $v(t)$ satisfy $u(0) = 0 = v(0)$, $u'(t)$ and $v'(t)$ exist for every t , and $u'(t)$ and $v'(t)$ do not both vanish at the same value of t . Setting $g(t) = f(u(t), v(t))$, we find that $g(t)$ is obviously differentiable at any value of t where $u(t)$ and $v(t)$ are not both zero. Now suppose $u(t_0) = v(t_0) = 0$. Then, since one of $u(t)$ and $v(t)$ is one-to-one on a neighborhood of t_0 , it follows that, for small non-zero values of $t - t_0$ we have $(u(t))^2 + (v(t))^2 > 0$, and then

$$\begin{aligned} \frac{g(t) - g(t_0)}{t - t_0} &= \frac{f(u(t), v(t)) - f(u(t_0), v(t_0))}{t - t_0} \\ &= \frac{\left(\frac{u(t) - u(t_0)}{t - t_0}\right)^3}{\left(\frac{u(t) - u(t_0)}{t - t_0}\right)^2 + \left(\frac{v(t) - v(t_0)}{t - t_0}\right)^2}, \end{aligned}$$

so that

$$g'(t_0) = \lim_{t \rightarrow t_0} \frac{g(t) - g(t_0)}{t - t_0} = \frac{(u'(t_0))^3}{(u'(t_0))^2 + (v'(t_0))^2}.$$

Thus $g(t)$ is differentiable. Observe that if $\gamma(t) \neq (0, 0)$, then

$$g'(t) = \frac{(u(t))^4 u'(t) + 3(u(t)v(t))^2 u'(t) - 2(u(t))^3 v(t) v'(t)}{((u(t))^2 + (v(t))^2)^2}.$$

The same argument used above to prove that $g'(t_0)$ exists shows that

$$\lim_{t \rightarrow t_0} g'(t) = \frac{(u'(t_0))^5 + (u'(t_0))^3 (v'(t_0))^2}{((u'(t_0))^2 + (v'(t_0))^2)^2} = \frac{(u'(t_0))^3}{(u'(t_0))^2 + (v'(t_0))^2} = g'(t_0),$$

so that g' is continuous at t_0 if u' and v' are. Continuity of g' at other points follows from the chain rule.

If f is differentiable at $(0,0)$, we necessarily have

$$f(x, y) = f(0, 0) + [xD_1f(0, 0) + yD_2f(0, 0)] + \varepsilon(x, y),$$

where

$$\lim_{(x,y) \rightarrow (0,0)} \frac{\varepsilon(x, y)}{\sqrt{x^2 + y^2}} = 0.$$

Since $D_1f(0, 0) = 1$ and $D_2f(0, 0) = 0$, it follows that

$$\varepsilon(x, y) = \frac{-xy^2}{x^2 + y^2},$$

and so we must have

$$\lim_{(x,y) \rightarrow (0,0)} \frac{-xy^2}{(x^2 + y^2)^{3/2}} = 0.$$

But this is clearly not the case, as we see by taking $x = y$. (The limit is then $-2^{-3/2}$.)

Exercise 9.15 Define $f(0, 0) = 0$, and put

$$f(x, y) = x^2 + y^2 - 2x^2y - \frac{4x^6y^2}{(x^4 + y^2)^2}$$

if $(x, y) \neq (0, 0)$.

(a) Prove, for all $(x, y) \in \mathbb{R}^2$, that

$$4x^4y^2 \leq (x^4 + y^2)^2.$$

Conclude that f is continuous.

(b) For $0 \leq \theta \leq 2\pi$, $-\infty < t < \infty$, define

$$g_\theta(t) = f(t \cos \theta, t \sin \theta).$$

Show that $g_\theta(0) = 0$, $g'_\theta(0) = 0$, $g''_\theta(0) = 2$. Each g_θ has therefore a strict local minimum at $t = 0$.

In other words, the restriction of f to each line through $(0, 0)$ has a strict local minimum at $(0, 0)$.

(c) Show that $(0, 0)$ is nevertheless not a local minimum for f , since $f(x, x^2) = -x^4$.

Solution. (a) This inequality follows by squaring the inequality $2x^2|y| \leq x^4 + y^2$, which in turn is equivalent to the inequality $(x^2 - |y|)^2 \geq 0$. Then, since $f(x, y)$

is obviously continuous except at $(0, 0)$, the continuity at the remaining point follows from the inequality

$$|f(x, y) - f(0, 0)| \leq 2x^2 + y^2 + 2x^2|y|,$$

which is easily derived from the inequality just proved and the definition of $f(x, y)$.

(b) We observe that for $t \neq 0$ we have

$$g_\theta(t) = t^2 - 2t^3 \cos^2 \theta \sin \theta - 4t^4 \frac{\cos^6 \theta \sin^2 \theta}{(t^2 \cos^4 \theta + \sin^2 \theta)^2},$$

from which it is routine computation to show that $g_\theta(0) = 0 = g'_\theta(0)$ and $g''_\theta(0) = 2$.

(c) The assertion that $f(x, x^2) = -x^4$ is routine computation. It implies that $f(x, y)$ assumes negative values in any neighborhood of $(0, 0)$, and hence that the $f(x, y)$ does not have a local minimum at $(0, 0)$.

Exercise 9.16 Show that the continuity of \mathbf{f}' at the point \mathbf{a} is needed in the inverse function theorem, even in the case $n = 1$: If

$$f(t) = t + 2t^2 \sin\left(\frac{1}{t}\right)$$

for $t \neq 0$, and $f(0) = 0$, then $f'(0) = 1$, f' is bounded in $(-1, 1)$ but f is not one-to-one in any neighborhood of 0.

Solution. The assertion that $f'(0) = 1$ is proved by direct computation: $\frac{f(t)}{t} = 1 + 2t \sin\left(\frac{1}{t}\right) \rightarrow 1$ as $t \rightarrow 0$. Since $f'(t) = 1 + 4t \sin\left(\frac{1}{t}\right) - 2 \cos\left(\frac{1}{t}\right)$ for $t \neq 0$, it follows that $|f'(t)| \leq 7$ for all $t \in (-1, 1)$. To show that f is not one-to-one in any neighborhood of 0, we observe that $f'\left(\frac{1}{k\pi}\right) = 1 + 2(-1)^k$, so that $f(t)$ is decreasing at $t = \frac{1}{k\pi}$ if k is odd and increasing if k is even. It follows that the minimum value of $f(t)$ on the interval $\left[\frac{1}{(2k+1)\pi}, \frac{1}{2k\pi}\right]$ is assumed at an interior point, so that $f(t)$ cannot be one-to-one on this interval.

Exercise 9.17 Let $\mathbf{f} = (f_1, f_2)$ be the mapping of R^2 into R^2 given by

$$f_1(x, y) = e^x \cos y, \quad f_2(x, y) = e^x \sin y.$$

(a) What is the range of \mathbf{f} ?

(b) Show that the Jacobian of \mathbf{f} is not zero at any point of R^2 . Thus every point of R^2 has a neighborhood in which \mathbf{f} is one-to-one. Nevertheless, \mathbf{f} is not one-to-one on R^2 .

(c) Put $\mathbf{a} = (0, \pi/3)$, $\mathbf{b} = f(\mathbf{a})$, let \mathbf{g} be the continuous inverse of \mathbf{f} , defined in a neighborhood of \mathbf{b} , such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$. Find an explicit formula for \mathbf{g} , compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, and verify the formula (52).

(d) What are the images under \mathbf{f} of lines parallel to the coordinate axes?

Solution. (a) The range of \mathbf{f} is all of R^2 except the point $(0,0)$. Indeed if $(u,v) \neq (0,0)$, choose y so that

$$\cos y = \frac{u}{\sqrt{u^2 + v^2}}, \quad \sin y = \frac{v}{\sqrt{u^2 + v^2}},$$

and let $x = \ln \sqrt{u^2 + v^2}$, so that $e^x = \sqrt{u^2 + v^2}$. It is then obvious from the equations defining y and x that $u = e^x \cos y$ and $v = e^x \sin y$. Hence every point except $(0,0)$ is in the range of \mathbf{f} . The point $(0,0)$ is not in the range, since $u^2 + v^2 = e^{2x} > 0$ for any point $(u,v) = \mathbf{f}(x,y)$.

(b) The Jacobian of $\mathbf{f}(x,y)$ is e^{2x} , which is never zero. However, since $\mathbf{f}(x,y+2\pi) = \mathbf{f}(x,y)$, it follows that \mathbf{f} is not one-to-one.

(c) By our definition $\mathbf{b} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. We can therefore take $y = \arctan(\frac{v}{u})$ for (u,v) near \mathbf{b} , the arctangent being between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Thus we have $\mathbf{g}(u,v) = (\ln \sqrt{u^2 + v^2}, \arctan(\frac{v}{u}))$. We then have

$$\mathbf{f}'(x,y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}, \quad \mathbf{g}'(u,v) = \begin{pmatrix} \frac{u}{u^2+v^2} & \frac{v}{u^2+v^2} \\ \frac{-v}{u^2+v^2} & \frac{u}{u^2+v^2} \end{pmatrix}.$$

When we take $u = e^x \cos y$ and $v = e^x \sin y$, we find that

$$\mathbf{g}'(\mathbf{f}(x,y)) = \begin{pmatrix} e^{-x} \cos y & e^{-x} \sin y \\ -e^{-x} \sin y & e^{-x} \cos y \end{pmatrix}.$$

It is then a routine computation to verify that $\mathbf{g}'(\mathbf{f}(x,y))\mathbf{f}'(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Likewise we find

$$\mathbf{f}'(\mathbf{g}(u,v)) = \begin{pmatrix} u & -v \\ v & u \end{pmatrix},$$

and a routine computation shows that $\mathbf{f}'(\mathbf{g}(u,v))\mathbf{g}'(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

(d) The family of lines $x = c$ maps to the family of concentric circles $u^2 + v^2 = e^{2c}$. The lines $y = c$ map to half-lines $v = Ku$, $u \geq 0$, where $K = \tan y$. (If y is an odd multiple of $\frac{\pi}{2}$, the half-line is either the positive or negative u -axis.)

Exercise 9.18 Answer analogous questions for the mapping defined by

$$u = x^2 - y^2, \quad v = 2xy.$$

Solution. (a) the range of the mapping $\mathbf{f}(x,y) = (x^2 - y^2, 2xy)$ is the entire plane R^2 . Indeed, every point (u,v) except $(0,0)$ has two distinct preimages, one of which is

$$x = \sqrt{\frac{\sqrt{u^2 + v^2} + u}{2}}, \quad y = (\operatorname{sgn} v) \sqrt{\frac{\sqrt{u^2 + v^2} - u}{2}}.$$

(The other preimage is $-x, -y$, with this x and this y .)

(b) The Jacobian of \mathbf{f} vanishes only at $x = y = 0$. Indeed,

$$\mathbf{f}'(x, y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}.$$

Hence the Jacobian is $4(x^2 + y^2)$.

(c) Taking $\mathbf{a} = (3, 4)$, so that $\mathbf{b} = (-7, 24)$, we can take, locally

$$g(u, v) = \left(\sqrt{\frac{\sqrt{u^2 + v^2} + u}{2}}, \sqrt{\frac{\sqrt{u^2 + v^2} - u}{2}} \right).$$

We then have

$$\mathbf{g}'(u, v) = \begin{pmatrix} \frac{1}{4} \sqrt{\frac{2}{\sqrt{u^2 + v^2} + u}} \left(1 + \frac{u}{\sqrt{u^2 + v^2}} \right) & \frac{1}{4} \sqrt{\frac{2}{\sqrt{u^2 + v^2} + u}} \left(\frac{v}{\sqrt{u^2 + v^2}} \right) \\ \frac{1}{4} \sqrt{\frac{2}{\sqrt{u^2 + v^2} - u}} \left(-1 + \frac{u}{\sqrt{u^2 + v^2}} \right) & \frac{1}{4} \sqrt{\frac{2}{\sqrt{u^2 + v^2} - u}} \left(\frac{v}{\sqrt{u^2 + v^2}} \right) \end{pmatrix}$$

Noting that the defining relations imply $u^2 + v^2 = (x^2 + y^2)^2$, we see that

$$\mathbf{g}'(\mathbf{f}(x, y)) = \begin{pmatrix} \frac{x}{2(x^2 + y^2)} & \frac{y}{2(x^2 + y^2)} \\ \frac{-y}{2(x^2 + y^2)} & \frac{x}{2(x^2 + y^2)} \end{pmatrix},$$

from which we see easily that $\mathbf{g}'(\mathbf{f}(x, y))\mathbf{f}'(x, y) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The corresponding equality with \mathbf{g} and \mathbf{f} interchanged is likewise simple, though more cumbersome to write out.

Exercise 9.19 Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

can be solved for x, y, u in terms of z ; for x, z, u in terms of y ; for y, z, u in terms of x ; but not for x, y, z in terms of u .

Solution. Adding the last two equations and subtracting the first yields $3u - u^2 = 0$, whence either $u = 0$ or $u = 3$. Hence unless u has one of these two values, there are no solutions at all. Therefore the system cannot generally be solved for x, y, z in terms of u . If one of these two equations holds, we can solve just the last two equations for any two of the variables x, y, z in terms of the third. The remaining equation will then automatically be satisfied. For example,

$$x = -\frac{z}{4}, \quad y = \frac{7z}{4}, \quad u = 0; \quad x = -\frac{9+z}{4}, \quad y = \frac{3+7z}{4}, \quad u = 3.$$

We could also have

$$x = -\frac{y}{7}, \quad z = \frac{4y}{7}, \quad u = 0; \quad x = \frac{60 + 4y}{7}, \quad z = \frac{4y - 3}{7}, \quad u = 3.$$

Finally, we could also have

$$y = -7x, \quad z = -4x, \quad u = 0; \quad y = \frac{7x - 60}{4}, \quad z = 9 - 4x, \quad u = 3.$$

Note that the matrix of the derivative of the transformation $\mathbf{f}(x, y, z, u) = (3x + y - z + u^2, x - y + 2z + u, 2x + 2y - 3z + 2u)$ is

$$\mathbf{f}'(x, y, z, u) = \begin{pmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{pmatrix}$$

and any 3×3 submatrix containing the last column is invertible when $u = 0$ or $u = 3$. However, the first three columns of this matrix does not form an invertible matrix.

Exercise 9.20 Take $n = m = 1$ in the implicit function theorem, and interpret the theorem (as well as its proof) graphically.

Solution. The theorem asserts that if $f(x, y)$ is continuously differentiable in a neighborhood of (x_0, y_0) , $f(x_0, y_0) = 0$, and $D_2f(x_0, y_0) \neq 0$, then there exist 1) an interval $I = (x_0 - \delta, x_0 + \delta)$, 2) an interval $J = (y_0 - \eta, y_0 + \eta)$, and 3) a continuously differentiable function $\varphi : I \rightarrow J$ such that for all $(x, y) \in I \times J$ the equation $f(x, y) = 0$ holds if and only if $y = \varphi(x)$.

The proof amounts to the argument that, since $D_2f(x_0, y_0) \neq 0$ and f is continuously differentiable, it must be that $D_2f(x, y) \neq 0$ for all (x, y) near (x_0, y_0) . Hence the function $g(y) = f(x_0, y)$ is strictly monotonic near $y = y_0$. Therefore, since $g(y_0) = 0$, there is a small interval $[y_0 - \eta, y_0 + \eta]$ such that $g(y_0 - \eta)$ and $g(y_0 + \eta)$ have opposite signs. By the continuity of $f(x, y)$, it follows that $f(x, y_0 - \eta)$ has the same sign as $f(x_0, y_0 - \eta)$ if x is near x_0 , and similarly $f(x, y_0 + \eta)$ has the same sign as $f(x_0, y_0 + \eta)$ for x near x_0 . That is, $f(x, y_0 - \eta)$ and $f(x, y_0 + \eta)$ have opposite signs if x is near x_0 . It follows that there is a point $\varphi(x) \in (y_0 - \eta, y_0 + \eta)$ such that $f(x, \varphi(x)) = 0$. By restricting the neighborhood so that $D_2f(x, y)$ is of constant sign, we assure that $g_x(y) = f(x, y)$ is monotonic on $[y_0 - \eta, y_0 + \eta]$ for each x near x_0 . It then follows that there can be at most one value of y in $(y_0 - \eta, y_0 + \eta)$ satisfying the equation $f(x, y) = 0$. That is, the function $\varphi(x)$ is unique. This proves all but the differentiability of φ .

The graphical interpretation is that, near a point on a smooth curve $f(x, y) = 0$ where the tangent is not vertical ($D_2f(x_0, y_0) \neq 0$) the curve intersects each vertical line exactly once.

Exercise 9.21 Define f in R^2 by

$$f(x, y) = 2x^3 - 3x^2 + 2y^3 + 3y^2.$$

(a) Find the four points in R^2 at which the gradient of f is zero. Show that f has exactly one local maximum and one local minimum in R^2 .

(b) Let X be the set of all $(x, y) \in R^2$ at which $f(x, y) = 0$. Find those points of S that have no neighborhoods in which the equation $f(x, y) = 0$ can be solved for y in terms of x (or for x in terms of y). Describe S as precisely as you can.

Solution. (a) We have $\nabla f(x, y) = 6(x^2 - x)\mathbf{i} + 6(y^2 + y)\mathbf{j}$. Hence $\nabla f(x, y) = \mathbf{0}$ precisely at the four points $(0, 0)$, $(1, 0)$, $(0, -1)$, $(1, -1)$. Since the Hessian matrix of f is

$$\begin{pmatrix} 12x - 2 & 0 \\ 0 & 12y + 2 \end{pmatrix}$$

this matrix has a positive determinant when $x > \frac{1}{6}$ and $y > -\frac{1}{6}$ or when $x < \frac{1}{6}$ and $y < -\frac{1}{6}$. Thus $(1, 0)$ and $(0, -1)$ are possible extrema. Since $12x - 2 > 0$ at $(1, 0)$, that point is a minimum. Likewise $(0, -1)$ is a maximum.

(b) Since $f(x, y) = (x+y)[2x^2 - 2xy + 2y^2 - 3x + 3y]$, the equation $f(x, y) = 0$ has the real solution $y = -x$ for every real value of x . In addition, if $-\frac{1}{2} \leq x \leq \frac{3}{2}$, it has the real solutions

$$y = \frac{2x - 3 + \sqrt{9 + 12x - 12x^2}}{4}, \quad y = \frac{2x - 3 - \sqrt{9 + 12x - 12x^2}}{4}.$$

According to the implicit function theorem, the only possible points near which there might not be a unique solution are for y in terms of x are those where $y = 0$ or $y = -1$. The corresponding values of x are $x = 0$ and $x = \frac{3}{2}$ for $y = 0$ and $x = 1$ and $x = -\frac{1}{2}$ for $y = -1$.

We observe that both solutions $y = -x$ and $y = \frac{2x - 3 + \sqrt{9 + 12x - 12x^2}}{4}$ tend to 0 as $x \rightarrow 0$. Hence there is no unique solution for y near $(0, 0)$. As $x \uparrow \frac{3}{2}$, the quantity under the radical sign tends to zero, and hence these two solutions converge toward the common value $y = 0$. Hence the point $(\frac{3}{2}, 0)$ is another point around which the solution for y is not unique. The two radicals also tend to zero as $x \downarrow -\frac{1}{2}$, causing the two values of y both to tend toward -1 , so that $(-\frac{1}{2}, -1)$ is not a point of unique solvability. Finally, as $x \rightarrow 1$, the three y values tend toward -1 , $\frac{1}{2}$, and -1 . Since two of these values are identical, there is no unique solution around the point $(1, -1)$.

Finally, the three x -values corresponding to any y are

$$x = -y, \quad x = \frac{2y + 3 \pm \sqrt{9 - 12y - 12y^2}}{4},$$

where the quantity under the radical is nonnegative in the range $-\frac{3}{2} \leq y \leq \frac{1}{2}$. The values where $D_1 f(x, y) = 0$ are $x = 0$ and $x = 1$, and the four points near which a solution for x might not be unique are $(0, 0)$, $(0, -\frac{3}{2})$, $(1, -1)$, and

$(1, \frac{1}{2})$. As y tends to zero, two of these tend to zero. Hence $(0, 0)$ is not a point of unique solvability for x in terms of y . As y tends to -1 , two of the x -values tend to 1, so that $(1, -1)$ is not a point of unique solvability for x . Finally, as y tends to $-\frac{3}{2}$ or $\frac{1}{2}$, the radical disappears, and so once again two of the x values tend to the same value, namely 1 as $y \rightarrow \frac{1}{2}$ and 0 as $y \rightarrow -\frac{3}{2}$. Thus these four points are not points of unique solvability for x .

In sum, the points near which the equation $f(x, y) = 0$ does not define either y as a function of x or x as a function of y are $(0, 0)$ and $(1, -1)$.

Exercise 9.22 Give a similar discussion for

$$f(x, y) = 2x^3 + 6xy^2 - 3x^2 + 3y^2.$$

Solution. The gradient is

$$\nabla 6(x^2 + y^2 - x)\mathbf{i} + 6(2xy + y)\mathbf{j}$$

As we see from solving the appropriate equations, this gradient vanishes at the points $(0, 0)$ and $(1, 0)$. The point $(0, 0)$ is a saddle point, since $f(x, 0)$ is negative for $x < 0$ and $f(0, y)$ is positive for y near zero. The Hessian determinant is positive at $(1, 0)$, and the upper left-hand entry is also; hence $(1, 0)$ is a minimum.

Because the equation $f(x, y) = 0$ can be written as

$$(6x + 3)y^2 = (3 - 2x)x^2,$$

there will be real solutions y if and only if $-\frac{1}{2} < x \leq \frac{3}{2}$. (When $x = -\frac{1}{2}$, the equation does not contain y .) In this range there are two distinct values of y except for $x = 0$ and $x = \frac{3}{2}$. Hence the two points on the locus of $f(x, y) = 0$ at which the equation cannot be solved for y are $(0, 0)$ and $(\frac{3}{2}, 0)$.

Since the equation is cubic in x , its solvability is more complicated from this point of view. Every value of y gives at least one value of x (but those x -values always lie between $-\frac{1}{2}$ and $\frac{3}{2}$). To find the points where two of the three (complex) x -roots coincide, we observe that at such points $D_1 f(x, y) = 0$, and hence also $3f(x, y) - xD_1 f(x, y) = 0$. This last equation says $x^2 - 4xy^2 + 3y^2 = 0$, i.e., $y^2 = \frac{x^2}{4x + 3}$. Substituting this value of y^2 into $f(x, y) = 0$, we get either $x = 0$ and $y = 0$ or

$$x^2 = \frac{3}{4}.$$

Since we have to have $-\frac{1}{2} < x$, we must have $x = \frac{\sqrt{3}}{2}$, and this gives $y^2 = \frac{2\sqrt{3}-3}{4}$. Hence the points near which $f(x, y) = 0$ cannot be solved uniquely for x are $(0, 0)$ and $(\frac{\sqrt{3}}{2}, \pm \frac{\sqrt{2\sqrt{3}-3}}{4})$.

Exercise 9.23 Define f in R^3 by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2.$$

Show that $f(0, 1, -1) = 0$, $(D_1 f)(0, 1, -1) \neq 0$, and that there exists therefore a differentiable function g in some neighborhood of $(1, -1)$ in R^2 such that $g(1, -1) = 0$ and

$$f(g(y_1, y_2), y_1, y_2) = 0.$$

Find $(D_1 g)(1, -1)$ and $(D_2 g)(1, -1)$.

Solution. The proof that $f(0, 1, -1) = 0$ is a routine computation. We have $(D_1 f)(x, y_1, y_2) = 2xy_1 + e^x$, so that $(D_1 f)(0, 1, -1) = 1 \neq 0$. To find the partial derivatives of g we use the chain rule. Let $\psi(y_1, y_2) = f(g(y_1, y_2), y_1, y_2) \equiv 0$. Then

$$0 = D_1 \psi(y_1, y_2) = D_1 f(g(y_1, y_2), y_1, y_2) D_1 g(y_1, y_2) + D_2 f(g(y_1, y_2), y_1, y_2),$$

so that

$$0 = (2y_1 g(y_1, y_2) + e^{g(y_1, y_2)}) D_1 g(y_1, y_2) + (g(y_1, y_2))^2.$$

Similarly, setting

$$0 = D_2 \psi(y_1, y_2) = D_1 f(g(y_1, y_2), y_1, y_2) D_2 g(y_1, y_2) + D_3 f(g(y_1, y_2), y_1, y_2),$$

we find

$$0 = (2y_1 g(y_1, y_2) + e^{g(y_1, y_2)}) D_2 g(y_1, y_2) + 1.$$

Taking $y_1 = 1$, $y_2 = -1$, $g(y_1, y_2) = 0$, we get

$$D_1 g(1, -1) = 0, \quad D_2 g(1, -1) = -1.$$

Exercise 9.24 For $(x, y) \neq (0, 0)$, define $\mathbf{f} = (f_1, f_2)$ by

$$f_1(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad f_2(x, y) = \frac{xy}{x^2 + y^2}.$$

Compute the rank of $\mathbf{f}'(x, y)$, and find the range of \mathbf{f} .

Solution. The matrix of $\mathbf{f}'(x, y)$ is

$$\begin{pmatrix} \frac{4xy^2}{(x^2 + y^2)^2} & \frac{-4x^2y}{(x^2 + y^2)^2} \\ \frac{y(y^2 - x^2)}{(x^2 + y^2)^2} & \frac{x(x^2 - y^2)}{(x^2 + y^2)^2} \end{pmatrix}.$$

Its determinant is 0 at every point. Hence its rank is either 0 or 1 at every point. Since the point $(0, 0)$ is excluded from the domain, the rank is 1 at every point. The range must therefore be 1-dimensional, i.e., there is some non-trivial

relation connecting f_1 and f_2 . Indeed, it is easy to verify that if $u = f_1(x, y)$ and $v = f_2(x, y)$, then

$$u^2 + 4v^2 = 1.$$

Thus the range of \mathbf{f} is a subset of this ellipse. In fact, it is all of this ellipse. The point $(1, 0)$ is its own image, and the point $(-1, 0)$ is the image of $(0, 1)$. For any other point (u, v) on this ellipse we have $-1 < u < 1$ and $v = \pm \frac{1}{2}\sqrt{1 - u^2}$. The point (u, v) is the image of the point $\left(1, \pm \sqrt{\frac{1-u}{1+u}}\right)$ (and, of course, many other points as well).

Exercise 9.25 Suppose $A \in L(R^n, R^m)$, let r be the rank of A .

(a) Define S as in the proof of Theorem 9.32. Show that SA is a projection in R^n whose nullspace is $\mathcal{N}(A)$ and whose range is $\mathcal{R}(S)$. *Hint:* By (68), $SASA = SA$.

(b) Use (a) to show that

$$\dim \mathcal{N}(A) + \dim \mathcal{R}(A) = n.$$

Solution. We recall that S is defined by first choosing a basis for the range of A , say $\{\mathbf{y}_1, \dots, \mathbf{y}_r\}$, then choosing vectors $\{\mathbf{z}_1, \dots, \mathbf{z}_r\}$ such that $A\mathbf{z}_i = \mathbf{y}_i$ for $i = 1, 2, \dots, r$. We then define $S\mathbf{y}_i = \mathbf{z}_i$ on the vectors \mathbf{y}_i (and S arbitrary on any set of vectors $\mathbf{y}_{r+1}, \dots, \mathbf{y}_m$ that can be adjoined to $\{\mathbf{y}_1, \dots, \mathbf{y}_r\}$ so as to make a basis of R^m). Thus S is a left inverse of the restriction of A to the subspace spanned by $\mathbf{z}_1, \dots, \mathbf{z}_r$. Since $A\mathbf{x}$ belongs to the range of A , it follows, as in (68), that $ASAx = Ax$, from which we conclude that $SASAx = SAx$, i.e., SA is a projection. Then every vector \mathbf{x} has the unique decomposition $\mathbf{x} = SAx + (\mathbf{x} - SAx)$, where the first vector on the right belongs to the range of SA and the second to the nullspace of this projection. The two subspaces have only the zero vector in common. Since S is an isomorphism of the range of A , the range of SA has the same dimension as the range of A . Since $A = ASA$, the nullspace of SA is the same as the nullspace of A . Thus $n = \dim \mathcal{N}(SA) + \dim \mathcal{R}(SA) = \dim \mathcal{N}(A) + \dim \mathcal{R}(A)$.

Exercise 9.26 Show that the existence (and even the continuity) of $D_{12}f$ does not imply the existence of D_1f . For example, let $f(x, y) = g(x)$, where g is nowhere differentiable.

Solution. The second sentence in the exercise is its solution. Since D_2f is identically zero, $D_{12}f$ is also identically zero, hence certainly continuous.

Exercise 9.27 Put $f(0, 0) = 0$, and

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

if $(x, y) \neq (0, 0)$. Prove that

- (a) f , D_1f , and D_2f are continuous in R^2 ;
- (b) $D_{12}f$ and $D_{21}f$ exist at every point of R^2 , and are continuous except at $(0, 0)$;
- (c) $(D_{12}f)(0, 0) = 1$, and $(D_{21}f)(0, 0) = -1$.

Solution. (a) The continuity of f is obvious at every point except $(0, 0)$; at $(0, 0)$ it follows from the inequality $|f(x, y)| \leq \frac{1}{2}(x^2 + y^2)$. It is also clear that $D_1f(0, 0) = 0 = D_2f(0, 0)$. For $(x, y) \neq (0, 0)$ we have $D_1f(x, y) = \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2}$ and $D_2f(x, y) = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2}$. The continuity of the partial derivatives at every point except $(0, 0)$ is obvious. It is easy to see that these derivatives satisfy the inequalities $|D_1f(x, y)| \leq 2|y|$ and $|D_2f(x, y)| \leq 2|x|$, so that D_1f and D_2f are also continuous at $(0, 0)$.

(b) Since $f(x, y)$ is a rational function with non-zero denominator for $(x, y) \neq (0, 0)$, it has continuous partial derivatives of all orders on this set.

(c) Since $D_1f(0, y) = -y$ and $D_2f(x, 0) = x$, it follows that $D_{21}f(0, y) = -1$ for all y and $D_{12}f(x, 0) = 1$ for all x .

Exercise 9.28 For $t \geq 0$ put

$$\varphi(x, t) = \begin{cases} x & (0 \leq x \leq \sqrt{t}) \\ -x + 2\sqrt{t} & (\sqrt{t} \leq x \leq 2\sqrt{t}) \\ 0 & (\text{otherwise}), \end{cases}$$

and put $\varphi(x, t) = -\varphi(x, |t|)$ if $t < 0$.

Show that φ is continuous on R^2 , and

$$(D_2\varphi)(x, 0) = 0$$

for all x . Define

$$f(t) = \int_{-1}^1 \varphi(x, t) dx.$$

Show that $f(t) = t$ if $|t| < \frac{1}{4}$. Hence

$$f'(0) \neq \int_{-1}^1 (D_2\varphi)(x, 0) dx.$$

Solution. This function is zero in the (closed) left half-plane of the xt -plane and on the positive x -axis. Since the functions by which it is defined are continuous, we need only verify that they agree on the boundary curves $x = \sqrt{t}$ and $x = 2\sqrt{t}$ in the first quadrant that separate the three different regions of definition. This is a routine computation.

Likewise the computation showing that $(D_2\varphi)(x, 0) = 0$ is routine, since for each $x > 0$ $\varphi(x, t) = 0$ for $0 \leq t \leq \frac{1}{4}x^2$, while $\varphi(x, t) = 0$ for all t if $x \leq 0$.

If $0 < t < \frac{1}{4}$, then

$$\begin{aligned} f(t) &= \int_0^{\sqrt{t}} x \, dx + \int_{\sqrt{t}}^{2\sqrt{t}} -x + 2\sqrt{t} \, dt \\ &= \frac{1}{2}t - \frac{1}{2}(4t - t) + 2\sqrt{t}(2\sqrt{t} - \sqrt{t}) \\ &= \frac{t}{2} - \frac{3t}{2} + 4t - 2t = t. \end{aligned}$$

Obviously $f(0) = 0$, and if $t < 0$, then $f(t) = -f(-t) = t$. Therefore $f'(0) = 1$. However

$$\int_{-1}^1 (D_2\varphi)(x, 0) \, dx = 0.$$

Note: This result is possible only because $D_2\varphi(x, t)$ is not bounded on $[-1, 1] \times [-a, a]$ for any $a > 0$. Also note that having -1 as the lower limit of the integral was a needless complication. The problem would have been more effective if the lower limit had been 0 .

Exercise 9.29 Let E be an open set in R^n . The classes $C'(E)$ and $C''(E)$ are defined in the text. By induction $C^{(k)}(E)$ can be defined as follows for all positive integers k : To say that $f \in C^{(k)}(E)$ means that the partial derivatives D_1f, \dots, D_nf belong to $C^{(k-1)}(E)$.

Assume $f \in C^{(k)}(E)$, and show (by repeated application of Theorem 9.41) that the k th-order derivative

$$D_{i_1 i_2 \dots i_k} f = D_{i_1} D_{i_2} \dots D_{i_k} f$$

is unchanged if the subscripts i_1, \dots, i_k are permuted.

For instance, if $n \geq 3$, then

$$D_{1213} f = D_{3112} f$$

for every $f \in C^{(4)}$.

Solution. If the permutation leaves i_k fixed, this follows from the result for $k-1$ applied to $D_{i_k} f$. To get the general result, we observe that by the case $k=2$ we have $D_{i_{k-1} i_k} f = D_{i_k i_{k-1}} f$. Hence the result holds for any permutation that maps i_{k-1} to i_k . But any permutation that maps i_j to i_k ($j \neq k, k-1$) can be written as the composition of a permutation that maps i_j to i_{k-1} , leaving i_k fixed, followed by the interchange of i_{k-1} and i_k , followed by a second permutation that leaves i_k fixed. Therefore the result applies to all permutations whatsoever.

Exercise 9.30 Let $f \in \mathcal{C}^{(m)}(E)$, where E is an open subset of R^n . Fix $\mathbf{a} \in E$, and suppose $\mathbf{x} \in R^n$ is so close to $\mathbf{0}$ that the points

$$\mathbf{p}(t) = \mathbf{a} + t\mathbf{x}$$

lie in E whenever $0 \leq t \leq 1$. Define

$$h(t) = f(\mathbf{p}(t))$$

for all $t \in R^1$ for which $\mathbf{p}(t) \in E$.

(a) For $1 \leq k \leq m$, show (by repeated application of the chain rule) that

$$h^{(k)}(t) = \sum (D_{i_1, \dots, i_k} f)(\mathbf{p}(t)) x_{i_1} \dots x_{i_k}.$$

The sum extends over all ordered k -tuples (i_1, \dots, i_k) in which each i_j is one of the integers $1, \dots, n$.

(b) By Taylor's theorem (5.15)

$$h(1) = \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!} + \frac{h^{(m)}(t)}{m!}$$

for some $t \in (0, 1)$. Use this to prove Taylor's theorem in n variables by showing that the formula

$$f(\mathbf{a} + \mathbf{x}) = \sum_{k=0}^{m-1} \frac{1}{k!} \sum (D_{i_1 \dots i_k} f)(\mathbf{a}) x_{i_1} \dots x_{i_k} + r(\mathbf{x})$$

represents $f(\mathbf{a} + \mathbf{x})$ as the sum of its so-called "Taylor polynomial of degree $m - 1$," plus a remainder that satisfies

$$\lim_{\mathbf{x} \rightarrow \mathbf{0}} \frac{r(\mathbf{x})}{|\mathbf{x}|^{m-1}} = 0.$$

Each of the inner sums extends over all ordered k -tuples (i_1, \dots, i_k) , as in part (a); as usual, the zero-order derivative of f is simply f , so that the constant term of the Taylor polynomial of f at \mathbf{a} is $f(\mathbf{a})$.

(c) Exercise 29 shows that repetition occurs in the Taylor polynomial as written in part (b). For instance D_{113} occurs three times, as D_{113} , D_{131} , D_{311} . The sum of the corresponding three terms can be written in the form

$$3(D_1^2 D_3 f)(\mathbf{a}) x_1^2 x_3.$$

Prove (by calculating how often each derivative occurs) that the Taylor polynomial in (b) can be written in the form

$$\sum \frac{(D_1^{s_1} \dots D_n^{s_n} f)(\mathbf{a})}{s_1! \dots s_n!} x_1^{s_1} \dots x_n^{s_n}.$$

Here the summation extends over all ordered n -tuples (s_1, \dots, s_n) such that each s_i is a nonnegative integer, and $s_1 + \dots + s_n \leq m - 1$.

Solution. (a) This formula is a simple application of the chain rule together with the fact that $D\mathbf{p}^{(i)}(t) = x_i$. The proof proceeds by induction on k .

(b) The formula is an immediate application of the fact that $\mathbf{p}(1) = \mathbf{a} + \mathbf{x}$, so that $h(1) = f(\mathbf{a} + \mathbf{x})$. The right-hand side is then an immediate application of the fact that $\mathbf{p}(0) = \mathbf{a}$. The only assertion that requires verification is that on the order of the remainder. The one-variable Taylor's theorem gives $r(\mathbf{x}) = \sum (D_{i_1, \dots, i_m} f(\mathbf{p}(t)) x_{i_1} \cdots x_{i_m})$ for some $t \in (0, 1)$, so that $|r(\mathbf{x})| \leq K|\mathbf{x}|^m$ for some constant K . The assertion as to the order of r follows from this fact.

(c) If $s_1 + \dots + s_n \leq m - 1$, the number of terms having the derivative combination $D_1^{s_1} \cdots D_n^{s_n} f$ is $\binom{s_1 + \dots + s_n}{s_1, \dots, s_n} = \frac{(s_1 + \dots + s_n)!}{s_1! \cdots s_n!}$. Thus the $k!$ that occurs in the one-variable Taylor's theorem is $(s_1 + \dots + s_n)!$; and when the terms are consolidated, this factor cancels the numerator of the multinomial symbol, effectively being replaced by $s_1! \cdots s_n!$.

Exercise 9.31 Suppose $f \in \mathcal{C}^{(3)}$ in some neighborhood of a point $\mathbf{a} \in R^2$, the gradient of f is $\mathbf{0}$ at \mathbf{a} , but not all second-order derivatives of f are 0 at \mathbf{a} . Show how one can then determine from the Taylor polynomial of f at \mathbf{a} (of degree 2) whether f has a local maximum or a local minimum, or neither, at the point \mathbf{a} .

Extend this to R^n in place of R^2 .

Solution. Let us simply do R^n in the first place and save the trouble of doing R^2 . According to Taylor's theorem

$$f(\mathbf{a} + \mathbf{x}) - f(\mathbf{a}) = \frac{1}{2} \sum_{i_1, i_2} (D_{i_1 i_2} f)(\mathbf{a}) x_{i_1} x_{i_2} + r(\mathbf{x}),$$

where $|\mathbf{x}|^{-2} r(\mathbf{x}) \rightarrow 0$ as $\mathbf{x} \rightarrow \mathbf{0}$. Note that the Taylor polynomial can be concisely written as $\frac{1}{2} \langle A\mathbf{x}, \mathbf{x} \rangle$, where A is the $n \times n$ Hessian matrix whose i, j entry is $D_{ij} f(\mathbf{a})$ and the angle brackets denote the inner product. If A is positive-definite, i.e., if $\langle A\mathbf{x}, \mathbf{x} \rangle > 0$ when $\mathbf{x} \neq \mathbf{0}$, there is a positive constant c such that $\langle A\mathbf{x}, \mathbf{x} \rangle \geq c|\mathbf{x}|^2$. (The constant c is the minimum value of $\langle A\mathbf{x}, \mathbf{x} \rangle$ on the unit sphere $|\mathbf{x}| = 1$.) Hence if $\delta > 0$ is chosen so that $|r(\mathbf{x})| < c|\mathbf{x}|^2$ when $0 < |\mathbf{x}| < \delta$, we see that $f(\mathbf{a} + \mathbf{x}) - f(\mathbf{a}) > 0$ if $0 < |\mathbf{x}| < \delta$, i.e., \mathbf{a} is a local minimum of f . Likewise if A is negative-definite, then \mathbf{a} is a local maximum of f .

It is well-known from linear algebra that a necessary and sufficient condition for positive-definiteness of the matrix A is that the principal minors be positive, i.e., the $k \times k$ -submatrix consisting of the elements in the first k rows and columns of A has a positive determinant. For negative-definiteness the corresponding criterion is that this minor have the same sign as $(-1)^k$.

There are no other reasonably regular cases that guarantee a maximum or minimum. A nonnegative-definite or nonpositive-definite matrix may well fail

to guarantee a maximum or minimum, even in R^1 . If the quadratic form $\langle Ax, x \rangle$ assumes both signs, then the point \mathbf{a} is definitely not either a maximum or a minimum. (If $\langle Ax, x \rangle > 0$, then $f(\mathbf{a} + t\mathbf{x}) - f(\mathbf{a}) > 0$ for small values of t , while if $\langle Ax, x \rangle < 0$, then $f(\mathbf{a} + t\mathbf{x}) - f(\mathbf{a}) < 0$ for small values of t .)

Chapter 10

Integration of Differential Forms

Exercise 10.1 Let H be a compact convex set in R^k with nonempty interior. Let $f \in C(H)$, put $f(\mathbf{x}) = 0$ in the complement of H and define $\int_H f$ as in Definition 10.3.

Prove that $\int_H f$ is independent of the order in which the integrations are carried out.

Hint: Approximate f by functions that are continuous on R^k and whose supports are in H , as was done in Example 10.4.

Solution. We first give the definition of $\int_H f$, namely $\int_I f$, where I is any k -cell containing H . This definition is unambiguous, since if I and J are both k -cells containing H , each of the single integrals carried out is an integral over the same line segment for both cells, namely the intersection of the path of integration with H .

There seems to be no way to avoid somehow proving that the boundary of H , denoted ∂H , has “measure zero.” The definition of the integral as an iterated integral makes that problem slightly more difficult than it would be otherwise, although we can show how to avoid this approach in two dimensions. We shall reserve that discussion until after the proof, which is rather lengthy. The length of the proof is due to the fact that integrals are really defined only over parallelepipeds. The point of the exercise is to enlarge the class of sets over which one can integrate. Our challenge is to show that the boundary of H can be enclosed in a finite set of parallelepipeds whose total volume can be arbitrarily small.

Our first job is to show that the hypersphere in R^k has measure zero. As the proof of that fact involves some work with $(k-2)$ -dimensional hyperspheres in k -dimensional space, we need to make several definitions in order to express these ideas properly.

First, for each real number z and each positive number r , $S_r^{k-2}(z)$ denotes the $(k-2)$ -dimensional hypersphere in R^k having radius r and center at the

point $(0, 0, \dots, 0, z)$, that is,

$$S_r^{k-2}(z) = \{\mathbf{x} : x_1^2 + \dots + x_{k-1}^2 = r^2, x_k = z\}.$$

When $z = 0$, we shall write simply S_r^{k-2} and identify this sphere with the same $(k-2)$ -dimensional hypersphere in R^{k-1} . We observe that $S_r^0(z)$ consists of the two points (r, z) and $(-r, z)$ in R^2 . Another way of defining the set $S_r^{k-2}(z)$ is as the intersection of the $(k-1)$ -dimensional sphere of radius r centered at $(0, 0, \dots, 0, z)$ in R^k with its equatorial hyperplane $P_z = \{(x_1, \dots, x_k) : x_k = z\}$.

Second, for each $\mathbf{a} \in R^k$ and each $\delta > 0$, $I_{\mathbf{a}}^k(\delta)$ is the closed hypercube of side δ in R^k whose "lower left" corner is \mathbf{a} , that is

$$I_{\mathbf{a}}^k(\delta) = \{\mathbf{x} : a_j \leq x_j \leq a_j + \delta, j = 1, \dots, k\}.$$

Third, the set of points $(m_1, \dots, m_k) \in R^k$ having integer coordinates will be denoted \mathcal{Z}^k .

Fourth, for all real numbers r and δ such that $0 < \delta < r$, $C_{r,\delta}^k$ is the set of lattice points $\mathbf{m} \in \mathcal{Z}^k$ for which the closed hypercube of side δ with lower left corner $\delta\mathbf{m}$ intersects the hypersphere S_r^{k-1} in R^k . That is,

$$C_{r,\delta}^k = \{\mathbf{m} : I_{\delta\mathbf{m}}^k(\delta) \cap S_r^{k-1} \neq \emptyset\}.$$

Fifth, $N(r, \delta, k)$ is the number of points in $C_{r,\delta}^k(z)$, that is, the number of hypercubes of side δ with lower left hand corner at a point $\delta\mathbf{m}$, $\mathbf{m} \in \mathcal{Z}^k$, that intersect the hypersphere S_r^{k-1} . Our main goal in the first stage of the proof will be to prove the estimate $N(r, \delta, k) \leq 6^{k^2} \left(\frac{r}{\delta}\right)^{k-1}$. (A smaller constant than 12^{k^2} could easily be attained, but we have no need of any improvement, and this constant seems to be the one that makes the argument simplest.)

Sixth, and finally, $A_{r,\delta}^k$ is the union of all the hypercubes $I_{\delta\mathbf{m}}^k(\delta)$ that intersect the hypersphere S_r^{k-1} , that is, for which $\mathbf{m} \in C_{r,\delta}^k$. This set is a finite union of compact sets, hence is compact. Obviously it contains the hypersphere S_r^{k-1} . What is slightly less obvious is that its interior contains this sphere. In fact no point of the sphere can be a limit point of points exterior to $A_{r,\delta}^k$, since if $\{\mathbf{x}_n\}$ is a sequence of points such that each \mathbf{x}_n belongs to a hypercube $I_{\delta\mathbf{m}_n}^k(\delta)$ not contained in $A_{r,\delta}^k$, and $\mathbf{x}_n \rightarrow \mathbf{x}$, some set $I_{\delta\mathbf{m}_{n_0}}^k(\delta)$ must occur infinitely often. (Any bounded neighborhood of \mathbf{x} intersects only finitely many of these hypercubes.) Since $I_{\delta\mathbf{m}_{n_0}}^k(\delta)$ is closed, this implies that \mathbf{x} belongs to $I_{\delta\mathbf{m}_{n_0}}^k(\delta)$. Since $I_{\delta\mathbf{m}_{n_0}}^k(\delta)$ is not contained in $A_{r,\delta}^k$ it follows that $\mathbf{x} \notin S_r^{k-1}$. Thus no sequence of points exterior to $A_{r,\delta}^k$ can approach a point of S_r^{k-1} . It follows that S_r^{k-1} contains no points of the boundary of $A_{r,\delta}^k$ and is therefore contained in the interior of this set.

With these definitions out of the way we can proceed to the proof, which we break into several stages, each broken into several steps, in order to make navigating easier.

Stage 1. Establish that the sphere S_r^{k-1} in R^k has k -dimensional content 0.

Step 1. Establish that $A_{r,\delta}^k \cup A_{r+\delta}^k$ contains the closed k -dimensional annulus consisting of the region between S_r^{k-1} and $S_{r+\delta}^{k-1}$, that is, all the points $\mathbf{x} \in R^k$ such that $r \leq |\mathbf{x}| \leq r + \delta$.

To this end, let \mathbf{x} belong to this annulus, so that $r \leq |\mathbf{x}| \leq r + \delta$. Since $S_s^{k-1} \subset A_{s,\delta}^k$ for each s , we can assume $r < |\mathbf{x}| < r + \delta$. Let $\mathbf{m} = (m_1, \dots, m_k)$ be a lattice point in \mathcal{Z}^k such that $\mathbf{x} \in I_{\delta\mathbf{m}}^k(\delta)$. Let $n_j = m_j$ if $m_j \geq 0$ and $n_j = m_j + 1$ if $m_j < 0$, so that $(\delta n_1, \dots, \delta n_k)$ is a corner of $I_{\delta\mathbf{m}}(\delta)$. Since $|n_j| = \min(|m_j|, |m_j + 1|)$ for all j , $(\delta\mathbf{n})$ is the unique point of $I_{\delta\mathbf{m}}^k(\delta)$ closest to the origin. In particular $|\delta\mathbf{n}| \leq |\mathbf{x}|$. The lattice point $\mathbf{n}' = (n_1 + \varepsilon_1(n_1), \dots, n_k + \varepsilon_k(n_k))$, where $\varepsilon_j(t)$ is 1 if $t \geq 0$ and -1 if $t < 0$, is such that $\delta\mathbf{n}'$ is the corner of $I_{\delta\mathbf{m}}(\delta)$ opposite to $\delta\mathbf{n}$ and is the unique point of $K_{\delta\mathbf{m}}(\delta)$ farthest from the origin. In particular $|\delta\mathbf{n}'| \geq |\mathbf{x}|$.

We claim first that $|\mathbf{n}'| - |\mathbf{n}| \geq 1$. Indeed, we have

$$\begin{aligned} |\mathbf{n}'|^2 &= (n_1^2 + \dots + n_k^2) + 2(n_1\varepsilon_1(n_1) + \dots + n_k\varepsilon_k(n_k)) + ((\varepsilon_1(n_1))^2 + \dots + (\varepsilon_k(n_k))^2) \\ &= (n_1^2 + \dots + n_k^2) + 2(|n_1| + \dots + |n_k|) + k, \end{aligned}$$

and

$$(|\mathbf{n}| + 1)^2 = (n_1^2 + \dots + n_k^2) + 2\sqrt{n_1^2 + \dots + n_k^2} + 1,$$

so that the desired inequality follows from the two inequalities $k \geq 1$ and $\sqrt{n_1^2 + \dots + n_k^2} \leq |n_1| + \dots + |n_k|$. This argument shows in general that, for any $\mathbf{m} \in \mathcal{Z}^k$, if \mathbf{b} and \mathbf{c} are the points in $I_{\delta\mathbf{m}}^k(\delta)$ of minimal and maximal absolute value respectively, then $|\mathbf{c}| - |\mathbf{b}| \geq \delta$.

From this we deduce a corollary: *Let r be any positive real number larger than δ . If $r \leq s \leq r + \delta$ and $\mathbf{m} \in C_{s,\delta}^k$, then either $\mathbf{m} \in C_{r,\delta}^k$ or $\mathbf{m} \in C_{r+\delta,\delta}^k$.* In plain words, if $I_{\delta\mathbf{m}}^k(\delta)$ meets S_s^{k-1} for some $s \in [r, r + \delta]$, it must meet either S_r^{k-1} or $S_{r+\delta}^{k-1}$.

To prove this corollary, we note that the assumption $\mathbf{m} \in C_{s,\delta}^k$ says that there exists $\mathbf{x} \in R^k$ such that $\mathbf{x} \in S_s^{k-1} \cap I_{\delta\mathbf{m}}^k(\delta)$. Now suppose \mathbf{m} belongs to neither of the sets $C_{r,\delta}^k$ and $C_{r+\delta,\delta}^k$. Then the set $I_{\delta\mathbf{m}}^k(\delta)$ contains no points of S_r^{k-1} . If \mathbf{b} is the point of $I_{\delta\mathbf{m}}^k(\delta)$ of smallest norm, it follows that $|\mathbf{b}| > r$. (For $I_{\delta\mathbf{m}}^k(\delta)$ contains the point \mathbf{x} of norm $s \geq r$. Since $I_{\delta\mathbf{m}}^k(\delta)$ is a connected set, if it contained a point of norm less than or equal to r it would also contain a point of S_r^{k-1} .) Similarly, if the set $I_{\delta\mathbf{m}}^k(\delta)$ contains no points of $S_{r+\delta}^{k-1}$, then $|\mathbf{c}| < r + \delta$. But then it follows that $|\mathbf{c}| - |\mathbf{b}| < r + \delta - r = \delta$, contrary to what has been proved.

Another way of stating what was just proved is that if $\mathbf{x} \in R^k$ is such that $r \leq |\mathbf{x}| \leq r + \delta$, then $\mathbf{x} \in A_{r,\delta}^k \cup A_{r+\delta,\delta}^k$. That is, the union $A_{r,\delta}^k \cup A_{r+\delta,\delta}^k$ contains the entire annulus of points \mathbf{x} such that $r \leq |\mathbf{x}| \leq r + \delta$. Step 1 of the proof is now complete.

Step 2. Assuming $k > 1$, estimate the number of k -dimensional hypercubes $I_{\delta\mathbf{m}}^k(\delta)$ that intersect various zones on the $(k-1)$ -sphere S_r^{k-1} in R^k .

We divide the upper hemisphere, consisting of \mathbf{x} such that $|\mathbf{x}| = r$ and $x_k \geq 0$ into half-open zones

$$Z_p = \{(x_1, \dots, x_{k-1}, x_k) : x_1^2 + \dots + x_k^2 = r^2, p\delta \leq x_k < (p+1)\delta\},$$

for $0 \leq p \leq \lceil \frac{r}{\delta} \rceil - 1$. Here $[a]$ denotes the integer part of a , that is, the integer q such that $q \leq a < q+1$, and we assume $0 < \delta < r$. Between the top zone $Z_{\lceil \frac{r}{\delta} \rceil - 1}$ and the "north pole" (the point $(0, 0, \dots, r)$, there is a closed "cap" of height η for some $\eta \in [0, \delta)$. The hypercubes $I_{\delta \mathbf{m}}^k(\delta)$ (where $m_k = \lceil \frac{r}{\delta} \rceil - 1$ or $m_k = \lceil \frac{r}{\delta} \rceil$) intersecting this cap must be handled separately from those intersecting the other zones.

We shall prove that the lattice points $\mathbf{m} \in \mathbb{Z}^k$ for which $m_k = p$ and $I_{\delta \mathbf{m}}^k(\delta)$ intersects Z_p are precisely those whose bottom face $x_k = p\delta$ intersects the half-closed $(k-1)$ -dimensional annulus between $S_s^{k-2}(p\delta)$ and $S_t^{k-2}(p\delta)$ in the hyperplane $x_k = p\delta$. (This annulus is closed at t and open at s , where $s = \sqrt{r^2 - ((p+1)\delta)^2}$ and $t = \sqrt{r^2 - (p\delta)^2}$.)

Indeed, this fact is nearly obvious, as the zone Z_p is the union of the $(k-2)$ -dimensional spheres $S_{\sqrt{r^2 - u^2}}^{k-2}(u)$ for $p\delta \leq u < (p+1)\delta$. If $(x_1, \dots, x_{k-1}, u) \in I_{\delta \mathbf{m}}^k(\delta) \cap Z_p$ and $m_k = p$, then $p\delta \leq u < (p+1)\delta$ and $x_1^2 + \dots + x_{k-1}^2 = r^2 - u^2$, so that $s < \sqrt{x_1^2 + \dots + x_{k-1}^2} \leq t$. Thus the point $(x_1, \dots, x_{k-1}, p\delta)$ belongs to both $I_{\delta \mathbf{m}}^k(\delta)$ and to the annulus. Conversely, if $I_{\delta \mathbf{m}}^k(\delta)$ with $m_k = p$ intersects the annulus, then this hypercube contains a point $(x_1, \dots, x_{k-1}, p\delta)$ with $s^2 < x_1^2 + \dots + x_{k-1}^2 \leq t^2$. Setting $u = \sqrt{r^2 - (x_1^2 + \dots + x_{k-1}^2)}$, we have $p\delta \leq u < (p+1)\delta$, and therefore $(x_1, \dots, x_{k-1}, u) \in Z_p \cap I_{\delta \mathbf{m}}^k(\delta)$.

To estimate the total number of hypercubes $I_{\delta \mathbf{m}}^k(\delta)$ that intersect the hypersphere S_r^{k-1} , we need an estimate of the number that intersect each zone Z_p . If $I_{\delta \mathbf{m}}^k(\delta)$ intersects Z_p , then $m_k = p$ or $m_k = p-1$. If $I_{(\delta m_1, \dots, \delta m_{k-1}, \delta(p-1))}^k(\delta)$ intersects Z_p , the intersection must be in the hyperplane $x_k = p\delta$, and hence $I_{(\delta m_1, \dots, \delta m_{k-1}, \delta p)}^k(\delta)$ also intersects Z_p . Hence we can get a (loose, but safe) upper bound on the number of hypercubes $I_{\delta \mathbf{m}}^k(\delta)$ that intersect Z_p by counting those for which $m_k = p$ and doubling. (The case of the bottom layer Z_0 is special, and the "northern cap" mentioned above will be handled separately.)

The fact that a hypercube $I_{\delta \mathbf{m}}^k(\delta)$ intersecting Z_p must intersect the annulus shows that we need only estimate of the width of the annulus, that is, the number $t - s = \sqrt{r^2 - (p\delta)^2} - \sqrt{r^2 - ((p+1)\delta)^2}$. For that width we have the following simple result:

$$\sqrt{r^2 - (p\delta)^2} - \sqrt{r^2 - ((p+1)\delta)^2} \leq \frac{(2p+1)\delta^2}{\sqrt{r^2 - (p\delta)^2}}.$$

The proof of this inequality is straightforward:

$$\begin{aligned} \sqrt{r^2 - (p\delta)^2} - \sqrt{r^2 - ((p+1)\delta)^2} &= \frac{(r^2 - (p\delta)^2) - (r^2 - ((p+1)\delta)^2)}{\sqrt{r^2 - (p\delta)^2} + \sqrt{r^2 - ((p+1)\delta)^2}} \\ &\leq \frac{(2p+1)\delta^2}{\sqrt{r^2 - (p\delta)^2}}. \end{aligned}$$

Now let j be the largest integer not larger than $\sqrt{\left(\frac{r}{\delta}\right)^2 - (p+1)^2}$ and l the smallest integer not smaller than $\sqrt{\left(\frac{r}{\delta}\right)^2 - p^2}$. We have just shown that

$$0 < l - j < 2 + \frac{(2p+1)\delta}{\sqrt{r^2 - (p\delta)^2}} = 2 + \frac{2p+1}{\sqrt{\left(\frac{r}{\delta}\right)^2 - p^2}}.$$

It is this inequality that provides the required estimate of the width of the annulus corresponding to the zone Z_p .

Step 3. Prove the estimate $N_{r,\delta}^k \leq 6^{k^2} \left(\frac{r}{\delta}\right)^{k-1}$.

We first do the case $k = 1$. This case is very straightforward. If $I_{(\delta m, z)}^1(\delta) \cap S_r^0$, then either $m\delta \leq r \leq (m+1)\delta$ or $m\delta \leq -r \leq (m+1)\delta$. If $r = k\delta$ for some integer k , there are four values of m for which one of these two sets of inequalities hold, namely $-k-1, -k, k-1$, and k . Otherwise there are only two such integers m , namely $\left[\frac{r}{\delta}\right]$ and $\left[\frac{-r}{\delta}\right]$. Thus we actually have $N_{r,\delta}^0 \leq 4 < 6^1 \left(\frac{r}{\delta}\right)^0$.

We now proceed by induction, supposing the theorem proved for dimensions less than k , and we assume $k \geq 2$. We consider all the lattice points $\mathbf{m} \in R^k$ such that $m_k = p$ and $I_{\delta\mathbf{m}}^k(\delta)$ intersects Z_p . From what we have shown above in Step 1 and Step 2, if \mathbf{m} has this property, then the bottom face of $I_{\delta\mathbf{m}}^k(\delta)$ intersects one of the spheres $S_{s\delta}^{k-2}(p\delta)$, where s is an integer such that $j \leq s \leq l$. The number of such \mathbf{m} is at most $N_{s\delta,\delta}^{k-1}$ and hence is at most $6^{(k-1)^2} s^{k-2}$, which is certainly no larger than

$$6^{(k-1)^2} \left(1 + \sqrt{\left(\frac{r}{\delta}\right)^2 - p^2}\right)^{k-2}.$$

Because of our estimate of $l - j$, we see that the total number of \mathbf{m} for which $m_k = p$ and $I_{\delta\mathbf{m}}^k(\delta)$ intersects Z_p is at most

$$6^{(k-1)^2} \left(2 + \frac{2p+1}{\sqrt{\left(\frac{r}{\delta}\right)^2 - p^2}}\right) \left(2^{k-2} + 2^{k-2} \left(\left(\frac{r}{\delta}\right)^2 - p^2\right)^{\frac{k-2}{2}}\right),$$

where we have used the inequality $(1+t)^q \leq 2^q + (2t)^q$ with $q = k-2$. We expand this last product into a sum of six terms:

$$\begin{aligned} & 6^{(k-1)^2} \left(2^{k-1} + 2^{k-1} \left(\left(\frac{r}{\delta}\right)^2 - p^2\right)^{\frac{k-2}{2}} + 2^{k-1} \frac{p}{\sqrt{\left(\frac{r}{\delta}\right)^2 - p^2}} \right. \\ & \quad \left. + \frac{2^{k-2}}{\sqrt{\left(\frac{r}{\delta}\right)^2 - p^2}} + 2^{k-1} p \left(\left(\frac{r}{\delta}\right)^2 - p^2\right)^{\frac{k-3}{2}} + 2^{k-2} \left(\left(\frac{r}{\delta}\right)^2 - p^2\right)^{\frac{k-3}{2}}\right) \\ & = 6^{(k-1)^2} (I_1(p) + I_2(p) + I_3(p) + I_4(p) + I_5(p) + I_6(p)). \end{aligned}$$

We need to estimate the sum of each of these terms over p from 0 to $\left[\frac{r}{\delta}\right] - 1$.

For $I_1(p)$ we have the simple estimate

$$\sum_{p=0}^{\lceil \frac{r}{\delta} \rceil - 1} I_1(p) = 2^{k-1} \lceil \frac{r}{\delta} \rceil \leq 2^{k-1} \left(\frac{r}{\delta} \right) < 2^{k-1} \left(\frac{r}{\delta} \right)^{k-1},$$

since $k > 1$ and $r > \delta$.

For $I_2(p)$ we have

$$\begin{aligned} \sum_{p=0}^{\lceil \frac{r}{\delta} \rceil - 1} I_2(p) &= 2^{k-1} \left(\frac{r}{\delta} \right)^{k-2} \sum_{p=0}^{\lceil \frac{r}{\delta} \rceil - 1} \left(1 - \frac{p\delta}{r} \right)^{\frac{k-2}{2}} \\ &< 2^{k-1} \left(\frac{r}{\delta} \right)^{k-2} \left(\frac{r}{\delta} \right) = 2^{k-1} \left(\frac{r}{\delta} \right)^{k-1}. \end{aligned}$$

Here we have used the fact that $0 < 1 - \left(\frac{p\delta}{r} \right)^2 \leq 1$ for the values of p in the range of summation, as we shall do twice more below.

For $I_3(p)$ we have

$$\begin{aligned} \sum_{p=0}^{\lceil \frac{r}{\delta} \rceil - 1} I_3(p) &= 2^{k-1} \left(\frac{\delta}{r} \right) \sum_{p=0}^{\lceil \frac{r}{\delta} \rceil - 1} \frac{p}{\sqrt{1 - \left(\frac{p\delta}{r} \right)^2}} \\ &\leq 2^{k-1} \left(\frac{\delta}{r} \right) \int_1^{\frac{r}{\delta}} \frac{x}{\sqrt{1 - \left(\frac{\delta x}{r} \right)^2}} dx \\ &= 2^{k-1} \left(\frac{r}{\delta} \right) \int_{\frac{\delta}{r}}^1 \frac{y}{\sqrt{1 - y^2}} dy \\ &< 2^{k-1} \left(\frac{r}{\delta} \right) \int_0^1 \frac{y}{\sqrt{1 - y^2}} dy \\ &= 2^{k-1} \frac{r}{\delta} \leq 2^{k-1} \left(\frac{r}{\delta} \right)^{k-1}. \end{aligned}$$

Since $I_4(p) \leq \frac{1}{2} I_3(p)$ for $p = 1, 2, \dots, \lceil \frac{r}{\delta} \rceil - 1$, it is clear that

$$\sum_{p=0}^{\lceil \frac{r}{\delta} \rceil - 1} I_4(p) < 2^{k-2} + 2^{k-2} \left(\frac{r}{\delta} \right)^{k-1} < 2^{k-1} \left(\frac{r}{\delta} \right)^{k-1}.$$

For $I_5(p)$ we have

$$\begin{aligned} \sum_{p=0}^{\lceil \frac{r}{\delta} \rceil - 1} I_5(p) &= 2^{k-1} \left(\frac{r}{\delta} \right)^{k-3} \sum_{p=0}^{\lceil \frac{r}{\delta} \rceil - 1} p \left(1 - \frac{p\delta}{r} \right)^{\frac{k-3}{2}} \\ &< 2^{k-2} \left(\frac{r}{\delta} \right)^{k-3} \left(\frac{r}{\delta} \right)^2 = 2^{k-2} \left(\frac{r}{\delta} \right)^{k-1}. \end{aligned}$$

For $I_6(p)$ we have

$$\sum_{p=0}^{\lceil \frac{r}{\delta} \rceil - 1} I_6(p) = 2^{k-2} \left(\frac{r}{\delta} \right)^{k-3} \sum_{p=0}^{\lceil \frac{r}{\delta} \rceil - 1} \left(1 - \frac{p\delta}{r} \right)^{\frac{k-3}{2}} < 2^{k-2} \left(\frac{r}{\delta} \right)^{k-3} \left(\frac{r}{\delta} \right) < 2^{k-2} \left(\frac{r}{\delta} \right)^{k-1}.$$

Adding all these estimates, we find a sum that is at most

$$6^{(k-1)^2} (2^{k-1} + 2^{k-1} + 2^{k-1} + 2^{k-2} + 2^{k-2} + 2^{k-2}) \left(\frac{r}{\delta} \right)^{k-1} \leq 6^{(k-1)^2+1} 2^{k-1} \left(\frac{r}{\delta} \right)^{k-1}.$$

If we wish to count the total number of hypercubes $I_{\delta \mathbf{m}}^k(\delta)$ that intersect the zones $Z_1, \dots, Z_{\lceil \frac{r}{\delta} \rceil - 1}$, we recall our previous observation that such a hypercube can intersect Z_p only if $m_k = p$ or $m_k = p - 1$, and if a hypercube $I_{\delta \mathbf{m}}^k(\delta)$ with $m_k = p - 1$ intersects Z_p , then so does the hypercube $I_{\delta(\mathbf{m} + \mathbf{e}_k)}^k(\delta)$, and the latter has already been counted. Hence in estimating the total number of hypercubes that intersect one of these zones we are more than safe in simply doubling the estimate we have already obtained.

As for the zone Z_0 , any hypercube $I_{\delta \mathbf{m}}^k(\delta)$ with $m_k = -1$ that intersects its bottom edge (the hypersphere S_r^{k-2}) also meets the reflection of Z_0 through the plane $x_k = 0$, and hence the reflection of that hypercube has already been counted among those that meet Z_0 . When we double our count to include the hypercubes meeting the "southern" hemisphere, all these hypercubes will automatically be counted. Thus it remains only to estimate the hypercubes that meet the "northern arctic zone," then double the count.

Hence we now consider the cap at the top of the hemisphere, whose boundary is the $(k-2)$ -dimensional hypersphere

$$S_s^{k-2} \left(\left\lceil \frac{r}{\delta} \right\rceil \delta \right),$$

where

$$s = \delta \sqrt{\left(\frac{r}{\delta} \right)^2 - \left\lceil \frac{r}{\delta} \right\rceil^2}.$$

If \mathbf{m} is such that $I_{\delta \mathbf{m}}^k(\delta)$ meets this set, then we must have $\lceil \frac{r}{\delta} \rceil - 1 \leq m_k \leq \lceil \frac{r}{\delta} \rceil$, so that there are only two possible values for m_k . As for m_j , $j < k$, we certainly have $-\frac{s}{\delta} - 1 \leq m_j \leq \frac{s}{\delta}$, so that there are at most $2^k \left(\frac{s}{\delta} + 1 \right)^{k-1}$ such hypercubes not already counted.

Since $\lceil \frac{r}{\delta} \rceil \leq \frac{r}{\delta} < \lceil \frac{r}{\delta} \rceil + 1$, we easily find that

$$s \leq 2\sqrt{r\delta},$$

so that $\frac{s}{\delta} + 1 \leq 2\sqrt{\frac{r}{\delta}} + 1$. Once more using the inequality $(1+t)^q \leq 2^q(1+t^q)$ for positive t , with $q = k-1$, we find that the number of hypercubes meeting the northern polar cap is at most $2^{2k-1} \left(1 + \left(\frac{r}{\delta} \right)^{\frac{k-1}{2}} \right)$, which is less than $2^{2k} \left(\frac{r}{\delta} \right)^{k-1}$

Adding the numbers up and then doubling to count the hypercubes that meet the southern hemisphere, we find that the hypersphere S_r^{k-1} meets at most

$$\left[6^{(k-1)^2+1}2^k + 2^{2k+1}\right]\left(\frac{r}{\delta}\right)^{k-1}.$$

The constant coefficient here is less than 6^{k^2} . This is directly computable for $k = 2$ and $k = 3$. Indeed, the quantity 2^{2k+1} is less than 6^k , while $2^k < 6^{\frac{k}{2}}$. The extremely weak inequality $a + b < ab$, which is valid for positive integers a and b both larger than 1, then implies that for $k \geq 4$ the coefficient is at most

$$6^{(k-1)^2+1+(k/2)+k} = 6^{k^2-k/2+2} \leq 6^{k^2}.$$

Stage 1 in the proof is now complete. We have shown that the $(k-1)$ -sphere S_r^{k-1} intersects at most $6^{k^2}\left(\frac{r}{\delta}\right)^{k-1}$ hypercubes from the family $I_{\delta\mathbf{m}}^k(\delta)$. As the volume of each hypercube is δ^k , it follows that the $(k-1)$ -sphere is contained in a finite union of cubes of total k -dimensional volume $6^{k^2}r^{k-1}\delta$. Since δ is an arbitrary positive number, the k -dimensional volume of S_r^{k-1} is zero.

We now move on to the second stage of the proof.

Stage 2. Given any convex set H in R^k with non-empty interior, construct a homeomorphism \mathbf{T} of R^k onto itself that maps $S_1^{k-1} = S_1^{k-1}$ to ∂H , the inside of the unit ball to the interior of H , and the outside to the exterior of H , and satisfies a Lipschitz condition on a neighborhood of S^{k-1} . To get this result we need some more background work on general convex sets in R^k .

Let C be a bounded convex set in R^k , and let \mathbf{z} be an interior point of C . For each point \mathbf{x} on the unit sphere S^{k-1} in R^k , let $\psi(\mathbf{x})$ be the distance from \mathbf{z} to the complement of C in the direction of \mathbf{x} , that is,

$$\psi(\mathbf{x}) = \inf\{t > 0 : \mathbf{z} + t\mathbf{x} \notin C\} = \sup\{t > 0 : \mathbf{z} + t\mathbf{x} \in C\}.$$

Step 1. Prove that the function $\psi : S^{k-1} \rightarrow (0, +\infty)$ is continuous, in fact, that it satisfies a Lipschitz condition: for some constant K , $|\psi(\mathbf{x}) - \psi(\mathbf{y})| \leq K|\mathbf{x} - \mathbf{y}|$.

There exist positive numbers a and b such that $a \leq \psi(\mathbf{x}) \leq b$ for all $\mathbf{x} \in S^{k-1}$. Indeed we can let a be the radius of the largest open ball about \mathbf{z} that is contained in C and b the radius of the smallest closed ball about \mathbf{z} containing C .

The result now follows from a lemma.

Let $0 \leq s \leq \psi(\mathbf{x})$. Then C contains the open ball of radius $\left(1 - \frac{s}{\psi(\mathbf{x})}\right)a$ about $\mathbf{z} + s\mathbf{x}$.

Proof: If $s = \psi(\mathbf{x})$, this ball is empty, and if $s = 0$ the assertion is merely the definition of a . Hence assume $0 < s < \psi(\mathbf{x})$. Now suppose $|\mathbf{y} - (\mathbf{z} + s\mathbf{x})| < \left(1 - \frac{s}{\psi(\mathbf{x})}\right)a$. Let $t = 1 - \frac{s}{\psi(\mathbf{x})}$, so that $0 < t < 1$, and let $r = \frac{1}{2} \min\left(a - \frac{|\mathbf{y} - (\mathbf{z} + s\mathbf{x})|}{t}, \frac{s}{t}\right)$, so that $r > 0$. Let $\mathbf{w} = \frac{\mathbf{y} - (\mathbf{z} + s\mathbf{x})}{t} + r\mathbf{x}$,

and let $u = \frac{s-tr}{1-t}$. We claim that $\mathbf{z} + \mathbf{w} \in C$ and that $\mathbf{z} + u\mathbf{x} \in C$, so that $\mathbf{y} = t(\mathbf{z} + \mathbf{w}) + (1-t)(\mathbf{z} + u\mathbf{x}) \in C$.

The first claim will follow if we show that $|\mathbf{w}| < a$. In fact

$$\begin{aligned} |\mathbf{w}| &\leq \frac{|\mathbf{y} - (\mathbf{z} + s\mathbf{x})|}{t} + r \quad (\text{since } |\mathbf{x}| = 1), \\ &\leq \frac{|\mathbf{y} - (\mathbf{z} + s\mathbf{x})|}{t} + \frac{1}{2} \left(a - \frac{|\mathbf{y} - (\mathbf{z} + s\mathbf{x})|}{t} \right) \\ &< \frac{|\mathbf{y} - (\mathbf{z} + s\mathbf{x})|}{t} + a - \frac{|\mathbf{y} - (\mathbf{z} + s\mathbf{x})|}{t} = a. \end{aligned}$$

The second claim will follow if we prove $0 < u < \psi(\mathbf{x})$. In fact $u = (s - tr) \cdot \frac{1}{1-t} = (s - tr) \cdot \frac{\psi(\mathbf{x})}{s} = \left(1 - \frac{tr}{s}\right) \psi(\mathbf{x}) < \psi(\mathbf{x})$. Since $r \leq \frac{s}{2t}$, we have $u \geq \frac{1}{2} \psi(\mathbf{x}) > 0$.

Finally, the last claim is a routine computation:

$$\begin{aligned} t(\mathbf{z} + \mathbf{w}) + (1-t)(\mathbf{z} + u\mathbf{x}) &= \mathbf{z} + t\mathbf{w} + (1-t)u\mathbf{x} \\ &= \mathbf{z} + \mathbf{y} - (\mathbf{z} + s\mathbf{x}) + tr\mathbf{x} + (1-t)u\mathbf{x} \\ &= \mathbf{y} + (tr - s + (1-t)u)\mathbf{x} \\ &= \mathbf{y} \quad (\text{since } (1-t)u = s - tr). \end{aligned}$$

The lemma is now proved.

Taking $\mathbf{y} = \mathbf{z} + s\mathbf{v}$ in this lemma (where $|\mathbf{v}| = 1$), we see that $\mathbf{y} \in C$ (and hence $\psi(\mathbf{v}) \geq s$) if

$$|\mathbf{v} - \mathbf{x}| < \left(\frac{1}{s} - \frac{1}{\psi(\mathbf{x})} \right) a.$$

Now let $t > \psi(\mathbf{x})$, $|\mathbf{v}| = 1$, and $|\mathbf{v} - \mathbf{x}| < \left(\frac{1}{\psi(\mathbf{x})} - \frac{1}{t} \right) a$. Choose $t' \in (\psi(\mathbf{x}), t)$ such that

$$|\mathbf{v} - \mathbf{x}| < \left(\frac{1}{t'} - \frac{1}{t} \right) a.$$

If $\psi(\mathbf{v}) \geq t$, we have *a fortiori* $\psi(\mathbf{v}) > t'$ and

$$|\mathbf{x} - \mathbf{v}| < \left(\frac{1}{t'} - \frac{1}{\psi(\mathbf{v})} \right) a,$$

which, as already shown, implies $\psi(\mathbf{x}) \geq t'$, contradicting the choice of t' . Therefore $\psi(\mathbf{v}) < t$.

To summarize, if $s < \psi(\mathbf{x}) < t$, then $s \leq \psi(\mathbf{v}) < t$ provided

$$|\mathbf{x} - \mathbf{v}| < \min \left(\left(\frac{1}{s} - \frac{1}{\psi(\mathbf{x})} \right) a, \left(\frac{1}{\psi(\mathbf{x})} - \frac{1}{t} \right) a \right).$$

This proves that ψ is continuous. Specializing to the case where $s = \psi(\mathbf{x}) - \varepsilon$ and $t = \psi(\mathbf{x}) + \varepsilon$, we see that $|\psi(\mathbf{x}) - \psi(\mathbf{v})| \leq \varepsilon$ provided $|\mathbf{x} - \mathbf{v}| < \frac{a\varepsilon}{\psi(\mathbf{x})(\psi(\mathbf{x}) + \varepsilon)}$.

Again, *a fortiori*,

$$|\mathbf{x} - \mathbf{v}| < \frac{a\varepsilon}{b(b + \varepsilon)} \Rightarrow |\psi(\mathbf{x}) - \psi(\mathbf{v})| \leq \varepsilon.$$

We now claim that

$$|\mathbf{x} - \mathbf{v}| \leq \frac{a\varepsilon}{2b^2} \Rightarrow |\psi(\mathbf{x}) - \psi(\mathbf{v})| \leq \varepsilon.$$

This follows from the previous statement and the continuity of ψ (together with the fact that a closed ball on the sphere S is the closure of the open ball with the same center and radius) if $\varepsilon \leq b - a$. If $\varepsilon > b - a$, the second inequality automatically holds because $\psi(\mathbf{x})$ and $\psi(\mathbf{v})$ differ by at most $b - a$. Specializing to equality in the hypothesis, we deduce the Lipschitz inequality

$$|\psi(\mathbf{x}) - \psi(\mathbf{v})| \leq \frac{2b^2}{a} |\mathbf{x} - \mathbf{v}|.$$

We remark that the statement that \mathbf{y} belongs to the interior, boundary, and exterior of C is equivalent to $|\mathbf{y} - \mathbf{z}| < \psi\left(\frac{\mathbf{y} - \mathbf{z}}{|\mathbf{y} - \mathbf{z}|}\right)$, $|\mathbf{y} - \mathbf{z}| = \psi\left(\frac{\mathbf{y} - \mathbf{z}}{|\mathbf{y} - \mathbf{z}|}\right)$, or $|\mathbf{y} - \mathbf{z}| > \psi\left(\frac{\mathbf{y} - \mathbf{z}}{|\mathbf{y} - \mathbf{z}|}\right)$.

Step 2. Use the function $\psi(\mathbf{x})$ to define a homeomorphism of R^k onto itself that maps S^{k-1} to ∂H and is Lipschitz in a neighborhood of S^{k-1} . Such a homeomorphism $\mathbf{T}(\mathbf{x})$ is defined for all $\mathbf{x} \in R^k$ as follows. We set $\mathbf{T}(\mathbf{0}) = \mathbf{z}$ and

$$\mathbf{T}(\mathbf{x}) = \mathbf{z} + \psi\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right)\mathbf{x}$$

if $\mathbf{x} \neq \mathbf{0}$. Since $|\mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{0})| \leq M|\mathbf{x}|$, where $M = \sup\{\psi(\mathbf{y}) : |\mathbf{y}| = 1\}$, it is clear that \mathbf{T} is continuous at $\mathbf{0}$. At all other points it is a composition of continuous functions, hence continuous. Since $\frac{\mathbf{T}(\mathbf{x}) - \mathbf{z}}{|\mathbf{T}(\mathbf{x}) - \mathbf{z}|} = \frac{\mathbf{x}}{|\mathbf{x}|}$, we have the continuous inverse function

$$\mathbf{x} = \frac{\mathbf{T}(\mathbf{x}) - \mathbf{z}}{\psi((\mathbf{T}(\mathbf{x}) - \mathbf{z})/|\mathbf{T}(\mathbf{x}) - \mathbf{z}|)},$$

That is, for $\mathbf{y} \neq \mathbf{z}$,

$$\mathbf{T}^{-1}(\mathbf{y}) = \frac{\mathbf{y} - \mathbf{z}}{\psi((\mathbf{y} - \mathbf{z})/|\mathbf{y} - \mathbf{z}|)},$$

which is not $\mathbf{0}$. Thus the mapping is one-to-one and onto.

The mapping also satisfies a Lipschitz condition on the exterior of each ball about $\mathbf{0}$; that is, on the set $E_\eta = \{\mathbf{x} : |\mathbf{x}| \geq \eta\}$ for each $\eta > 0$. To see this we observe that for any \mathbf{x} and \mathbf{y} in this set,

$$|\mathbf{T}(\mathbf{x}) - \mathbf{T}(\mathbf{y})| = \left| \psi\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right)\mathbf{x} - \psi\left(\frac{\mathbf{y}}{|\mathbf{y}|}\right)\mathbf{y} \right|$$

$$\begin{aligned}
&\leq \left| \psi\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) - \psi\left(\frac{\mathbf{y}}{|\mathbf{y}|}\right) \right| |\mathbf{y}| + \psi\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right) |\mathbf{x} - \mathbf{y}| \\
&\leq K \left| \frac{\mathbf{x}}{|\mathbf{x}|} - \frac{\mathbf{y}}{|\mathbf{y}|} \right| + M |\mathbf{x} - \mathbf{y}| \\
&\leq \left(\frac{2K}{\eta} + M \right) |\mathbf{x} - \mathbf{y}|.
\end{aligned}$$

Here we have used the fact that

$$\begin{aligned}
\left| \frac{\mathbf{x}}{|\mathbf{x}|} - \frac{\mathbf{y}}{|\mathbf{y}|} \right| &= \frac{1}{|\mathbf{x}||\mathbf{y}|} |\mathbf{y}|\mathbf{x} - |\mathbf{x}|\mathbf{y}| \\
&= \frac{1}{|\mathbf{x}||\mathbf{y}|} (|\mathbf{y}| - |\mathbf{x}|)\mathbf{x} + |\mathbf{x}|(\mathbf{x} - \mathbf{y})| \\
&\leq \frac{1}{|\mathbf{y}|} ||\mathbf{y}| - |\mathbf{x}|| + \frac{1}{|\mathbf{y}|} |\mathbf{x} - \mathbf{y}| \\
&= \frac{2}{|\mathbf{y}|} |\mathbf{y} - \mathbf{x}|.
\end{aligned}$$

The statements about the images of the inside of the unit ball, the unit sphere, and the outside are now obvious. For example, as remarked above, $\mathbf{y} \in \partial H$ if and only if $|\mathbf{y} - \mathbf{z}| = \psi\left(\frac{\mathbf{y} - \mathbf{z}}{|\mathbf{y} - \mathbf{z}|}\right)$. But this is equivalent to the statement that $\mathbf{T}^{-1}(\mathbf{y}) = \frac{\mathbf{y} - \mathbf{z}}{|\mathbf{y} - \mathbf{z}|}$, which says precisely that $\mathbf{T}^{-1}(\mathbf{y})$ belongs to the unit sphere.

We have now finished Stage 2 of the proof and are ready for the third and final stage.

Stage 3. For each $\delta > 0$, approximate a function $f(\mathbf{x})$ that is continuous on H by a function $f_\delta(\mathbf{x})$ that is continuous on all of R^k and such that the iterated integrals of f and f_δ differ by at most a fixed multiple of δ no matter what order they are taken in.

To that end, we first let $\delta \in (0, 1/\sqrt{k})$ be given. According to what was proved in Stage 1, the hypersphere S^{k-1} is contained in the interior of the set of hypercubes $I_{\delta\mathbf{m}}^k(\delta)$ that intersect it, and there are at most $6^{k^2} \delta^{1-k}$ of these hypercubes. In each hypercube $I_{\delta\mathbf{m}}^k(\delta)$ from this family we choose and keep fixed one point \mathbf{x}_m belonging to S^{k-1} . The image of these hypercubes under \mathbf{T} is a compact set containing ∂H in its interior, and each of them is contained in a hypercube of side at most $2L\sqrt{k}\delta$ centered at $\mathbf{T}(\mathbf{x}_m) \in \partial H$, where L is the Lipschitz constant for the mapping \mathbf{T} on the set $E_{1-\delta}$, so that the total volume of these hypercubes is at most $6^{k^2} (2L\sqrt{k})^k \delta$. Let $c > 0$ be the distance from H to the complement of the union of these hypercubes.

We define $f_\delta(\mathbf{x})$ as a continuous function that equals $f(\mathbf{x})$ for $\mathbf{x} \in H$, while for \mathbf{x} not in the interior of H we set $f_\delta(\mathbf{x}) = \max\left(0, 1 - \frac{d(\mathbf{x}, H)}{c}\right) f(\theta(\mathbf{x}))$. Here $\theta(\mathbf{x})$ is the unique point of H closest to \mathbf{x} and $d(\mathbf{x}, H)$ is the distance from \mathbf{x} to H . On the boundary of H , where we have apparently given two definitions of f_δ we have $d(\mathbf{x}, H) = 0$, so that the two definitions are consistent. Hence the piecewise-defined function will be continuous if each of the pieces is. The piece defined

on H is continuous by assumption, so that we need only concern ourselves with the second definition. It is well-known that $d(\mathbf{x}, H)$ is a continuous function of \mathbf{x} . It is somewhat less obvious that $\theta(\mathbf{x})$ is continuous, so that we must prove that fact.

First we show that there is a unique point $\theta(\mathbf{x})$ in H closest to \mathbf{x} . This is obvious if $\mathbf{x} \in H$, so we assume $\mathbf{x} \notin H$. Let $c = \min\{|\mathbf{x} - \mathbf{z}| : \mathbf{z} \in H\}$, and suppose \mathbf{z} and \mathbf{w} are two points of H such that $|\mathbf{x} - \mathbf{z}| = c = |\mathbf{x} - \mathbf{w}|$. Then the point $\mathbf{w} + t(\mathbf{z} - \mathbf{w})$ belongs to H for $0 \leq t \leq 1$, and so the quadratic function $|\mathbf{x} - \mathbf{w} - t(\mathbf{z} - \mathbf{w})|^2 = |\mathbf{x} - \mathbf{w}|^2 - 2t(\mathbf{x} - \mathbf{w}) \cdot (\mathbf{z} - \mathbf{w}) + t^2|\mathbf{z} - \mathbf{w}|^2$ has its minimum value c on $[0, 1]$ at both endpoints. But this is impossible for a non-constant quadratic function whose leading coefficient is positive. Hence the function is constant, that is, $\mathbf{z} = \mathbf{w}$. Now suppose $\mathbf{x}_n \rightarrow \mathbf{x}$. We claim $\theta(\mathbf{x}_n) \rightarrow \theta(\mathbf{x})$.

Since H is compact, we can pass to a subsequence if necessary and assume that $\theta(\mathbf{x}_n) \rightarrow \mathbf{z}$ for some point $\mathbf{z} \in H$. Certainly $|\mathbf{x}_n - \theta(\mathbf{x}_n)| \rightarrow |\mathbf{x} - \mathbf{z}|$. But $|\mathbf{x}_n - \theta(\mathbf{x}_n)| = d(\mathbf{x}_n, H) \rightarrow d(\mathbf{x}, H)$, so that $|\mathbf{x} - \mathbf{z}| = d(\mathbf{x}, H) = |\mathbf{x} - \theta(\mathbf{x})|$. As H contains only one point satisfying this equality, we must have $\mathbf{z} = \theta(\mathbf{x})$. Thus $\theta(\mathbf{x})$ is a continuous function, and therefore $f_\delta(\mathbf{x})$ is continuous on all of R^k .

It is now clear that $|f(\mathbf{x})|$ and $|f_\delta(\mathbf{x})|$ have the same maximum value, say J , and that f and f_δ differ only on the finite set of hypercubes covering ∂H . The iterated integrals of the two functions, taken in any order, over this finite set of hypercubes differ by at most $6^{k^2} J L (2\sqrt{k})^k \delta$. Thus the iterated integral of f differs from the iterated integral of f_δ by at most this amount, and since all the iterated integrals of f_δ are equal, it follows that any two iterated integrals of f differ by arbitrarily small amounts, hence are equal.

The proof is, at long last, complete.

Because this proof is so long and involved, it may be worthwhile to look at an alternative proof that works only for the case $k = 2$ and does not generalize to higher dimensions. To this end, let $k = 2$. we define two functions $m(x)$ and $M(x)$, as follows: The domain of both functions is the projection of H on the x -axis, that is, the set $\prod(H)$ consisting of x such that there exists y for which $(x, y) \in H$. By definition $m(x)$ is the minimal y for which $(x, y) \in H$, and $M(x)$ is the maximal y for which $(x, y) \in H$. We claim that these functions are continuous on $\prod(H)$. Indeed, suppose $(x^{(n)}, y^{(n)}) \in H$ and $x^{(n)} \rightarrow x$. Without loss of generality we can assume that $x^{(n)} > x$ for all x . (By passing to a subsequence if necessary, we can have either $x^{(n)} < x$ for all n or $x^{(n)} > x$ or $x^{(n)} = x$ for all n . The last case is trivial, and the other two cases are handled by identical arguments.) Some subsequence of $M(x^{(n)})$ converges to a value z . Since $(x^{(n)}, M(x^{(n)})) \in H$, and H is closed, it follows that (x, z) belongs to H . It is clear then that the assumption $z > M(x)$ contradicts the definition of $M(x)$ as the maximal number y for which $(x, y) \in H$. Hence it suffices to prove that $z \geq M(x)$. This will certainly be the case if $M(x^{(n)}) \geq M(x)$ for all n . Hence assume that n_0 is an index for which $M(x^{(n_0)}) < M(x)$. Now $x^{(n_0)} > x$, since if the two were equal, $M(x^{(n_0)})$ would equal $M(x)$. We observe that if $t \in [0, 1]$, then the point $(tx^{(n_0)} + (1-t)x, tM(x^{(n_0)}) + (1-t)M(x))$ belongs

to H . In particular, taking $t = \frac{x^{(n)} - x}{x^{(n_0)} - x}$, we find that $tx^{(n_0)} + (1-t)x = x^{(n)}$. It therefore follows that $M(x^{(n)}) \geq \frac{x^{(n)} - x}{x^{(n_0)} - x} M(x^{(n_0)}) + \frac{x^{(n_0)} - x}{x^{(n_0)} - x} M(x) \rightarrow M(x)$. Therefore $z \geq M(x)$.

(It is this part of the argument that does not generalize to R^3 , as shown by the the convex set

$$H = \{(1-t, ty, tz) : 0 \leq t \leq 1, -1 \leq y \leq 1, y^2 \leq z \leq 1\}.$$

On this set, if we define $M(y, z) = \sup\{x : (x, y, z) \in H\}$, we have $M(s, s^2) = 0$ for $s \neq 0$, but $M(0, 0) = 1$.)

It now follows that $M(x)$ is continuous on H , and the proof that $m(x)$ is continuous is similar.

Now let H be a convex closed set in R^2 containing an interior point. For each $\delta > 0$, we let H_δ be the δ -neighborhood of H , that is, the set of points whose distance from H is at most δ . It is clear that H_δ is a convex set containing H in its interior. If f is a continuous function on H , we extend f to a function f_δ defined on all of R^2 , as above.

By our definition

$$\int_H f(x, y) dy dx = \int_a^b \int_{m(x)}^{M(x)} f(x, y) dy dx,$$

where $[a, b]$ is the projection of H on the x -axis and for each $x \in [a, b]$

$$m(x) = \min\{t : (x, t) \in H\}$$

and

$$M(x, y) = \max\{t : (x, t) \in H\}.$$

We intend to show that when these integrals are evaluated, the resulting value is the limit of the same integrals evaluated for f_δ , and of course the same for the integrals in reverse order. Hence these two iterated integrals are equal.

To that end, let A be the maximal value of $|f(x, y)|$, which is also the maximal value of $|f_\delta(x, y)|$. As we have set $f(x, y) = 0$ on the complement of H , the two functions $f(x, y)$ and $f_\delta(x, y)$ differ only on the set $H_\delta \setminus H$, and by no more than A at any point.

Let $P_\delta = [a - \lambda(\delta), b + \mu(\delta)]$ be the projection of H_δ on the x -axis. We claim that $\lambda(\delta)$ and $\mu(\delta)$ both tend to zero as δ tends to zero. For certainly $\lambda(\delta)$ decreases as δ decreases. Let its limit be c . There is a point $(a - \lambda(\delta), y_\delta) \in H_\delta$ for each $\delta > 0$. If y is a limit point of y_δ as $\delta \rightarrow 0$, then, since $(a - \lambda(\eta), y_\eta) \in H_\eta \subset H_\delta$ for $\eta < \delta$ and H_δ is closed, it follows that $(a - c, y) \in H_\delta$ for all $\delta > 0$, and therefore, since $\bigcap_{\delta > 0} H_\delta = H$, that $(a - c, y) \in H$. By definition of a , it then follows that $a \leq a - c \leq a$, and so $c = 0$. The proof that $\mu \rightarrow 0$ is similar.

Let $m_\delta(x)$ and $M_\delta(x)$ be the functions corresponding to $m(x)$ and $M(x)$ for H_δ . For $x \in [a, b]$ we have $m_\delta(x) < m(x) \leq M(x) < M_\delta(x)$. An argument similar to the one just given shows that $m(x) - m_\delta(x)$ and $M_\delta(x) - M(x)$ tend

monotonically to zero for each $x \in [a, b]$. Since these are continuous functions, this convergence is *uniform*. let $\varphi(\delta) = \max_{x \in [a, b]} \{m(x) - m_\delta(x), M_\delta(x) - M(x)\}$, so that $\varphi(\delta) \rightarrow 0$ as $\delta \rightarrow 0$.

Now the double integrals that we wish to evaluate are

$$\int_a^b \int_{m(x)}^{M(x)} f(x, y) dy dx$$

and

$$\int_{a-\lambda(\delta)}^{b+\mu(\delta)} \int_{m_\delta(x)}^{M_\delta(x)} f_\delta(x, y) dy dx.$$

Now fix a number N larger than twice the absolute value of any coordinate of any point in H_1 , and assume $\delta < \min(1, N)$. We observe that the difference between the two integrals is

$$\begin{aligned} \int_{a-\lambda(\delta)}^a \int_{-N}^N f_\delta(x, y) dy dx + \int_a^b \int_{m_\delta(x)}^{m(x)} f_\delta(x, y) dy dx \\ + \int_a^b \int_{M(x)}^{M_\delta(x)} f_\delta(x, y) dy dx + \int_b^{b+\mu(\delta)} \int_{-N}^N f_\delta(x, y) dy dx. \end{aligned}$$

This expression is assuredly not larger than

$$2AN(\lambda(\delta) + \varphi(\delta) + \mu(\delta)),$$

and hence it tends to zero as $\delta \rightarrow 0$. The same is true of the integral in the reverse order, and for the same reasons. Since the integral of f_δ is the same in either order, it follows that the integral of f is also the same in either order.

Exercise 10.2 For $i = 1, 2, 3, \dots$, let $\varphi_i \in \mathcal{C}(R^1)$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put

$$f(x, y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)] \varphi_i(y).$$

The f has compact support in R^2 , f is continuous except at $(0, 0)$, and

$$\int dy \int f(x, y) dx = 0 \text{ but } \int dx \int f(x, y) dy = 1.$$

Observe that f is unbounded in every neighborhood of $(0, 0)$.

Solution: The computation is straightforward:

$$\int f(x, y) dx = \sum_{i=1}^{\infty} \varphi_i(y) [1 - 1] = 0;$$

$$\int f(x, y) dy = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)] = \varphi_1(x).$$

To justify the first of these, we observe that the sum is finite for each fixed y , since $\varphi_i(y) = 0$ for $i > -\log_2(y)$ if $y > 0$. Likewise the second sum is finite for each fixed x . The result now follows. The function must be unbounded, since the integral of φ_i must be 1, even though the support of that function has length 2^{-i} .

Exercise 10.3 (a) If \mathbf{F} is as in Theorem 10.7, put $\mathbf{A} = \mathbf{F}'(\mathbf{0})$, $\mathbf{F}_1(\mathbf{x}) = \mathbf{A}^{-1}\mathbf{F}(\mathbf{x})$. Then $\mathbf{F}'_1(\mathbf{0}) = I$. Show that

$$\mathbf{F}_1(\mathbf{x}) = \mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x})$$

in some neighborhood of $\mathbf{0}$, for certain primitive mappings $\mathbf{G}_1, \dots, \mathbf{G}_n$. This gives another version of Theorem 10.7:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}'(\mathbf{0})\mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x}).$$

(b) Prove that the mapping $(x, y) \rightarrow (y, x)$ of R^2 onto R^2 is not the composition of any two primitive mappings, in any neighborhood of the origin. (This shows that the flips B_1 cannot be omitted from the statement of Theorem 10.7.)

Solution: (a) According to the proof of Theorem 10.7, the flips are needed only to interchange m and k , where k is the first index not less than m for which $D_m\alpha_k(\mathbf{0}) \neq 0$. Here

$$\mathbf{F}'_m(\mathbf{0})\mathbf{e}_m = \sum_{i=m}^n (D_m\alpha_i)(\mathbf{0})\mathbf{e}_i.$$

But in that proof $\mathbf{F}_1 = \mathbf{F}$, and since in the present case $\mathbf{F}'(\mathbf{0})$ is the identity, B_1 is the identity. But then the definition of $\mathbf{G}_1(\mathbf{x})$ as

$$\mathbf{G}_1(\mathbf{x}) = \mathbf{x} + [\alpha_1(\mathbf{x}) - x_1]\mathbf{e}_1$$

implies that $\mathbf{G}'_1(\mathbf{0})$ is also the identity. Suppose we know that B_j , $\mathbf{F}'_m(\mathbf{0})$, and $\mathbf{G}'_j(\mathbf{0})$ are all equal to the identity for $j \leq m$. Then the inductive definition of \mathbf{F}_{m+1} as $\mathbf{F}_{m+1}(\mathbf{y}) = \mathbf{F}_m \circ \mathbf{G}_m^{-1}(\mathbf{y})$ implies that $\mathbf{F}'_{m+1}(\mathbf{0})$ is also the identity, from which it then follows that $\mathbf{F}'_{m+1}(\mathbf{0})$, B_{m+1} , and $\mathbf{G}'_{m+1}(\mathbf{0})$ are all equal to the identity. Thus the decomposition of \mathbf{F}_1 involves no flips, as asserted.

(b) If this map were a composition of two primitive maps, its derivative at $(0, 0)$ would be the product of two matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} = \begin{pmatrix} a+bc & bd \\ c & d \end{pmatrix}.$$

Since this matrix must be $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, it follows that $c = 1$, $d = 0$. But then the second column of the product of the two matrices is zero, which is a contradiction.

Exercise 10.4 For $(x, y) \in \mathbb{R}^2$, define

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y).$$

Prove that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$, where

$$\begin{aligned}\mathbf{G}_1(x, y) &= (e^x \cos y - 1, y) \\ \mathbf{G}_2(x, y) &= (u, (1 + u) \tan v)\end{aligned}$$

are primitive in some neighborhood of $(0, 0)$.

Compute the Jacobians of \mathbf{G}_1 , \mathbf{G}_2 , and \mathbf{F} at $(0, 0)$. Define

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$ in some neighborhood of $(0, 0)$.

Solution: The equation $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$ is a routine computation, and the fact that \mathbf{G}_1 and \mathbf{G}_2 are primitive is immediate.

The Jacobians of \mathbf{G}_1 and \mathbf{G}_2 are

$$\mathbf{G}'_1(x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ 0 & 1 \end{pmatrix}, \quad \mathbf{G}'_2(u, v) = \begin{pmatrix} 1 & 0 \\ \tan v & (1 + u) \sec^2 v \end{pmatrix},$$

so that each of them equals the identity at $(0, 0)$. It therefore follows that $\mathbf{F}'(0, 0) = I$ also.

If we take $h(u, v) = (\sqrt{e^{2u} - v^2} - 1, v)$, the primitive mapping $\mathbf{H}_1(u, v) = (h(u, v), v)$ will yield $\mathbf{H}_1 \circ \mathbf{H}_2 = \mathbf{F}$.

Exercise 10.5 Formulate and prove an analogue of Theorem 10.8, in which K is a compact subset of an arbitrary metric space. (Replace the functions φ_i that occur in the proof of Theorem 10.8 by functions of the type constructed in Exercise 22 of Chap. 4.)

Solution: We are given a compact set K in a metric space X (say with metric d) and a cover of K by open sets V_i , $i = 1, 2, \dots, n$. (We may as well assume a finite number of sets, since we can find a finite subcover of any infinite cover.)

We need to construct continuous functions ψ_i , $i = 1, 2, \dots, n$ such that $0 \leq \psi_i(x) \leq 1$ for all i and all $x \in X$, the support of $\psi_i(x)$ is contained in V_i , and $\sum_{i=1}^n \psi_i(x) = 1$ for all $x \in K$.

To do this, let $\eta > 0$ be a Lebesgue number for the covering of K by the sets V_i , that is such that the η -neighborhood of every point $x \in K$ is contained in some V_i . Let $\varepsilon \in (0, \eta)$, and let U_i be the set of points whose distance from the complement of V_i is larger than ε and W_i the set of points whose distance

from the complement of V_i is larger than $\frac{\varepsilon}{2}$. Since the distance from x to the complement of V_i is a continuous function of x , it follows that U_i and W_i are open sets. It is obvious that the closure of U_i is contained in W_i and the closure of W_i is contained in V_i . We note that $K \subset \bigcup_{i=1}^n U_i$. For if $x \in K$ there exists V_i such that the η -neighborhood of x is contained in V_i , and hence the distance from x to the complement of that V_i is at least η .

Now let A_i be the closure of U_i , and B_i the complement of W_i . Define

$$\varphi_i(x) = \frac{d(x, B_i)}{d(x, A_i) + d(x, B_i)}.$$

Then $\varphi_i(x)$ is 1 on A_i (and hence certainly on U_i) and 0 on B_i , $\varphi_i(x)$ is continuous, and $0 \leq \varphi_i(x) \leq 1$ for all x . Since the support of $\varphi_i(x)$ is the closure of W_i , it is contained in V_i . Since $\varphi_i(x) > 0$ for x in W_i , the sum $\varphi(x) = \sum_{i=1}^n \varphi_i(x)$ is positive on the open set $W = \bigcup_{i=1}^n W_i$, which contains K . Now let L be the complement of W , and define a continuous function $\psi(x)$ by

$$\psi(x) = \frac{d(x, L)}{d(x, K) + d(x, L)},$$

so that $0 \leq \psi(x) \leq 1$ for all x , $\psi(x) = 1$ if $x \in K$, and $\psi(x) = 0$ if $x \in L$. If we now define $\psi_i(x) = 0$ for $x \notin W$ and

$$\psi_i(x) = \frac{\varphi_i(x)\psi(x)}{\varphi(x)},$$

then $\psi_i(x)$ is continuous on the entire space. Its restriction to L is continuous. If we can show that its restriction to the closure of W is continuous, we shall be done. But it is obvious that it is continuous on W itself, and so we need only show that it is continuous at a point of ∂W . Hence let $x_n \rightarrow x \in \partial W$. Since $\varphi_i(x)/\varphi(x)$ is bounded, and $\psi(x_n) \rightarrow 0$, it follows that $\psi_i(x_n) \rightarrow 0 = \psi_i(x)$, and hence ψ_i is continuous at x .

The construction is now complete.

Exercise 10.6 Strengthen the conclusion of Theorem 10.8 by showing that the functions ψ_i can be made differentiable, and even infinitely differentiable. (Use Exercise 1 of Chap. 8 in the construction of the auxiliary functions φ_i .)

Solution: The function $\varphi_i(\mathbf{x})$ is required to have only three properties: 1) $\varphi_i(\mathbf{x}) = 1$ for $|\mathbf{x} - \mathbf{a}_i| \leq r_i$; 2) $\varphi_i(\mathbf{x}) = 0$ for $|\mathbf{x} - \mathbf{a}_i| \geq s_i$; 3) $0 \leq \varphi_i(\mathbf{x}) \leq 1$ for all \mathbf{x} . These properties can be achieved with an infinitely differentiable function $\varphi_i(\mathbf{x})$. To construct such a function, we go to the function $f(t)$ in Exercise 1 of Chapter 8, namely

$$f(t) = e^{-\frac{1}{t^2}}$$

for $t \neq 0$ and $f(0) = 0 = f^{(n)}(0)$ for all positive integers n , $f^{(n)}(t)$ being the n th derivative of $f(t)$. It was established in that exercise that $f(t)$ is infinitely differentiable, and it is obvious that $f(t)$ is strictly increasing for nonnegative values of t .

Let

$$g(t) = \frac{f(f(1) - f(t))}{f(f(1))}.$$

Then it is obvious that $g(t)$ is an infinitely differentiable function that decreases from 1 to 0 as x increases from 0 to 1. If we show that $g^{(n)}(0) = 0 = g^{(n)}(1)$ for all positive integers n , it will follow that the function

$$h(t) = \begin{cases} 1, & t \leq 0, \\ g(t), & 0 \leq t \leq 1, \\ 0, & 1 \leq t \end{cases}$$

is also a C^∞ function, and we can then take

$$\varphi(\mathbf{x}) = h\left(\frac{|\mathbf{x}|^2 - r_i^2}{s_i^2 - r_i^2}\right).$$

But it is easy to prove these properties by showing inductively that for all integers j and k with $0 \leq j \leq n - k$ and $1 \leq k \leq n$ there exist infinitely differentiable functions $\theta_{j,k,n}(t)$ such that

$$g^{(n)}(t) = \sum_{\substack{0 \leq j \leq n-k \\ 1 \leq k \leq n}} \theta_{j,k,n}(t) f^{(k)}(f(1) - f(t)) f^{(n-k-j+1)}(t).$$

In fact the chain rule shows that

$$\theta_{0,1,1}(t) = -\frac{1}{f(f(1))}.$$

Then, assuming there exist such functions $\theta_{j,k,n}(t)$, we find

$$\begin{aligned} g^{(n+1)}(t) = & \sum_{\substack{0 \leq j \leq n-k \\ 1 \leq k \leq n}} \left\{ \theta'_{j,k,n}(t) f^{(k)}(f(1) - f(t)) f^{(n-k-j+1)}(t) \right. \\ & + \theta_{j,k,n}(t) (-f'(t)) f^{(k+1)}(f(1) - f(t)) f^{(n-k-j+1)}(t) \\ & \left. + \theta_{j,k,n}(t) f^{(k)}(f(1) - f(t)) f^{(n-k-j+2)}(t) \right\} \end{aligned}$$

Each term in this expression contains a factor $f^{(s)}(f(1) - f(t)) f^{(n+1-s-r+1)}(t)$ with $0 \leq r \leq n + 1 - s$, $1 \leq s \leq n + 1$ and with a coefficient that is infinitely differentiable. Thus when suitably rearranged, this sum has the appropriate form

$$g^{(n+1)}(t) = \sum_{\substack{0 \leq j \leq n+1-k \\ 1 \leq k \leq n+1}} \theta_{j,k,n+1}(t) f^{(k)}(f(1) - f(t)) f^{(n-k-j+1)}(t)$$

with infinitely differentiable functions $\theta_{j,k,n+1}$. Since each term contains a factor $f^{(k)}(f(1)-f(t))f^{(l)}(t)$ with $k \geq 1$, it follows that each term vanishes when $t = 0$ or $t = 1$, and hence that $g^{(n)}(1) = 0 = g^{(n)}(0)$ for $n = 1, 2, \dots$

Exercise 10.7 (a) Show that the simplex Q^k is the smallest convex subset of R^k that contains $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$.

(b) Show that affine mappings take convex sets to convex sets.

Solution: (a) By definition $Q^k = \{\mathbf{x} : x_1 + \dots + x_k \leq 1, x_j \geq 0, j = 1, \dots, k\}$. It is obvious that Q^k contains all the points $\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_k$. It is nearly obvious that Q^k is convex. Indeed, if \mathbf{x} and \mathbf{y} are points of Q^k and $0 < t < 1$, then $t\mathbf{x} + (1-t)\mathbf{y} = \mathbf{z}$, where $z_j = tx_j + (1-t)y_j$. Since $x_j \geq 0$ and $y_j \geq 0$ and $0 < t < 1$, it is clear that $z_j \geq 0$. Simple algebra shows that $z_1 + \dots + z_k \leq t + (1-t) = 1$, so that $\mathbf{z} \in Q^k$ also. Thus Q^k is convex.

Now let C be any convex set containing these points, and let $\mathbf{x} \in Q^k$. We need to show that $\mathbf{x} \in C$. We shall show by induction that the point $x_1\mathbf{e}_1 + \dots + x_j\mathbf{e}_j$ is in C whenever $x_1 \geq 0, \dots, x_j \geq 0$ and $x_1 + \dots + x_j \leq 1$. If $j = 1$, this is obvious, since $x_1\mathbf{e}_1 = x_1\mathbf{e}_1 + (1-x_1)\mathbf{0}$ and by assumption $0 \leq x_1 \leq 1$.

Suppose the theorem is true for j , and let $c = x_1 + \dots + x_{j+1} \leq 1, x_1 \geq 0, \dots, x_{j+1} \geq 0$. If $c = 0$, the point $x_1\mathbf{e}_1 + \dots + x_{j+1}\mathbf{e}_{j+1}$ is $\mathbf{0}$, and hence belongs to C . Therefore we assume $c > 0$. Since $\mathbf{e}_{j+1} \in C$, we need only consider the case $x_{j+1} < 1$. By the induction assumption, taking $x'_l = \frac{x_l}{1-x_{j+1}}$ for $l = 1, \dots, j$, we find that the point $\mathbf{y} = x'_1\mathbf{e}_1 + \dots + x'_j\mathbf{e}_j$ belongs to C , and therefore the point $(1-x_{j+1})\mathbf{y} + x_{j+1}\mathbf{e}_{j+1} = x_1\mathbf{e}_1 + \dots + x_j\mathbf{e}_j + x_{j+1}\mathbf{e}_{j+1}$ does also.

(b) Let $\mathbf{A}(\mathbf{x})$ be an affine mapping, that is, $\mathbf{A}(\mathbf{x}) = \mathbf{x}_0 + \mathbf{T}(\mathbf{x})$, where $\mathbf{T}(\mathbf{x})$ is a linear transformation, let C be any convex set, and let $\mathbf{u} \in \mathbf{A}(C), \mathbf{v} \in \mathbf{A}(C)$. We need to show that $t\mathbf{u} + (1-t)\mathbf{v} \in \mathbf{A}(C)$ for all $t \in (0, 1)$. But this is trivial, since if $\mathbf{u} = \mathbf{A}(\mathbf{x})$ and $\mathbf{v} = \mathbf{A}(\mathbf{y})$, then $t\mathbf{u} + (1-t)\mathbf{v} = \mathbf{A}(t\mathbf{x} + (1-t)\mathbf{y})$ and $t\mathbf{x} + (1-t)\mathbf{y} \in C$.

Exercise 10.8 Let H be the parallelogram in R^2 whose vertices are $(1, 1), (3, 2), (4, 5), (2, 4)$. Find the affine map T which sends $(0, 0)$ to $(1, 1), (1, 0)$ to $(3, 2), (0, 1)$ to $(2, 4)$. Show that $J_T = 5$. Use T to convert the integral

$$\alpha = \int_H e^{x-y} dx dy$$

to an integral over I^2 and thus compute α .

Solution: Clearly the constant term in an affine mapping is the image of $(0, 0)$, which in the present case is to be $(1, 1)$. Thus we are looking for a linear transformation L such that $(3, 2) = (1, 1) + L(1, 0)$ and $(2, 4) = (1, 1) + L(0, 1)$,

which is to say $L(1,0) = (2,1)$ and $L(0,1) = (1,3)$. Obviously $L(x,y) = (2x+y, x+3y)$. Then $J_T = 2 \cdot 3 - 1 \cdot 1 = 5$. The inverse of T is given by $T^{-1}(u,v) = L^{-1}((u,v) - (1,1))$. Simple algebra then reveals that

$$T^{-1}(u,v) = \left(\frac{-2+3u-v}{5}, \frac{-1-u+2v}{5} \right).$$

The parallelogram H is the image of the unit square S under T , and so

$$\alpha = \int_{T(S)} e^{x-y} dx dy = \int_S e^{T^{-1}(u,v)} |J_{T^{-1}}| du dv.$$

Thus

$$\begin{aligned} \alpha &= \int_0^1 \int_0^1 e^{\frac{-1+4u-3v}{5}} \frac{1}{5} du dv \\ &= \frac{e^{-\frac{1}{5}}}{5} \int_0^1 e^{\frac{4u}{5}} du \int_0^1 e^{\frac{-3v}{5}} dv \\ &= e^{-\frac{1}{5}} \cdot \frac{5}{4} \cdot (e^{\frac{4}{5}} - 1) \cdot \left(\frac{-5}{3} \right) \cdot (e^{\frac{-3}{5}} - 1). \end{aligned}$$

Exercise 10.9 Define $(x,y) = T(r,\theta)$ on the rectangle

$$0 \leq r \leq a, \quad 0 \leq \theta \leq 2\pi$$

by the equations

$$x = r \cos \theta, \quad y = r \sin \theta.$$

Show that T maps this rectangle onto the closed disc D with center at $(0,0)$ and radius a , that T is one-to-one on the interior of the rectangle, and that $J_T(r,\theta) = r$. If $f \in \mathcal{C}(D)$, prove the formula for integration in polar coordinates:

$$\int_D f(x,y) dx dy = \int_0^a \int_0^{2\pi} f(T(r,\theta)) r dr d\theta.$$

Hint: Let D_0 be the interior of D , minus the interval from $(0,0)$ to $(0,a)$. As it stands, Theorem 10.9 applies to continuous functions whose support lies in D_0 . To remove this restriction, proceed as in Example 10.4.

Solution: The simple geometry of this transformation allows a fairly straightforward proof. Let $\varepsilon \in (0, \min(\pi, a/2))$. Let $H_\varepsilon = \{(r,\theta) : \varepsilon \leq r \leq a - \varepsilon, \varepsilon \leq \theta \leq 2\pi - \varepsilon\}$. The transformation T is one-to-one on H_ε . Let $\varphi_\varepsilon(x,y)$ be a continuous function on all of R^2 such that $\varphi_\varepsilon(x,y) = 1$ for $(x,y) \in T(H_\varepsilon)$, $\varphi_\varepsilon(x,y) = 0$ for $(x,y) \notin T(H_{\varepsilon/2})$ and $0 \leq \varphi_\varepsilon(x,y) \leq 1$. Define $f_\varepsilon(x,y) = f(x,y)\varphi_\varepsilon(x,y)$ for $(x,y) \in D$ and $f_\varepsilon(x,y) = 0$ for $(x,y) \notin D$. Then $f_\varepsilon(x,y) = f(x,y)$ except for $(x,y) \in D \setminus T(H_\varepsilon)$. Hence $f_\varepsilon(T(x,y)) = f(x,y)$ on $[0,a] \times [0,2\pi] \setminus H_\varepsilon$. Let

M be the maximum of $|f(x, y)|$ on D . Since the support of f_ε is contained in $T(H_{\varepsilon/2})$, which in turn is contained in D_0 , we certainly have

$$\int_{R^2} f_\varepsilon(x, y) dx dy = \int_{R^2} f_\varepsilon(r \cos \theta, r \sin \theta) r dr d\theta.$$

We need to see how much each of these integrals differs from the corresponding integral of f . We first look at $f_\varepsilon(x, y)$. In evaluating its integral we can confine ourselves to the square $-a \leq x \leq a$, $-a \leq y \leq a$, since D is contained in that square. We first exclude the three intervals $-a \leq y \leq -a + \varepsilon$, $\min(-\varepsilon, -a \sin \varepsilon) \leq y \leq \max(\varepsilon, a \sin \varepsilon)$, and $a - \varepsilon \leq y \leq a$. When y is not in these intervals, we have $\varepsilon^2 \leq y^2 \leq (a - \varepsilon)^2$, and $f(x, y)$ and $f_\varepsilon(x, y)$ can differ only on the two intervals where $\sqrt{(a - \varepsilon)^2 - y^2} \leq |x| \leq \sqrt{a^2 - y^2}$, each of which has length

$$\begin{aligned} \sqrt{a^2 - y^2} - \sqrt{(a - \varepsilon)^2 - y^2} &= \frac{(a^2 - y^2) - ((a - \varepsilon)^2 - y^2)}{\sqrt{a^2 - y^2} + \sqrt{(a - \varepsilon)^2 - y^2}} \\ &\leq \frac{2a\varepsilon}{\sqrt{a^2 - y^2}} \leq \frac{2a\varepsilon}{\sqrt{2a\varepsilon + \varepsilon^2}} \leq \frac{2a}{\sqrt{2a + \varepsilon}} \sqrt{\varepsilon} \leq \sqrt{2a\varepsilon}. \end{aligned}$$

Since the maximum possible difference between $f(x, y)$ and $f_\varepsilon(x, y)$ is M , we see that

$$\left| \int f(x, y) dx - \int f_\varepsilon(x, y) dx \right| \leq 2M\sqrt{2a\varepsilon}$$

if y is not in one of the three excluded intervals.

If y is in one of the three excluded intervals, since f and f_ε can differ by at most M , we have

$$\left| \int f(x, y) dx - \int f_\varepsilon(x, y) dx \right| \leq 2Ma.$$

Since the total length of the excluded y -intervals is at most $(2a + 3)\varepsilon$, and the total length of the interval over which y varies is at most $2a$, we see that

$$\left| \iint f(x, y) dx dy - \iint f_\varepsilon(x, y) dx dy \right| \leq 4Ma\sqrt{2a\varepsilon} + 2Ma(2a + 3)\varepsilon.$$

Thus this approximation can be made arbitrarily good by taking ε sufficiently small.

As for the integral with respect to r, θ , we observe that we can confine ourselves to the rectangle $0 \leq r \leq a$, $0 \leq \theta \leq 2\pi$, and that $f_\varepsilon(r \cos \theta, r \sin \theta) = f(r \cos \theta, r \sin \theta)$ for $\varepsilon \leq r \leq a - \varepsilon$ and $\varepsilon \leq \theta \leq 2\pi - \varepsilon$. Thus, excluding the intervals $0 \leq \theta \leq \varepsilon$, and $2\pi - \varepsilon \leq \theta \leq 2\pi$, we find that for θ not in these intervals $f(r \cos \theta, r \sin \theta)$ and $f_\varepsilon(r \cos \theta, r \sin \theta)$ can differ (by at most M) only on the two intervals $0 \leq r \leq \varepsilon$ and $a - \varepsilon \leq r \leq a$. Hence as before, if θ is not in one of these two intervals, then

$$\left| \int f(r \cos \theta, r \sin \theta) r dr - \int f_\varepsilon(r \cos \theta, r \sin \theta) r dr \right| \leq 2Ma\varepsilon.$$

On the other hand, if θ is in one of these two intervals, we have

$$\left| \int f(r \cos \theta, r \sin \theta) r dr - \int f_\varepsilon(r \cos \theta, r \sin \theta) r dr \right| \leq Ma^2.$$

Since the exceptional intervals have total length 2ε and the total length of the θ interval is 2π , we see that

$$\left| \iint f(r \cos \theta, r \sin \theta) r dr d\theta - \iint f_\varepsilon(r \cos \theta, r \sin \theta) r dr d\theta \right| \leq 4\pi Ma\varepsilon + 2Ma^2\varepsilon.$$

Hence these two integrals also can be made arbitrarily close together by choosing ε sufficiently small. Since the two integrals of f_ε are equal for each $\varepsilon > 0$, it follows that the other two are also equal.

Exercise 10.10 Let $a \rightarrow \infty$ in Exercise 9, and prove that

$$\int_{R^2} f(x, y) dx dy = \int_0^\infty \int_0^{2\pi} f(T(r, \theta)) r d\theta dr,$$

for continuous functions f that decrease sufficiently rapidly as $|x| + |y| \rightarrow \infty$. (Find a more precise formulation.) Apply this to

$$f(x, y) = \exp(-x^2 - y^2)$$

to derive formula (101) of Chap. 8.

Solution: Without striving for ultimate generality, we shall assume that there are positive numbers K and δ such that $|f(x, y)| \leq K(x^2 + y^2)^{-1-\delta}$ for all $(x, y) \neq (0, 0)$. (Such an estimate holds for (x, y) ranging over any bounded set merely because $f(x, y)$ is continuous.) Let $D_a = \{(x, y) : 0 \leq x^2 + y^2 \leq a^2\}$ and $S_a = \{(x, y) : |x| \leq a, |y| \leq a\}$. Since both D_a and S_a are convex sets, the functions $g_a(x, y) = \chi_{D_a}(x, y)f(x, y)$ and $h_a(x, y) = \chi_{S_a}(x, y)f(x, y)$ are both integrable over R^2 . We shall show that

$$\lim_{a \rightarrow \infty} \int_{R^2} g_a(x, y) dx dy = \int_{R^2} f(x, y) dx dy = \lim_{a \rightarrow \infty} \int_{R^2} h_a(x, y) dx dy.$$

Our job is simpler if we first show that

$$\lim_{a \rightarrow \infty} \left(\int_{R^2} g_a(x, y) dx dy - \int_{R^2} h_a(x, y) dx dy \right) = 0.$$

As before, we let $M = \sup\{|f(x, y)|\}$. Since $g_a(x, y) = h_a(x, y)$ except for $(x, y) \in S_a \setminus D_a$, and on this set $g_a(x, y) = 0$ and $|h_a(x, y)| \leq Ka^{-2-2\delta}$, the maximum possible difference in these two integrals is $4Ka^{-2-2\delta}$, which does indeed tend to zero as $a \rightarrow \infty$.

It now suffices to show only the second of the two equalities given above, i.e., that

$$\int_{R^2} f(x, y) dx dy = \lim_{a \rightarrow \infty} \int_{R^2} h_a(x, y) dx dy.$$

To that end, we fix y . We then have, if $|y| \geq a$, so that $h_a(x, y) = 0$,

$$\begin{aligned} \int_{-\infty}^{\infty} f(x, y) - h_a(x, y) dx &\leq K \int_{-\infty}^{\infty} \frac{1}{(x^2 + y^2)^{1+\delta}} dx \\ &\leq \int_{-\infty}^{-|y|} \frac{K}{(x^2)^{1+\delta}} dx + \int_{-|y|}^{|y|} \frac{K}{(y^2)^{1+\delta}} dx + \int_{|y|}^{\infty} \frac{K}{(x^2)^{1+\delta}} dx \\ &\leq \frac{2K|y|^{-1-2\delta}}{1+2\delta} + 2K|y|^{-1-2\delta} \leq 4K|y|^{-1-2\delta}. \end{aligned}$$

If $|y| \leq a$, we note that $f(x, y) = h_a(x, y)$ for $-a \leq x \leq a$, and so

$$\begin{aligned} \int_{-\infty}^{\infty} f(x, y) - h_a(x, y) dx &\leq \int_{-\infty}^{-a} \frac{K}{(x^2)^{1+\delta}} dx + \int_a^{\infty} \frac{K}{(x^2)^{1+\delta}} dx \\ &\leq \frac{2Ka^{-1-2\delta}}{1+2\delta} \leq 2Ka^{-1-2\delta}. \end{aligned}$$

Applying these two inequalities we find that

$$\begin{aligned} \left| \int_{R^2} f(x, y) - h_a(x, y) dx dy \right| &\leq 4K \int_{-\infty}^{-a} |y|^{-1-2\delta} dy + 4Ka^{-2\delta} + \\ &\quad + 4K \int_a^{\infty} y^{-1-2\delta} dy \leq 4K \left(1 + \frac{1}{\delta} \right) a^{-2\delta}. \end{aligned}$$

The desired formula is now proved by merely remarking that

$$\int_{R^2} h_a(x, y) dx dy = \int_0^a \int_0^{2\pi} f(r \cos \theta, r \sin \theta) r d\theta dr.$$

The fact that the limit on the right-hand side exists as $a \rightarrow \infty$ follows from the fact that the limit on the left-hand side does, but can also be proved directly, since $|f(r \cos \theta, r \sin \theta)r| \leq Kr^{-1-2\delta}$.

Applying this formula with $f(x, y) = e^{-x^2-y^2}$, we find that

$$\begin{aligned} \left(\int_{-\infty}^{\infty} e^{-t^2} dt \right)^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2} e^{-y^2} dx dy = \int_0^{\infty} \int_0^{2\pi} e^{-r^2} r d\theta dr = \\ &= 2\pi \int_0^{\infty} e^{-r^2} r dr = \pi \int_0^{\infty} e^{-u} du = \pi. \end{aligned}$$

In other words,

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}.$$

Exercise 10.11 Define $(u, v) = T(s, t)$ on the strip

$$0 < s < \infty, \quad 0 < t < 1$$

by setting $u = s - st$, $v = st$. Show that T is a 1-1 mapping of the strip onto the positive quadrant Q in R^2 . Show that $J_T(s, t) = s$.

For $x > 0$, $y > 0$ integrate

$$u^{x-1}e^{-u}v^{y-1}e^{-v}$$

over Q , use Theorem 10.9 to convert the integral to one over the strip, and derive formula (96) of Chap. 8 in this way.

(For this application, Theorem 10.9 has to be extended so as to cover certain improper integrals. Provide this extension.)

Solution: It is easy to compute the inverse of T , namely

$$s = u + v, \quad t = \frac{v}{u + v},$$

and this inverse is defined on the entire (u, v) -plane with the line $v = -u$ removed. It is obvious that v is positive if and only if s and t have the same sign, and that u is positive if and only if s and $1 - t$ have the same sign.

Thus if u and v are both positive, then t and $1 - t$ have the same sign, which happens if and only if $0 < t < 1$. In this case s must also be positive. Conversely, the equations that give s and t show that if u and v are both positive, then $s > 0$ and $0 < t < 1$. The Jacobian matrix of T is

$$\begin{pmatrix} 1-t & -s \\ t & s \end{pmatrix},$$

so that $J_T(s, t) = s$.

The integral of $u^{x-1}e^{-u}v^{y-1}e^{-v}$ over the quadrant is

$$\int_0^\infty u^{x-1}e^{-u} du \int_0^\infty v^{y-1}e^{-v} dv = \Gamma(x)\Gamma(y).$$

According to Theorem 10.9

$$\int_0^\infty \int_0^1 f(s - st, st) s dt ds = \int_0^\infty \int_0^\infty f(u, v) du dv$$

for any function $f(u, v)$ having compact support contained in the open quadrant. Assuming this theorem remains valid for the particular function we have in mind, we get

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^\infty s^{x+y-1}e^{-s} ds \int_0^1 t^{y-1}(1-t)^{x-1} dt = \\ &= \Gamma(x+y) \int_0^1 t^{x-1}(1-t)^{y-1} dt, \end{aligned}$$

which is indeed formula (96) of Chapter 8.

Thus we need only justify the use of Theorem 10.9 with the function f in the unbounded regions. To do this, we first show that Theorem 10.9 applies to the function $f(u, v)\varphi_\delta(u, v)$, where $\varphi_\delta(u, v)$ is the characteristic function of the set

$$E_\delta = \{(u, v) : \delta \leq u \leq \delta^{-1}, \delta \leq v \leq \delta^{-1}\}.$$

Since this function is positive on E_δ , it is easy to modify it and make it into a continuous nonnegative function f_η that vanishes outside the set $E_{\delta-\eta}$, for $\eta < \delta$ and this can be done without increasing its maximal value. Theorem 10.9 applies to f_η , and it is easy to see that the integral of f_η on both sides of the formula tends to the integral of $f\varphi_\delta$ as $\eta \rightarrow \delta$. (Indeed, there is a constant ε such that $T^{-1}(u, v) = (s, t)$ lies in the strip $\varepsilon \leq t \leq 1 - \varepsilon$ whenever $(u, v) \in E_{\delta-\eta}$ and $\eta \leq \frac{\delta}{2}$. In that case, for each fixed t , the distance between the rightmost points (s, t) in $T^{-1}(E_{\delta-\eta})$ and in $T^{-1}(E_\delta)$ is at most $\frac{\delta-\eta}{\varepsilon}$. A similar statement applies to the leftmost points in the two regions, showing that the usual argument applies: The integrals of f and $f\varphi_\delta$ over each horizontal line differ by at most $\frac{2M(\delta-\eta)}{\varepsilon}$, except for a small range of t whose length tends to zero with $\delta - \eta$, on which the difference is bounded. It then follows that both of the integrals of f_η tend toward the corresponding integrals of $f\varphi_\delta$.

It then remains only to prove that the integral of $f\varphi_\delta$ tends to the integral of f on both sides of the formula. Since these integrals increase as δ decreases, there is no question that the limit exists, and we need only show that in both cases the limit is the integral in the formula. This is nearly immediate in the case of the integral over the quadrant. As for the integral over the strip, the set $T^{-1}(E_\delta)$ contains the region $\delta^{1/2} \leq s \leq \frac{1}{\delta - \delta^{3/2}}, \delta^{1/2} \leq t \leq 1 - \delta^{1/2}$. For these inequalities imply that $\delta \leq st \leq \frac{1}{\delta}$, and since $1 - t$ satisfies the same inequalities as t , we also have $\delta \leq s(1 - t) \leq \frac{1}{\delta}$. The integral of $f(s - st, st)s$ over the two strips $0 \leq t \leq \sqrt{\delta}$ and $1 - \sqrt{\delta} \leq t \leq 1$ tends to zero with δ , and for each t with $\sqrt{\delta} \leq t \leq 1 - \sqrt{\delta}$ the integral

$$\int_{\sqrt{\delta}}^{1/(\delta - \delta^{3/2})} f(s - st, st)s \, ds$$

differs from the integral from 0 to ∞ by less than

$$t^{1-x}(1-t)^{1-y} \left(\int_0^{\sqrt{\delta}} s^{x+y-1} \, ds + \int_{1/(\delta - \delta^{3/2})}^{\infty} s^{x+y-1} e^{-s} \, ds \right).$$

The first of these integrals is explicitly calculable and tends to zero as $\delta \rightarrow 0$. In the second we use the fact that $e^{-s} < \frac{n!}{s^n}$ for all $s > 0$ and take $n \geq x + y + 1$. It then follows that the integral of $f(s - st, st)\varphi_\delta(s - st, st)s$ over each of these horizontal line differs from the integral of $f(s - st, st)s$ by an amount that tends to zero uniformly for $\sqrt{\delta} \leq t \leq 1 - \sqrt{\delta}$.

The proof is now complete.

Exercise 10.12 Let I^k be the set of all $\mathbf{u} = (u_1, \dots, u_k) \in R^k$ with $0 \leq u_i \leq 1$ for all i ; let Q^k be the set of all $\mathbf{x} = (x_1, \dots, x_k) \in R^k$ with $x_i \geq 0$, $\sum x_k \leq 1$. (I^k is the unit cube; Q^k is the standard simplex in R^k .) Define $\mathbf{x} = T(\mathbf{u})$ by

$$\begin{aligned} x_1 &= u_1 \\ x_2 &= (1 - u_1)u_2 \\ &\dots\dots\dots \\ x_k &= (1 - u_1) \cdots (1 - u_{k-1})u_k. \end{aligned}$$

Show that

$$\sum_{i=1}^k x_i = 1 - \prod_{i=1}^k (1 - u_i).$$

Show that T maps I^k onto Q^k , that T is 1-1 in the interior of I^k , and that its inverse S is defined in the interior of Q^k by $u_1 = x_1$ and

$$u_i = \frac{x_i}{1 - x_1 - \cdots - x_{i-1}}$$

for $i = 2, \dots, k$. Show that

$$J_T(\mathbf{u}) = (1 - u_1)^{k-1} (1 - u_2)^{k-2} \cdots (1 - u_{k-1}),$$

and

$$J_S(\mathbf{x}) = [(1 - x_1)(1 - x_1 - x_2) \cdots (1 - x_1 - \cdots - x_{k-1})]^{-1}.$$

Solution: The first identity is easily proved by induction on k . It is obvious for $k = 1$, and

$$\begin{aligned} \sum_{i=1}^{k+1} x_i &= x_{k+1} + \sum_{i=1}^k x_i \\ &= (1 - u_1) \cdots (1 - u_k)u_{k+1} + 1 - (1 - u_1) \cdots (1 - u_k) \\ &= 1 - (1 - u_1) \cdots (1 - u_k)(1 - u_{k+1}). \end{aligned}$$

The defining formulas and the formula just proved show that $\mathbf{x} \in Q^k$ whenever $\mathbf{u} \in I^k$. In the process of showing that T is onto, we shall prove the inverse formula. Let $\mathbf{x} \in Q^k$, and assume for the moment that $\sum_{i=1}^{k-1} x_i < 1$. Then all of the equations given as inverse equations are defined. We need only show that the defining equations yield \mathbf{x} when applied to the left-hand sides of these equations. Certainly we do have $x_1 = u_1$. Suppose that $x_r = (1 - u_1) \cdots (1 - u_{r-1})u_r$ for $r < j$. For the moment assume $u_r \neq 0$.

$$\begin{aligned} (1 - u_1) \cdots (1 - u_{r-1})(1 - u_r)u_{r+1} &= x_r \left(1 - \frac{1}{u_r}\right)u_{r+1} = \\ &= x_r \cdot \frac{1 - x_1 - \cdots - x_r}{x_r} \cdot \frac{x_{r+1}}{1 - x_1 - \cdots - x_r} = x_{r+1}. \end{aligned}$$

If $u_l \neq 0$, but $u_j = 0$ for $l < j \leq r$, then $x_j = 0$ also for these values, and $u_{r+1} = \frac{x_{r+1}}{1-x_1-\dots-x_l}$. We then have

$$\begin{aligned} x_{r+1} &= (1-u_1)\cdots(1-u_l)u_{r+1} \\ &= x_l\left(1-\frac{1}{u_l}\right)u_{r+1} \\ &= x_l \cdot \frac{1-x_1-\dots-x_l}{x_l} \cdot \frac{x_{r+1}}{1-x_1-\dots-x_l} \\ &= x_{r+1}. \end{aligned}$$

Finally, if $u_1 = u_2 = \dots = u_r = 0$, we have simply $u_{r+1} = x_{r+1}$ in both sets of equations. Thus in all cases the point $\mathbf{u} \in I^k$ is a preimage of the point $\mathbf{x} \in Q^k$.

It remains only to consider the case when $\sum_{i=1}^r x_i = 1$ for some $r < k$. For these points $x_{r+1} = \dots = x_k = 0$.

To find preimages of these points, let r be the first index for which $\sum_{i=1}^r x_i = 1$.

If $r = 1$, we have $x_2 = \dots = x_k = 0$, and this point is its own preimage. In general the preimage of the point \mathbf{x} for which $x_{r+1} = \dots = x_k = 0$ is \mathbf{u} , where u_1, \dots, u_r are given by the formulas for S . The formulas imply $u_r = 1$. The values of u_{r+1}, \dots, u_k are then arbitrary, since the formulas that define T will automatically make the remaining x_i equal to zero.

The Jacobian matrix is a triangular matrix whose diagonal consists of the entries $1, (1-u_1), (1-u_1)(1-u_2), \dots, (1-u_1)\cdots(1-u_{k-1})$, and this fact yields the formula for $J_T(\mathbf{u})$ immediately.

Likewise, the Jacobian of S is triangular and has diagonal entries $1, \frac{1}{1-x_1}, \frac{1}{1-x_1-x_2}, \dots, \frac{1}{1-x_1-x_2-\dots-x_{k-1}}$, from which again the formula for $J_S(\mathbf{x})$ is immediate.

Exercise 10.13 Let r_1, \dots, r_k be nonnegative integers, and prove that

$$\int_{Q^k} x_1^{r_1} \cdots x_k^{r_k} dx = \frac{r_1! \cdots r_k!}{(k+r_1+\dots+r_k)!}.$$

Hint: Use Exercise 12, Theorems 10.9 and 8.20.

Note that the special case $r_1 = \dots = r_k = 0$ shows that the volume of Q^k is $1/k!$.

Solution: Following the hint, we rewrite the integral in terms of \mathbf{u} , getting

$$\begin{aligned} \int_{I^k} u_1^{r_1} \cdots u_k^{r_k} (1-u_1)^{r_2+\dots+r_k} (1-u_2)^{r_3+\dots+r_k} \cdots \\ (1-u_{k-1})^{r_k} (1-u_1)^{k-1} (1-u_2)^{k-2} \cdots (1-u_{k-1}) du_1 \cdots du_k. \end{aligned}$$

This integral is the product

$$\prod_{i=1}^k \int_0^1 u_i^{r_i} (1-u_i)^{k-i+r_{i+1}+\dots+r_k} du_i,$$

which by formula (96) of Chapter 8 (just proved in Exercise 11 above) equals the product

$$\prod_{i=1}^k \frac{\Gamma(r_i + 1)\Gamma(k + 1 - i + r_{i+1} \cdots + r_k)}{\Gamma(k + 2 - i + r_i + r_{i+1} \cdots + r_k)}.$$

When this product is evaluated, the numerator $\Gamma(k + 1 - i + r_{i+1} \cdots + r_k)$ in each factor cancels the denominator $\Gamma(k + 2 - (i + 1) + r_{i+1} \cdots + r_k)$ in the next factor. Thus the product “telescopes” to the product of the factors $\Gamma(r_i + 1)$ in the numerators divided by the first denominator $\Gamma(k + 1 + r_1 + \cdots + r_k)$. Considering that $\Gamma(n + 1) = n!$ for integers n , we therefore get the required formula.

Theoretically we ought to be worried about the fact that T is not 1-1 on the entire cube I^k . This problem, however, is handled by the same reasoning used in Exercises 9, 10, and 11, and need not be repeated.

Exercise 10.14 Prove formula (46).

Solution: Formula (46) asserts that $\prod_{p < q} \operatorname{sgn}(j_q - j_p)$ is -1 if the permutation j_1, \dots, j_k is odd and 1 if the permutation is even. We observe that this product is $(-1)^k$, where k is the number of pairs (j_p, j_q) for which $j_p > j_q$. Since $\operatorname{sgn}(j_q - j_p) = 1$ if $j_p < j_q$ and $\operatorname{sgn}(j_q - j_p) = -1$ if $j_p > j_q$, we need to show that the parity of k is the same as the parity of the number of interchanges that will be used in converting this permutation to the identity. (As a corollary, that parity will be the same, no matter what particular sequence of interchanges is used to get to the identity.) This equality is obvious if the permutation is the identity to begin with. Suppose then that $j_m > j_n$ and $m < n$. The elements j_i , $m < i < n$ are of three kinds: Set A , those for which $j_i < j_n$; set B , those for which $j_n < j_i < j_m$; and set C , those for which $j_m < j_i$. Before j_m and j_n are interchanged, there is one out-of-order pair (j_m, j_i) for each $j_i \in A$, one out-of-order pair (j_i, j_n) for each $j_i \in C$, and two out-of-order pairs (j_m, j_i) and (j_i, j_n) for each $j_i \in B$. After the switch there is one out-of-order pair (j_n, j_i) for each $j_i \in A$, and one pair (j_i, j_m) for each $j_i \in C$. There are no pairs involving any $j_i \in B$. Hence, when an out-of-order pair (j_m, j_n) is put in the right order by interchanging its elements, the number of out-of-order pairs decreases by $2|B| + 1$, where B is the set of elements j_i between j_m and j_n that are in the wrong order relative to both j_m and j_n and $|B|$ is the number of elements in B .

Of course the number would *increase* by an odd number if we foolishly interchanged a pair that were not out-of-order relative to each other. (The number would increase by $2|B| + 1$, where $|B|$ is the number of elements between them that were in the correct order relative to both elements of the interchanged pair.) In any case, each interchange of two elements changes the number of inversions (out-of-order pairs) by an odd number, so that an odd number of interchanges, starting from the identity, will result in an odd number of inversions, and an even number of interchanges will result in an even number of inversions.

Exercise 10.15 If ω and λ are k - and m -forms, respectively, prove that

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega$$

Solution: Because of the associative and distributive laws, it suffices to prove this in the case when $\omega = f dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ and $\lambda = g dx_{i_{k+1}} \wedge \cdots \wedge dx_{i_{k+m}}$. In that case

$$\omega \wedge \lambda = fg dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{i_{k+1}} \wedge \cdots \wedge dx_{i_{k+m}}.$$

For each $j = 1, 2, \dots, k$ exactly m interchanges of adjacent basic one-forms will move $dx_{i_{k+1-j}}$ to the position just right of $dx_{i_{k+m}}$, if these moves are made in increasing order of j . Thus a total of km interchanges will exactly reverse λ and ω . The result now follows from the alternating property of the wedge product on basis elements.

Exercise 10.16 If $k \geq 2$ and $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_k]$ is an oriented affine k -simplex, prove that $\partial^2 \sigma = 0$, directly from the definition of the boundary operator ∂ . Deduce from this that $\partial^2 \Psi = 0$ for every chain Ψ .

Hint: For orientation, do it first for $k = 2$, $k = 3$. In general, if $i < j$, let σ_{ij} be the $(k-2)$ -simplex obtained by deleting \mathbf{p}_i and \mathbf{p}_j from σ . Show that each σ_{ij} occurs twice in $\partial^2 \sigma$ with opposite sign.

Solution. For $k = 2$ we have

$$\partial \sigma = [\mathbf{p}_1, \mathbf{p}_2] - [\mathbf{p}_0, \mathbf{p}_2] + [\mathbf{p}_0, \mathbf{p}_1],$$

so that

$$\partial^2 \sigma = (\mathbf{p}_2 - \mathbf{p}_1) - (\mathbf{p}_2 - \mathbf{p}_0) + (\mathbf{p}_1 - \mathbf{p}_0) = 0.$$

For $k = 3$ we have

$$\partial \sigma = [\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3] - [\mathbf{p}_0, \mathbf{p}_2, \mathbf{p}_3] + [\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_3] - [\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2],$$

so that

$$\begin{aligned} \partial^2 \sigma &= ([\mathbf{p}_2, \mathbf{p}_3] - [\mathbf{p}_1, \mathbf{p}_3] + [\mathbf{p}_1, \mathbf{p}_2]) \\ &\quad - ([\mathbf{p}_2, \mathbf{p}_3] - [\mathbf{p}_0, \mathbf{p}_3] + [\mathbf{p}_0, \mathbf{p}_2]) \\ &\quad + ([\mathbf{p}_1, \mathbf{p}_3] - [\mathbf{p}_0, \mathbf{p}_3] + [\mathbf{p}_0, \mathbf{p}_1]) \\ &\quad - ([\mathbf{p}_1, \mathbf{p}_2] - [\mathbf{p}_0, \mathbf{p}_2] + [\mathbf{p}_0, \mathbf{p}_1]) \\ &= 0. \end{aligned}$$

In general the order in which \mathbf{p}_i and \mathbf{p}_j are omitted from σ determines the sign that σ_{ij} will have. If \mathbf{p}_j is omitted first, the resulting $(k-1)$ -simplex $\sigma_j = [\mathbf{p}_0, \dots, \mathbf{p}_{j-1}, \mathbf{p}_{j+1}, \dots, \mathbf{p}_k]$ will acquire the sign $(-1)^j$. If \mathbf{p}_i is then omitted, the resulting $(k-2)$ -simplex will acquire a factor of $(-1)^i$, resulting in $(-1)^{i+j} \sigma_{ij}$.

However, if \mathbf{p}_i is omitted first, \mathbf{p}_j will move forward one position in the resulting $(k-1)$ -simplex σ_i , and when it is subsequently omitted, a factor of

$(-1)^{j-1}$ will be affixed, resulting in $(-1)^{i+j-1}\sigma_{ij}$. Hence the two occurrences of σ_{ij} in the second boundary will cancel each other.

The linearity of the boundary operator, operating on a base of simplexes, then shows that ∂^2 is the zero operator on all chains.

Exercise 10.17 Put $J^2 = \tau_1 + \tau_2$, where

$$\tau_1 = [0, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2], \quad \tau_2 = -[0, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1].$$

Explain why it is reasonable to call J^2 the positively oriented unit square in R^2 . Show that ∂J^2 is the sum of 4 oriented affine simplexes. Find these. What is $\partial(\tau_1 - \tau_2)$?

Solution: Although J^2 is really a collection of two affine mappings, the *ranges* of these mappings cover the unit square, the diagonal from $(0, 0)$ to $(1, 1)$ being covered twice with opposite orientations in the two mappings. In both cases, the sense of orientation is such that the cross product of the last two vertices of the simplex is \mathbf{e}_3 , which is a reasonable definition of the positive orientation on the unit square.

By routine computation,

$$\begin{aligned} \partial J^2 &= ([\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] \\ &\quad - [0, \mathbf{e}_1 + \mathbf{e}_2] + [0, \mathbf{e}_1]) - ([\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] - [0, \mathbf{e}_1 + \mathbf{e}_2] + [0, \mathbf{e}_2]) \\ &= [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] + [\mathbf{e}_2, 0] + [0, \mathbf{e}_1]. \end{aligned}$$

Again, by routine computation,

$$\begin{aligned} \partial(\tau_1 - \tau_2) &= ([\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [0, \mathbf{e}_1 + \mathbf{e}_2] + [0, \mathbf{e}_1]) \\ &\quad + ([\mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2] - [0, \mathbf{e}_1 + \mathbf{e}_2] + [0, \mathbf{e}_2]) \\ &= [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_2] - [\mathbf{e}_2, 0] + [0, \mathbf{e}_1] - 2[0, \mathbf{e}_1 + \mathbf{e}_2]. \end{aligned}$$

Exercise 10.18 Consider the oriented affine 3-simplex

$$\sigma_1 = [0, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]$$

in R^3 . Show that σ_1 (regarded as a linear transformation) has determinant 1. Thus σ_1 is positively oriented.

Let $\sigma_2, \dots, \sigma_6$ be five other oriented simplexes, obtained as follows: There are five permutations (i_1, i_2, i_3) of $(1, 2, 3)$ distinct from $(1, 2, 3)$. Associate with each (i_1, i_2, i_3) the simplex

$$s(i_1, i_2, i_3)[0, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$$

where s is the sign that occurs in the definition of the determinant. (This is how τ_2 was obtained from τ_1 in Exercise 17.)

Show that $\sigma_2, \dots, \sigma_6$ are positively oriented.

Put $J^3 = \sigma_1 + \dots + \sigma_6$. Then J^3 may be called the positively oriented unit cube in R^3 .

Show that ∂J^3 is the sum of 12 oriented affine 2-simplexes. (These 12 triangles cover the surface of the unit cube I_3 .)

Show that $\mathbf{x} = (x_1, x_2, x_3)$ is in the range of σ_1 if and only if $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$.

Show that the ranges of $\sigma_1, \dots, \sigma_6$ have disjoint interiors, and that their union covers I^3 . (Compare with Exercise 13; note that $3! = 6$.)

Solution. We first show that each of these simplexes is positively oriented. To that end, it is convenient to refer to the simplex $[0, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$ corresponding to the permutation (i_1, i_2, i_3) as $\sigma^{(i_1, i_2, i_3)}$.

The simplex $\sigma^{(i_1, i_2, i_3)}$, regarded as a linear transformation, maps (x, y, z) to $(x+y+z)\mathbf{e}_{i_1} + (y+z)\mathbf{e}_{i_2} + z\mathbf{e}_{i_3}$. Its matrix therefore has $(1 \ 1 \ 1)$ as row i_1 , $(0 \ 1 \ 1)$ as row i_2 , and $(0 \ 0 \ 1)$ as row i_3 . By interchanging rows in correspondence with the interchanges needed to convert the permutation (i_1, i_2, i_3) to the identity, we can convert this matrix to an upper-triangular matrix with 1's on the main diagonal. The determinant of the matrix is therefore $s(i_1, i_2, i_3)$, so that the simplex $s(i_1, i_2, i_3)[0, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$ is positively oriented.

The boundary of $\sigma^{(i_1, i_2, i_3)}$ consists of four terms, two of which $([\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$ and $-[0, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}])$ are not shared with any other $\sigma^{(i)}$. The other two terms $(-[0, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$ and $[0, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$ are shared with $\sigma^{(i_1, i_3, i_2)}$ and $\sigma^{(i_2, i_1, i_3)}$ respectively. As these two permutations each differ from (i_1, i_2, i_3) by a single interchange, the sign of each of these terms will be opposite in its two occurrences, and hence they will cancel out. Thus the boundary of J^3 will consist of a total of 12 oriented affine 2-simplexes.

A point $\mathbf{x} = (x_1, x_2, x_3)$ is in the range of σ_1 if and only if there are numbers $r, s, t \in [0, 1]$ such that $r + s + t \leq 1$ and $\mathbf{x} = r\mathbf{e}_1 + s(\mathbf{e}_1 + \mathbf{e}_2) + t(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$, that is, $x_1 = r + s + t$, $x_2 = s + t$, and $x_3 = t$. If such numbers r, s, t exist, obviously $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$. Conversely, if these conditions hold, there will be such numbers r, s, t , namely $t = x_3$, $s = x_2 - x_3$, and $r = x_1 - x_2$.

The interior of the range of $\sigma^{(i_1, i_2, i_3)}$ is the set of all $\mathbf{x} = (x_1, x_2, x_3)$ such that $0 < x_{i_3} < x_{i_2} < x_{i_1} < 1$. For the range of this simplex is the set of \mathbf{x} for which each of these inequalities or the corresponding equality holds. If equality holds in any of them, the point can be approached by points outside the range, as one can easily see. That the union covers I^3 is also obvious. Indeed, the characterization of the range of σ_1 applies to all $\sigma^{(i_1, i_2, i_3)}$ and shows that this range is contained in I^3 . Thus we need only show the reverse inclusion.

If $\mathbf{x} = (x_1, x_2, x_3) \in I^3$, let i_1 be the smallest subscript i for which $x_i = \max\{x_1, x_2, x_3\}$. Let i_2 be the first subscript for which $x_i = \max(\{x_1, x_2, x_3\} \setminus \{x_{i_1}\})$. Finally, let i_3 be such that $\{x_{i_3}\} = \{x_1, x_2, x_3\} \setminus \{x_{i_1}, x_{i_2}\}$. By construction $0 \leq x_{i_3} \leq x_{i_2} \leq x_{i_1} \leq 1$, and so, by the argument given above, \mathbf{x} belongs to the range of $\sigma^{(i_1, i_2, i_3)}$. Symmetry shows that all of these simplexes have the same volume, which must therefore be $1/6$. (Remember that we showed back in Exercise 1 that the boundary of a convex set in R^k has k -dimensional volume 0,

so that the volume of each of these sets equals the volume of its interior. As the interiors are disjoint, the sum of their volumes is at most 1. Since the simplexes together cover I^3 , the sum of their volumes is at least 1. Therefore it is exactly 1.)

Exercise 10.19 Let J^2 and J^3 be as in Exercise 17 and 18. Define

$$\begin{aligned} B_{01}(u, v) &= (0, u, v), & B_{11}(u, v) &= (1, u, v), \\ B_{02}(u, v) &= (u, 0, v), & B_{12}(u, v) &= (u, 1, v), \\ B_{03}(u, v) &= (u, v, 0), & B_{13}(u, v) &= (u, v, 1). \end{aligned}$$

These are affine and map R^2 into R^3 .

Put $\beta_{ri} = B_{ri}(J^2)$, for $r = 0, 1$, $i = 1, 2, 3$. Each β_{ri} is an affine-oriented 2-chain. (See Sec. 10.30.) Verify that

$$\partial J^3 = \sum_{i=1}^3 (-1)^i (\beta_{0i} - \beta_{1i}),$$

in agreement with Exercise 18.

Solution. Although we did not spell it out in our solution of Exercise 18, the boundary of J^3 is the 2-chain

$$\sum_{i_1, i_2, i_3} s(i_1, i_2, i_3) ([e_{i_1}, e_{i_1} + e_{i_2}, e_{i_1} + e_{i_2} + e_{i_3}] - [0, e_{i_1}, e_{i_1} + e_{i_2}]).$$

This sum can be rearranged as a sum of three terms, each of which consists of four terms. For example, the terms in the sum for which $i_1=1$ can be written as

$$\begin{aligned} &([e_1, e_1 + e_2, e_1 + e_2 + e_3] \\ &\quad - [e_1, e_1 + e_3, e_1 + e_2 + e_3]) - ([0, e_1, e_1 + e_2] - [0, e_1, e_1 + e_3]). \end{aligned}$$

For $i_1 = 2$ we get a similar set of four terms, namely,

$$\begin{aligned} &(-[e_2, e_2 + e_1, e_1 + e_2 + e_3] + [e_2, e_2 + e_3, e_1 + e_2 + e_3]) \\ &\quad + ([0, e_2, e_1 + e_2] - [0, e_2, e_2 + e_3]). \end{aligned}$$

Finally, for $i_1 = 3$ we have

$$\begin{aligned} &([e_3, e_3 + e_1, e_1 + e_2 + e_3] - [e_3, e_3 + e_2, e_1 + e_2 + e_3]) \\ &\quad - ([0, e_3, e_3 + e_1] - [0, e_3, e_3 + e_2]). \end{aligned}$$

Now consider the 2-chain β_{01} . According to the notation of Eq. (88), it is $B_{01}(\tau_1) + B_{01}(\tau_2)$. Letting $(u, v) = \tau_1(x, y) = (x + y)e_1 + ye_2$, and then $(u, v) = \tau_2(x, y) = (x + y)e_2 + ye_1$ (and keeping in mind the orientation assigned to τ_2), we see that $\beta_{01}(x, y) = B_{01}(x + y, y) - B_{01}(y, x + y) = (0, x + y, y) -$

$(0, y, x + y) = [0, e_2, e_2 + e_3] - [0, e_3, e_3 + e_2]$. Notice that these two terms occur in the expression for ∂J^3 , in the groupings for $i_1 = 2$ and $i_1 = 3$ respectively, but each occurs with the opposite sign. Hence these terms can be accounted for in ∂J^3 by being grouped together and written as $-\beta_{01}$. Similarly when we look at β_{11} , we find that it is the 2-chain whose points are $(1, x + y, y) - (1, y, x + y)$, which is $[e_1, e_1 + e_2, e_1 + e_2 + e_3] - [e_1, e_1 + e_3, e_1 + e_2 + e_3]$. Again these terms occur in the expression for e_1 , this time with exactly the same signs, so that they can be accounted for by grouping them and writing them as the term β_{11} . Thus four of the twelve simplexes in ∂J^3 are accounted for by the expression $(-1)^1(\beta_{01} - \beta_{11})$. The other 8 simplexes are accounted for similarly.

Exercise 10.20 State conditions under which the formula

$$\int_{\Phi} f d\omega = \int_{\partial\Phi} f\omega - \int_{\Phi} (df) \wedge \omega$$

is valid, and show that it generalizes the formula for integration by parts.

Hint: $d(f\omega) = (df) \wedge \omega + f d\omega$.

Solution. Given the formula in the hint, we need only invoke Stokes' Theorem. For any chain Φ satisfying the hypotheses of that theorem we shall have

$$\int_{\Phi} d(f\omega) = \int_{\partial\Phi} f\omega,$$

which is precisely the given theorem. The ordinary formula for integration by parts follows by considering a 0-form fg .

Exercise 10.21 As in Example 10.36, consider the 1-form

$$\eta = \frac{x dy - y dx}{x^2 + y^2}$$

in $R^2 \setminus \{0\}$.

(a) Carry out the computation that leads to formula (113), and prove that $d\eta = 0$.

(b) Let $\gamma(t) = (r \cos t, r \sin t)$, for some $r > 0$, and let Γ be a C'' -curve in $R^2 \setminus \{0\}$, with parameter interval $[0, 2\pi]$, with $\Gamma(0) = \Gamma(2\pi)$, such that the intervals $[\gamma(t), \Gamma(t)]$ do not contain 0 for any $t \in [0, 2\pi]$. Prove that

$$\int_{\Gamma} \eta = 2\pi.$$

Hint: For $0 \leq t \leq 2\pi$, $0 \leq u \leq 1$, define

$$\Phi(t, u) = (1 - u)\Gamma(t) + u\gamma(t).$$

then Φ is a 2-surface in $R^2 \setminus \{0\}$ whose parameter domain is the indicated rectangle. Because of cancellations (as in Example 10.32),

$$\partial\Phi = \Gamma - \gamma.$$

Use Stokes' theorem to deduce that

$$\int_{\Gamma} \eta = \int_{\gamma} \eta$$

because $d\eta = 0$.

(c) Take $\Gamma(t) = (a \cos t, b \sin t)$ where $a > 0$, $b > 0$ are fixed. Use part (b) to show that

$$\int_0^{2\pi} \frac{ab}{a^2 \cos^2 t + b^2 \sin^2 t} dt = 2\pi.$$

(d) Show that

$$\eta = d\left(\arctan \frac{y}{x}\right)$$

in any convex open set in which $x \neq 0$, and that

$$\eta = d\left(-\arctan \frac{x}{y}\right)$$

in any convex open set in which $y \neq 0$.

Explain why this justifies the notation $\eta = d\theta$, in spite of the fact that η is not exact in $R^2 \setminus \{0\}$.

(e) Show that (b) can be derived from (d).

(f) If Γ is any closed C' -curve in $R^2 \setminus \{0\}$, prove that

$$\frac{1}{2\pi} \int_{\Gamma} \eta = \text{Ind}(\Gamma).$$

(See Exercise 23 of Chap. 8 for the definition of the index of a curve.)

Solution. (a) By the rules for computing line integrals, given that $x = r \cos t$ and $y = r \sin t$,

$$\int_{\gamma} \eta = \int_0^{2\pi} \frac{(r \cos t)(r \cos t) dt - (r \sin t)(-r \sin t) dt}{r^2 \cos^2 t + r^2 \sin^2 t} = \int_0^{2\pi} dt = 2\pi.$$

(b) Let $\Gamma(t) = (X(t), Y(t))$ and $\gamma(t) = (x(t), y(t))$. Following the hint, observing that the hypothesis that the interval from $\Gamma(t)$ to $\gamma(t)$ does not pass through 0 , we find that $\Phi(t, u)$ is indeed a 2-surface in $R^2 \setminus \{0\}$, and making it into a singular 2-chain by regarding the domain as an affine 2-chain, as in Exercise 17, we find by Stokes' theorem that

$$\begin{aligned} 0 &= \int_{\Phi} d\eta = \int_{\partial\Phi} \eta \\ &= - \int_{\Gamma} \eta + \int_{\gamma} \eta + \int_{\delta} \eta - \int_{\epsilon} \eta, \end{aligned}$$

where δ is the curve $\delta(u) = \Phi(2\pi, u) = (1-u)\Gamma(2\pi) + u\gamma(2\pi)$ and ε is the curve $\varepsilon(u) = \Phi(0, u) = (1-u)\Gamma(0) + u\gamma(0)$. Since δ and ε are the same curve, the last two terms in this expression cancel each other, yielding the required result.

(c) We need only verify that $\Phi(t, u) \neq \mathbf{0} = (0, 0)$. But this is clear: If $((1-u)a + ur)\cos t = 0$, then $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$, since $(1-u)a + ur \geq \min(a, r) > 0$. But this means that $((1-u)b + ur)\sin t \neq 0$, since t is not a multiple of π . The result now follows.

(d) It is a routine computation that the differential of $\arctan \frac{y}{x}$ is η in the entire right or left half-plane, and similarly for $\pi - \arctan \frac{x}{y}$, which is after all $\operatorname{arccot} \frac{x}{y}$, which in turn is $\arctan \frac{y}{x}$ wherever both functions are defined. Thus *locally* we have $\eta = d\theta$, even though θ is not defined *globally* in $R^2 \setminus \{\mathbf{0}\}$.

(e) Break the integral over γ into five parts: $0 \leq t \leq \frac{\pi}{4}$, $\frac{\pi}{4} \leq t \leq \frac{3\pi}{4}$, $\frac{3\pi}{4} \leq t \leq \frac{5\pi}{4}$, $\frac{5\pi}{4} \leq t \leq \frac{7\pi}{4}$, $\frac{7\pi}{4} \leq t \leq 2\pi$. In the first, third, and fifth parts we have $\eta = d(\arctan \frac{y}{x})$, and in the second and fourth we have $\eta = d(-\arctan \frac{x}{y})$. Now in the first, third, and fifth parts, $\frac{y}{x} = \frac{\sin t}{\cos t} = \tan t$, so that either $t = \arctan \frac{y}{x}$ or $t = \pi + \arctan \frac{y}{x}$ on these arcs. In either case the integral over these parts is just the difference in t at the endpoints. Hence these three integrals contribute $\frac{\pi}{4} + \frac{\pi}{2} + \frac{\pi}{4} = \pi$ to the integral. On the other parts $\frac{x}{y} = \cot t = \tan(\frac{\pi}{2} - t)$. Hence, once again, $\arctan \frac{x}{y}$ is either $\frac{\pi}{2} - t$ or $\frac{3\pi}{2} - t$. In either case, these two integrals contribute $\frac{\pi}{2} + \frac{\pi}{2} = \pi$ to the integral, and provides the result of (b).

(f) The definition of $\operatorname{Ind}(\Gamma)$ is defined by regarding $\Gamma(t)$ as a curve $X(t) + Y(t)i$ in the complex plane, in which case

$$\operatorname{Ind}(\Gamma) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\Gamma'(t)}{\Gamma(t)} dt = \frac{1}{2\pi i} \int_0^{2\pi} \frac{(X(t) - Y(t)i)(X'(t) + Y'(t)i)}{(X(t))^2 + Y(t)^2} dt.$$

Since we know the imaginary part is zero, we consider only the real part, which is

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{X(t)Y'(t) - Y(t)X'(t)}{(X(t))^2 + Y(t)^2} dt = \frac{1}{2\pi} \int_{\Gamma} \eta.$$

(Incidentally, it follows from Stokes' theorem that the imaginary part of this complex integral is zero, since it is

$$-\frac{1}{2\pi} \int_0^{2\pi} \frac{X(t)X'(t) + Y(t)Y'(t)}{(X(t))^2 + (Y(t))^2} dt = -\frac{1}{4\pi} \int_{\Gamma} d\zeta = -\frac{1}{4\pi} \int_{\partial\Gamma} \zeta$$

where $\zeta(x, y) = \ln(x^2 + y^2)$. This last integral is zero, since Γ is a closed curve.)

Exercise 10.22 As in Example 10.37, define ζ in $R^3 - \mathbf{0}$ by

$$\zeta = \frac{x dy \wedge dz + y dz \wedge dx + z dx \wedge dy}{r^3},$$

where $r = (x^2 + y^2 + z^2)^{1/2}$, let D be the rectangle given by $0 \leq u \leq \pi$, $0 \leq v \leq 2\pi$, and let Σ be the 2-surface in R^3 , with parameter domain D , given by

$$x = \sin u \cos v, \quad y = \sin u \sin v, \quad z = \cos u.$$

(a) Prove that $d\zeta = 0$ in $R^3 \setminus \mathbf{0}$.

(b) Let S denote the restriction of Σ to a parameter domain $E \subset D$. Prove that

$$\int_S \zeta = \int_E \sin u \, du \, dv = A(S),$$

where A denotes area, as in Sec. 10.43. Note that this contains (115) as a special case.

(c) Suppose g, h_1, h_2, h_3 , are C'' -functions on $[0, 1]$, $g > 0$. Let $(x, y, z) = \Phi(s, t)$ define a 2-surface Φ , with parameter domain I^2 , by

$$x = g(t)h_1(s), \quad y = g(t)h_2(s), \quad z = g(t)h_3(s).$$

Prove that

$$\int_{\Phi} \zeta = 0,$$

directly from (35).

Note the shape of the range of Φ : For fixed s , $\Phi(s, t)$ runs over an interval on a line through $\mathbf{0}$. The range of Φ thus lies in a "cone" with vertex at the origin.

(d) Let E be a closed rectangle in D , with edges parallel to those of D . Suppose $f \in C''(D)$, $f > 0$. Let Ω be the 2-surface with parameter domain E , defined by

$$\Omega(u, v) = f(u, v)\Sigma(u, v).$$

Define S as in (b) and prove that

$$\int_{\Omega} \zeta = \int_S \zeta = A(S).$$

(Since S is the "radial projection" of Ω onto the unit sphere, this result makes it reasonable to call $\int_{\Omega} \zeta$ the "solid angle" subtended by the range of Ω at the origin.)

Hint: Consider the 3-surface Ψ given by

$$\Psi(t, u, v) = [1 - t + tf(u, v)]\Sigma(u, v),$$

where $(u, v) \in E$, $0 \leq t \leq 1$. For fixed v , the mapping $(t, u) \rightarrow \Psi(t, u, v)$ is a 2-surface Φ to which (c) can be applied to show that $\int_{\Phi} \zeta = 0$. The same thing holds when u is fixed. By (a) and Stokes' theorem,

$$\int_{\partial\Psi} \zeta = \int_{\Psi} d\zeta = 0.$$

(e) Put $\lambda = -(z/r)\eta$, where

$$\eta = \frac{x dy - y dx}{x^2 + y^2},$$

as in Exercise 21. Then λ is a 1-form in the open set $V \subset R^3$ in which $x^2 + y^2 > 0$. Show that ζ is exact in V by showing that

$$\zeta = d\lambda.$$

(f) Derive (d) from (e), without using (c).

Hint: To begin with, assume $0 < u < \pi$ on E . By (e),

$$\int_{\Omega} \zeta = \int_{\partial\Omega} \lambda \quad \text{and} \quad \int_S \zeta = \int_{\partial S} \lambda.$$

Show that the two integrals of λ are equal, by using part (d) of Exercise 21, and by noting that z/r is the same at $\Sigma(u, v)$ as at $\Omega(u, v)$.

(g) Is ζ exact in the complement of every line through the origin?

Solution. (a) We note that, since $\frac{\partial r}{\partial x} = xr^{-1}$, we have

$$\frac{\partial}{\partial x} \frac{x}{r^3} = r^{-3} - 3x^2 r^{-5} = r^{-5}(r^2 - 3x^2).$$

By symmetry we have analogous relations for the partial derivatives of yr^{-3} and zr^{-3} with respect to y and z respectively. Since $dx \wedge dy \wedge dz = dy \wedge dz \wedge dx = dz \wedge dx \wedge dy$, we find that

$$d\zeta = r^{-5}(r^2 - 3x^2 + r^2 - 3y^2 + r^2 - 3z^2) dx \wedge dy \wedge dz = 0.$$

(b) Since $r(\Sigma(u, v)) = 1$, we have only to note that the differentials pull back to D as $dy \wedge dz = \frac{\partial(y, z)}{\partial(u, v)} du \wedge dv = \sin^2 u \cos v du \wedge dv$, $dz \wedge dx = \sin^2 u \sin v du \wedge dv$ and $dx \wedge dy = \sin u \cos u du \wedge dv$. The integrand then pulls back as $(\sin^3 u + \sin u \cos^2 u) du \wedge dv = \sin u du \wedge dv$. The reference to Sec. 10.43 must be a misprint for Sec. 10.46.

(c) For the application to be made in part (d) below we actually need to allow the function $g(t)$ to depend on s also. Thus we consider $g(s, t)$ instead of $g(t)$. Using only the definition (35) for the integral, we need to get the pullbacks of the wedge products to the parameter domain $[0, 1] \times [0, 1]$. Since $dx = \frac{\partial g}{\partial t} h_1(s) dt + (g(s, t)h_1'(s) + h_1(s)\frac{\partial g}{\partial s} ds$, with similar expressions for dy and dz , we find that $dy \wedge dz = g(s, t)\frac{\partial g}{\partial t}(h_3(s)h_2'(s) - h_3'(s)h_2(s)) ds \wedge dt$, $dz \wedge dx = g(s, t)\frac{\partial g}{\partial t}(h_3'(s)h_1(s) - h_3(s)h_1'(s)) ds \wedge dt$, and $dx \wedge dy = g(s, t)\frac{\partial g}{\partial t}(h_1'(s)h_2(s) - h_1(s)h_2'(s)) ds \wedge dt$. Thus, assuming $h_1(t)$, $h_2(t)$, and $h_3(t)$ do not vanish simultaneously, we have

$$\int_{\Phi} \zeta = \int_0^1 \int_0^1 \frac{\frac{\partial g}{\partial t}}{g(s, t)} \frac{(h_1(s)h_2(s)h_3(s))' - (h_1(s)h_2(s)h_3(s))'}{(h_1(s))^1 + (h_2(s))^2 + (h_3(s))^2} ds dt = 0.$$

(d) Using part (c), as amended, we note that $\partial\Psi$ consists of six mappings $\Psi(1, u, v) = \Omega(u, v)$, $\Psi(0, u, v) = S(u, v)$, $\Psi(t, b, v)$, $\Psi(t, a, v)$, $\Psi(t, u, d)$, and $\Psi(t, u, c)$, where $E = [a, b] \times [c, d]$. By part (c) the integrals over each of the last 4 surfaces are all zero. Since $d\zeta = 0$, Stokes' theorem implies that

$$\int_{\Omega} \zeta - \int_S \zeta = 0.$$

(e) By straightforward computation,

$$\begin{aligned} d\lambda &= -d(z/r) \wedge \eta - (z/r) d\eta \\ &= \frac{xz dx + yz dy + (z^2 - r^2) dz}{r^3} \wedge \eta \\ &= \frac{(x^2 z + y^2 z) dx \wedge dy}{r^3(x^2 + y^2)} - \frac{x dz \wedge dy - y dz \wedge dx}{r^3} \\ &= \zeta. \end{aligned}$$

(f) Again by Stokes' theorem we must have

$$\int_{\Omega} \zeta = \int_{\Omega} d\lambda = \int_{\partial\Omega} \lambda.$$

But η is independent of z , and z/r is the same for both $S(u, v)$ and $\Omega(u, v)$. Therefore

$$\int_{\partial\Omega} \lambda = \int_{\partial S} \lambda.$$

(g) Yes, ζ is exact on the complement of every line through the origin. Indeed, for every line through the origin there is a rotation T that maps that line to the z -axis. By Theorem 10.22, part (c) we have $d(\lambda_T) = (d\lambda)_T = \zeta_T$. However, $\zeta_T = \zeta$, as one can easily compute. Indeed, since r is invariant under T , we need only show that $x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$ is rotation-invariant. To that end, suppose $(u, v, w) = T(x, y, z)$, say $u = t_{11}x + t_{12}y + t_{13}z$, so that $du = t_{11}dx + t_{12}dy + t_{13}dz$, etc. We then have $dv \wedge dw = (t_{22}t_{33} - t_{32}t_{23})dy \wedge dz + (t_{23}t_{31} - t_{33}t_{21})dz \wedge dx + (t_{21}t_{32} - t_{31}t_{22})dx \wedge dy$, etc. and so $u dv \wedge dw + v dw \wedge du + w du \wedge dv$ works out (after tedious computation) to precisely $x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$.

Exercise 10.23 Fix n . Define $r_k = (x_1^2 + \cdots + x_k^2)^{1/2}$ for $1 \leq k \leq n$, let E_k be the set of all $\mathbf{x} \in R^n$ at which $r_k > 0$, and let ω_k be the $(k-1)$ -form defined in E_k by

$$\omega_k = (r_k)^{-k} \sum_{i=1}^k (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k.$$

Note that $\omega_2 = \eta$, $\omega_3 = \zeta$ in the notation of Exercises 21 and 22. Note also that

$$E_1 \subset E_2 \subset \cdots \subset E_n = R^n \setminus \{\mathbf{0}\}.$$

(a) Prove that $d\omega_k = 0$ in E_k .

(b) For $k = 2, \dots, n$, prove that ω_k is exact in E_{k-1} , by showing that

$$\omega_k = d(f_k \omega_{k-1}) = (df_k) \wedge \omega_{k-1},$$

where $f_k(\mathbf{x}) = (-1)^k g_k(x_k/r_k)$ and

$$g_k(t) = \int_{-1}^t (1-s)^{(k-3)/2} ds \quad (-1 < t < 1).$$

Hint: f_k satisfies the differential equations

$$\mathbf{x} \cdot (\nabla f_k)(\mathbf{x}) = 0$$

and

$$(D_k f_k)(\mathbf{x}) = \frac{(-1)^k (r_{k-1})^{k-1}}{(r_k)^k}.$$

(c) Is ω_n exact in E_n ?

(d) Note that (b) is a generalization of part (e) of Exercise 22. Try to extend some of the other assertions of Exercises 21 and 22 to ω_n , for arbitrary n .

Solution. (a) Computation shows that $d\left(\sum_{i=1}^k (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_k\right) = k dx_1 \wedge \cdots \wedge dx_k$, and $\frac{\partial r_k}{\partial x_j} = \frac{x_j}{r_k}$ for $j \leq k$, so that $d(r_k) = -k(r_k)^{-k-2} \sum_{j=1}^k x_j dx_j$, we find that

$$\begin{aligned} d\omega_k &= k(r_k)^{-k} dx_1 \wedge \cdots \wedge dx_k - k(r_k)^{-k-2} \sum_{j=1}^k x_j^2 dx_1 \wedge \cdots \wedge dx_k = \\ &= k(r_k)^{-k-2} \left(r_k^2 - \sum_{j=1}^k x_j^2 \right) dx_1 \wedge \cdots \wedge dx_k = 0. \end{aligned}$$

This argument shows, incidentally, that $d\omega_k = 0$ in $E_n = R^n \setminus \{0\}$.

(b) We compute that

$$\begin{aligned} df_k &= (-1)^k (1 - x_k^2/r_k^2)^{(k-3)/2} \left((r_k^{-1} - x_k^2 r_k^{-3}) dx_k - \sum_{i=1}^{k-1} x_k x_i r_k^{-3} dx_i \right) \\ &= (-1)^k (r_{k-1}/r_k)^{k-3} \left((r_k^{-3} r_{k-1}^2) dx_k - r_k^{-3} \sum_{i=1}^{k-1} x_i x_k \right) dx_i \\ &= (-1)^k (r_k)^{-k} \left(r_{k-1}^{k-1} dx_k - r_{k-1}^{k-3} \sum_{i=1}^{k-1} x_i x_k dx_i \right). \end{aligned}$$

Hence, since $(df_k) \wedge \omega_{k-1} = (-1)^{k-2} \omega_{k-1} \wedge (df_k)$, the first term in this last expression contributes

$$(r_k)^{-k} \sum_{i=1}^{k-1} (-1)^{i-1} x_i dx_i \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{k-1} \wedge dx_k$$

to the wedge product. As this contribution is all of ω_k except the last term $r_k^{-k} (-1)^{k-1} x_k dx_1 \wedge \cdots \wedge dx_{k-1}$, we must endeavor to show that the contribution of the remaining terms amounts to this expression. Since any term containing a repeated factor dx_j is zero, we see that the rest of the expression is

$$\begin{aligned} & (-1)^{k-1} x_k (r_k)^{-k} (r_{k-1})^{k-3} \left(\sum_{i=1}^{k-1} x_i dx_i \right) \wedge (r_{k-1})^{k-1} \times \\ & \quad \times \sum_{i=1}^{k-1} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_{k-1}, \end{aligned}$$

which is easily seen to be the same as

$$(-1)^{k-1} r_k^{-k} x_k r_{k-1}^{-2} \sum_{i=1}^{k-1} x_i^2 dx_1 \wedge \cdots \wedge dx_{k-1} = (-1)^{k-1} r_k^{-k} x_k dx_1 \wedge \cdots \wedge dx_{k-1},$$

exactly as required. Thus we have computed this result by “brute force,” arrogantly ignoring the hint.

For the benefit of those who wish to *use* the hint, here is an alternative approach. The wedge product $(df_k) \wedge \omega_{k-1}$ is the sum of $D_k f_k(\mathbf{x}) dx_k \wedge \omega_{k-1}$ and

$$\begin{aligned} & r_{k-1}^{-k-1} \left(\sum_{i=1}^{k-1} x_i D_i f(\mathbf{x}) \right) dx_1 \wedge \cdots \wedge dx_k = \\ & \quad = r_{k-1}^{-k-1} (\mathbf{x} \cdot (\nabla f_k)(\mathbf{x}) - x_k D_k f_k(\mathbf{x})) dx_1 \wedge \cdots \wedge dx_k, \end{aligned}$$

and hence, by the first differential equation, equals

$$D_k f_k(\mathbf{x}) dx_k \wedge \omega_{k-1} - r_{k-1}^{-k-1} x_k D_k f_k(\mathbf{x}) dx_1 \wedge \cdots \wedge dx_{k-1},$$

so that the second equation yields the result immediately. The two differential equations themselves are routine computations.

(c) No, ω_n is not exact in E_n for any n , since its integral over the $(n-1)$ -sphere equals $\frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$, as will be shown below in the answer to part (d). (If it were exact, say the differential of λ , this integral would equal the integral of λ over the boundary of the $(n-1)$ -sphere, which is the 0 $(n-2)$ -chain.)

(d) We can parameterize the $(n-1)$ -sphere Σ^{n-1} by the mapping T_n defined by

$$x_1 = \sin t_1 \sin t_2 \cdots \sin t_{n-1},$$

$$\begin{aligned}
x_2 &= \cos t_1 \sin t_2 \cdots \sin t_{n-1}, \\
x_3 &= \cos t_2 \sin t_3 \cdots \sin t_{n-1}, \\
&\dots \dots \dots \\
x_{n-1} &= \cos t_{n-2} \sin t_{n-1} \\
x_n &= \cos t_{n-1},
\end{aligned}$$

where $0 \leq t_1 \leq 2\pi$ and $0 \leq t_j \leq \pi$ for $2 \leq j \leq n-1$. That is, the domain of T_n is the parallelepiped $D = [0, 2\pi] \times [0, \pi]^{n-2}$. This is known to be true for $n = 2$ and $n = 3$, and follows easily by induction on n . Suppose, for example, we know it is true for $n-1$, and suppose $x_1^2 + \cdots + x_n^2 = 1$. If $x_n = \pm 1$, we can take $t_{n-1} = 0$ or π , and the values of the other angles can be anything. If $-1 < x_n < 1$, there is precisely one angle $t_{n-1} \in (0, \pi)$ such that $x_n = \cos t_{n-1}$. But then the point $(x_1/\sin t_{n-1}, \dots, x_{n-1}/\sin t_{n-1})$ belongs to Σ^{n-2} , and hence, by induction, can be written as

$$\begin{aligned}
x_1/\sin t_{n-1} &= \sin t_1 \cdots \sin t_{n-2}, \\
x_2/\sin t_{n-1} &= \cos t_1 \cdots \sin t_{n-2}, \\
&\dots \dots \dots \\
x_{n-2}/\sin t_{n-1} &= \cos t_{n-3} \sin t_{n-2} \\
x_{n-1}/\sin t_{n-1} &= \cos t_{n-2}.
\end{aligned}$$

This completes the induction. Observe that the angle t_1 requires the entire range $[0, 2\pi]$. That is, all points on the unit circle in R^2 can be written as $(\cos t, \sin t)$ only if t is allowed to range from 0 to 2π . Otherwise put, the $(n-1)$ -sphere is parameterized by $n-2$ latitude angles and one longitude angle.

We can easily show by induction that the pullback of ω_n is

$$(\omega_n)_{T_n} = (-1)^{n-1} \sin t_2 \sin^2 t_3 \cdots \sin^{n-2} t_{n-1} dt_1 \wedge \cdots \wedge dt_{n-1}.$$

To make the induction work, we need to distinguish the x_i 's in various numbers of dimensions; hence let the transformation T_n be defined by giving its components $x_i^{(n)}$, $i \leq n$, by the equations

$$\begin{aligned}
x_1^{(n)} &= \sin t_1 \sin t_2 \cdots \sin t_{n-1}, \\
x_2^{(n)} &= \cos t_1 \sin t_2 \cdots \sin t_{n-1}, \\
x_3^{(n)} &= \cos t_2 \sin t_3 \cdots \sin t_{n-1}, \\
&\dots \dots \dots \\
x_{n-1}^{(n)} &= \cos t_{n-2} \sin t_{n-1}, \\
x_n^{(n)} &= \cos t_{n-1},
\end{aligned}$$

Thus we have $x_n^n = \cos t_{n-1}$ and $x_j^{(n)} = x_j^{(n-1)} \sin t_{n-1}$ for $j < n$. Suppose we have proved that

$$\left(\sum_{i=1}^{n-1} (-1)^{i-1} x_i^{(n-1)} dx_1^{(n-1)} \wedge \cdots \wedge dx_{i-1}^{(n-1)} \wedge dx_{i+1}^{(n-1)} \wedge \cdots \wedge dx_{n-1}^{(n-1)} \right)_{T_{n-1}} =$$

$$= (-1)^{n-2} (\sin t_1 \sin^2 t_2 \cdots \sin^{n-3} t_{n-2} dt_1 \wedge \cdots \wedge dt_{n-2}).$$

We observe that the Jacobian matrix of the transformation T_n is the $n \times (n-1)$ matrix

$$\frac{\partial(x_1^{(n)}, \dots, x_n^{(n)})}{\partial(t_1, \dots, t_{n-1})} = \begin{pmatrix} \frac{\partial x_1^{(n-1)}}{\partial t_1} \sin t_{n-1} & \cdots & \frac{\partial x_1^{(n-1)}}{\partial t_{n-2}} \sin t_{n-1} & x_1^{(n-1)} \cos t_{n-1} \\ \frac{\partial x_2^{(n-1)}}{\partial t_1} \sin t_{n-1} & \cdots & \frac{\partial x_2^{(n-1)}}{\partial t_{n-2}} \sin t_{n-1} & x_2^{(n-1)} \cos t_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_{n-1}^{(n-1)}}{\partial t_1} \sin t_{n-1} & \cdots & \frac{\partial x_{n-1}^{(n-1)}}{\partial t_{n-2}} \sin t_{n-1} & x_{n-1}^{(n-1)} \cos t_{n-1} \\ 0 & \cdots & 0 & -\sin t_{n-1} \end{pmatrix}$$

It follows immediately, when we expand the determinant of the first $n-1$ rows along the last column, that

$$\begin{aligned} (dx_1^{(n)} \wedge \cdots \wedge dx_{n-1}^{(n)})_{T_n} &= \sin^{n-2} t_{n-1} \cos t_{n-1} \times \\ &\times \sum_{i=1}^{n-1} (-1)^{n-1+i} x_i^{(n-1)} \frac{\partial(x_1^{(n-1)}, \dots, x_{n-1}^{(n-1)})}{\partial(t_1, \dots, t_{n-2})} dt_1 \wedge \cdots \wedge dt_{n-1} = \\ &= (-1)^n \sin^{n-2} t_{n-1} \cos t_{n-1} \left(\sum_{i=1}^{n-1} (-1)^{i-1} x_i^{(n-1)} dx_1^{(n-1)} \wedge \cdots \right. \\ &\quad \left. \cdots \wedge dx_{i-1}^{(n-1)} \wedge dx_{i+1}^{(n-1)} \wedge \cdots \wedge dx_{n-1}^{(n-1)} \right)_{T_{n-1}} \wedge dt_{n-1} \\ &= \sin^{n-2} t_{n-1} \cos t_{n-1} (\sin t_2 \sin^2 t_3 \cdots \sin^{n-3} t_{n-2}) dt_1 \wedge \cdots \wedge dt_{n-1}. \end{aligned}$$

Hence

$$\begin{aligned} (-1)^{n-1} (x_n^{(n)} dx_1^{(n)} \wedge \cdots \wedge dx_{n-1}^{(n)})_{T_n} &= \\ &= (-1)^{n-1} \cos^2 t_{n-1} \sin t_2 \sin^2 t_3 \cdots \sin^{n-2} t_{n-1} dt_1 \wedge \cdots \wedge dt_{n-1}. \end{aligned}$$

Next, omitting row i ($i < n$) and expanding the resulting determinant along the last row, we find that

$$\begin{aligned} (dx_1^{(n)} \wedge \cdots \wedge dx_{i-1}^{(n)} \wedge dx_{i+1}^{(n)} \wedge \cdots \wedge dx_n^{(n)})_{T_n} &= \\ &= -\sin^{n-1} t_{n-1} (dx_1^{(n-1)} \wedge \cdots \wedge dx_{i-1}^{(n-1)} \wedge dx_{i+1}^{(n-1)} \wedge \cdots \wedge dx_{n-1}^{(n-1)})_{T_{n-1}} \wedge dt_{n-1}, \end{aligned}$$

so that

$$\begin{aligned} \left(\sum_{i=1}^{n-1} (-1)^{i-1} x_i^{(n)} dx_1^{(n)} \wedge \cdots \wedge dx_{i-1}^{(n)} \wedge dx_{i+1}^{(n)} \wedge \cdots \wedge dx_n^{(n)} \right)_{T_n} &= \\ &= -\sin^n t_{n-1} \left(\sum_{i=1}^{n-1} (-1)^{i-1} x_i^{(n-1)} dx_1^{(n-1)} \wedge \cdots \wedge dx_{i-1}^{(n-1)} \wedge dx_{i+1}^{(n-1)} \wedge \cdots \right. \\ &\quad \left. \cdots \wedge dx_{n-1}^{(n-1)} \right)_{T_{n-1}} \wedge dt_{n-1}, \end{aligned}$$

and again by induction this is

$$(-1)^{n-1} \sin^2 t_{n-1} (\sin t_2 \sin^2 t_3 \cdots \sin^{n-2} t_{n-1}) dt_1 \cdots dt_{n-1}.$$

Combining these results we find that

$$\begin{aligned} \left(\sum_{i=1}^n (-1)^{i-1} x_i^{(n)} dx_1^{(n)} \wedge \cdots \wedge dx_{i-1}^{(n)} \wedge dx_{i+1}^{(n)} \wedge \cdots \wedge dx_n^{(n)} \right)_{T_n} = \\ = (-1)^{n-1} \sin t_2 \sin^2 t_3 \cdots \sin^{n-2} t_{n-1} dt_1 \wedge \cdots \wedge dt_{n-1}. \end{aligned}$$

The induction is now complete.

Except for the unimportant factor of -1 , this formula gives results consistent with the known results for the area of the $(n-1)$ -sphere, namely a total area of

$$A_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}.$$

This is easily verified for $n=2$ and $n=3$. In general

$$\begin{aligned} A_{n-1} &= A_{n-2} \int_0^\pi \sin^{n-2} t_{n-1} dt_{n-1} = \\ &= 2A_{n-2} \int_0^{\frac{\pi}{2}} \sin^{n-2} s ds = 2A_{n-2} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}. \end{aligned}$$

It easily follows, since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, that the formula for the surface area of $\Sigma^{(n-1)}$ is valid for all n .

Similarly we can show the analog of part (c) of Exercise 22, namely that

$$\int_{\Phi} \omega_n = 0$$

for any $(n-1)$ -dimensional surface given by a mapping of the form

$$\begin{aligned} \Phi(s_1, \dots, s_{n-2}, t) = (g(s_1, \dots, s_{n-2}, t)h_1(s_1, \dots, s_{n-2}), \dots \\ \dots, g(s_1, \dots, s_{n-2}, t)h_n(s_1, \dots, s_{n-2})). \end{aligned}$$

Indeed, the pullback of ω_n is

$$\begin{aligned} (\omega_n)_{\Phi} = \\ = \begin{vmatrix} gh_1 & \frac{\partial g}{\partial s_1} h_1 + g \frac{\partial h_1}{\partial s_1} & \cdots & \frac{\partial g}{\partial s_{n-2}} h_1 + g \frac{\partial h_1}{\partial s_{n-2}} & \frac{\partial g}{\partial t} h_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ gh_n & \frac{\partial g}{\partial s_1} h_n + g \frac{\partial h_n}{\partial s_1} & \cdots & \frac{\partial g}{\partial s_{n-2}} h_n + g \frac{\partial h_n}{\partial s_{n-2}} & \frac{\partial g}{\partial t} h_n \end{vmatrix} ds_1 \wedge \cdots \wedge ds_{n-2} \wedge dt. \end{aligned}$$

But this determinant is zero, since the first and last columns are proportional.

We can now prove that if $f(t_1, \dots, t_{n-1}) > 0$ and

$$\Omega(t_1, \dots, t_{n-1}) = f(t_1, \dots, t_{n-1}) \Sigma^{(n-1)}(t_1, \dots, t_{n-1}),$$

then

$$\int_{\Omega} \omega_n = \int_S \omega_n = \pm A_{n-1}(S).$$

To do so, we consider the n -surface in R^n given by

$$\Psi(t_1, \dots, t_{n-1}, t) = [1 - t + tf(t_1, \dots, t_{n-1})]\Sigma^{(n-1)}(t_1, \dots, t_{n-1}),$$

for $0 \leq t \leq 1$ and t_1, \dots, t_n ranging over a parallelepiped contained in the interior of D with boundary faces parallel to those of D . For each fixed t_j , this Ψ is an $(n-1)$ -surface of the form just considered, and hence the integral of ω_n over it is zero. This applies in particular to the faces of the closed parallelepiped E . Since $\int_{\partial\Psi} \omega_n = \int_{\Psi} d\omega_n = 0$, it then follows that, up to a factor of ± 1 ,

$$\int_{\Omega} \omega_n = \int_S \omega_n = A_{n-1}(S).$$

Finally, as in Exercise 22, ω_n is exact in the complement of every $(n-2)$ -hyperplane through the origin, since there is a rotation that maps the complement of that hyperplane to E_{n-1} , while ω_n is rotation-invariant.

Exercise 10.24 Let $\omega = \sigma a_i(\mathbf{x}) dx_i$ be a 1-form of class C'' in a convex open set $E \subset R^n$. Assume $d\omega = 0$ and prove that ω is exact in E by completing the following outline:

Fix $\mathbf{p} \in E$. Define

$$f(\mathbf{x}) = \int_{[\mathbf{p}, \mathbf{x}]} \omega \quad (\mathbf{x} \in E).$$

Apply Stokes' theorem to affine-oriented 2-simplexes $[\mathbf{p}, \mathbf{x}, \mathbf{y}]$ in E . Deduce that

$$f(\mathbf{y}) - f(\mathbf{x}) = \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y}) dt$$

for $\mathbf{x} \in E, \mathbf{y} \in E$. Hence $D_i f(\mathbf{x}) = a_i(\mathbf{x})$.

Solution. Because $d\omega = 0$, the integral of ω over the boundary of the oriented 2-simplex $[\mathbf{p}, \mathbf{x}, \mathbf{y}]$ is zero. That is

$$\int_{[\mathbf{x}, \mathbf{y}]} \omega - \int_{[\mathbf{p}, \mathbf{y}]} \omega + \int_{[\mathbf{p}, \mathbf{x}]} \omega = 0,$$

which can be rewritten as

$$f(\mathbf{y}) - f(\mathbf{x}) = \int_{[\mathbf{x}, \mathbf{y}]} \omega = \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y}) dt.$$

Differentiating with respect to y_i , we find

$$D_i f(\mathbf{y}) = \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y}) dt + \sum_{j=1}^n (y_j - x_j) \int_0^1 t D_i a_j((1-t)\mathbf{x} + t\mathbf{y}) dt.$$

The fact that $d\omega = 0$ says that $D_i a_j = D_j a_i$, so that we have

$$\begin{aligned} D_i f(\mathbf{y}) &= \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y}) dt + \int_0^1 \sum_{j=1}^n t (y_j - x_j) D_j a_i((1-t)\mathbf{x} + t\mathbf{y}) dt \\ &= \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y}) dt + \int_0^1 t \frac{d}{dt} a_i((1-t)\mathbf{x} + t\mathbf{y}) dt \\ &= \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y}) dt + t a_i((1-t)\mathbf{x} + t\mathbf{y}) \Big|_0^1 \\ &\quad - \int_0^1 a_i((1-t)\mathbf{x} + t\mathbf{y}) dt \\ &= a_i(\mathbf{y}). \end{aligned}$$

Thus $\omega = df$.

Exercise 10.25 Assume that ω is a 1-form in an open set $E \subset R^n$ such that

$$\int_{\gamma} \omega = 0$$

for every closed curve γ in E of class C' . Prove that ω is exact in E , by imitating part of the argument sketched in Exercise 24.

Solution. We first observe that Stokes' theorem and the argument of Theorem 10.15 show that $d\omega = 0$ in E . (Theorem 10.15 actually shows that if some component of $d\omega$ is nonzero at some point of E , then there is a 2-surface Φ in E whose domain is a 2-cell in R^2 for which $\int_{\Phi} d\omega \neq 0$. Then by Stokes' theorem, $\int_{\partial\Phi} \omega \neq 0$ also, contradicting the assumption of the problem.

In each connected component E_{α} of E , we choose a fixed point \mathbf{x}_{α} . There is a ball of some positive radius r_{α} centered at \mathbf{x}_{α} and contained in E . Let this ball be B_{α} . Exercise 24 shows that there is a function $f(\mathbf{x})$ such that $\omega(\mathbf{x}) = df(\mathbf{x})$ inside B_{α} . By subtracting a constant from f we can assume that $f(\mathbf{x}_{\alpha}) = 0$.

Now consider the set S of all points $\mathbf{x} \in E_{\alpha}$ having the property that there exist a connected open set $F_{\mathbf{x}}$ containing \mathbf{x} and \mathbf{x}_{α} and a function $f_{\mathbf{x}}$ defined on $F_{\mathbf{x}}$ such that $df_{\mathbf{x}} = \omega$ on $F_{\mathbf{x}}$ and $f_{\mathbf{x}_{\alpha}} = 0$. It is clear that S is an open connected subset of E_{α} , being the union of all the connected open sets $F_{\mathbf{x}}$, which have the common point \mathbf{x}_{α} . It is also clear that there is a function f defined on S such that $df = \omega$ on S . In fact we can define $f(\mathbf{x}) = f_{\mathbf{x}}(\mathbf{x})$, and this definition is unambiguous, since if $f_{\mathbf{x}}$ and $f_{\mathbf{y}}$ are both defined at \mathbf{z} , then

$$f_{\mathbf{x}}(\mathbf{z}) = \int_{\gamma} df_{\mathbf{x}} = \int_{\gamma} \omega = \int_{\delta} \omega = \int_{\delta} df_{\mathbf{y}} = f_{\mathbf{y}}(\mathbf{z}).$$

Here γ is a path in $E_{\mathbf{x}}$ from \mathbf{x}_{α} to \mathbf{z} , and δ is a path in $E_{\mathbf{y}}$ from \mathbf{x}_{α} to \mathbf{z} . The path $\gamma - \delta$ lies in E and is a closed loop, so that

$$\int_{\gamma-\delta} \omega = 0.$$

We need only show that $S = E_{\alpha}$. But if not, then E_{α} contains a boundary point $\mathbf{x} \in S$. Some open ball B about \mathbf{x} is contained in E , and this open ball contains a point $\mathbf{y} \in S$. But then there exists a function g such that $dg = \omega$ in B , and subtracting a constant makes it possible to ensure that $g(\mathbf{y}) = f_{\mathbf{y}}(\mathbf{y}) = f(\mathbf{y})$. We claim that $g(\mathbf{z}) = f(\mathbf{z})$ on the entire set $S \cap B$. In fact this argument merely repeats the argument just given to show that f is unambiguously defined. It then follows that \mathbf{y} is contained in the connected open set $S \cap B$ and that the function h defined to be f on S and g on B has the property that $dh = \omega$ on $S \cap B$. By definition, this means $\mathbf{y} \in S$, which contradicts the assumption that \mathbf{y} is a boundary point of S . Therefore $S = E_{\alpha}$.

Thus we can find a primitive for ω on each connected component of E . These primitives can be pieced together to provide a single primitive for ω on E .

Exercise 10.26 Assume ω is a 1-form in $R^3 \setminus \{\mathbf{0}\}$, of class \mathcal{C}' and $d\omega = 0$. Prove that ω is exact in $R^3 \setminus \{\mathbf{0}\}$.

Hint: Every closed continuously differentiable curve in $R^3 \setminus \{\mathbf{0}\}$ is the boundary of a 2-surface in $R^3 \setminus \{\mathbf{0}\}$. Apply Stokes' theorem and Exercise 25.

Solution. Given the assumption in the hint, the solution is easy. By Exercise 25 we need only show that the integral of ω over every closed curve is zero. By the assertion in the hint, this closed curve is the boundary of a two-surface. By Stokes' theorem, the integral of ω over the curve equals the integral of $d\omega$ over the 2-surface.

To prove the claim that every continuously differentiable curve in $R^3 \setminus \{\mathbf{0}\}$ is the boundary of a two-surface, we may assume that the curve is of the form $\mathbf{x}(t)$, $0 \leq t \leq 1$ and $\mathbf{x}(0) = \mathbf{x}(1)$. Let $\mathbf{x}(t) = (x(t), y(t), z(t))$. We shall show first of all that there is some line through the origin in R^3 that does not intersect the curve.

To that end, we observe that the intersection of a sphere of radius ρ in R^3 with a ball of radius r ($r \leq 2\rho$) about a point of the sphere is a spherical cap whose area is πr^2 . (Note that this result is independent of ρ . It is a remarkable fact, whose proof is a routine computation.) Since the area of the whole sphere is $4\pi\rho^2$, it follows that half of any given hemisphere cannot be covered by fewer than ρ^2/r^2 such spherical caps. Now, since $\mathbf{x}(t) \neq \mathbf{0}$ and $\mathbf{x}'(t)$ is continuous, it follows that $\mathbf{v}(t) = \mathbf{x}(t)/|\mathbf{x}(t)|$ is a Lipschitz function, that is, there exists a constant M such that $|\mathbf{v}(s) - \mathbf{v}(t)| \leq M|t - s|$ for all s and t . In particular the image of each interval $[k/n, (k+1)/n]$ is contained in a spherical cap of radius M/n . Thus the complete curve is contained in a set of n spherical caps of radius at most M/n . But to cover the half of any given hemisphere of the unit sphere

requires at least $\frac{n^2}{M^2}$ such caps. Hence, if $n > M^2$, the projection of the curve $\mathbf{x}(t)$ on the unit sphere is contained in a set of spherical caps covering less than half of the upper hemisphere and less than half of the lower hemisphere. Hence there are two antipodal points \mathbf{x}_0 and $-\mathbf{x}_0$ on the unit sphere not in its image. That means there is at least one line through the origin that the curve does not intersect.

This line through the origin gives us a sense of positive rotation from $\mathbf{x}(t)$ to $\mathbf{x}(t+\frac{1}{2})$ for each $t \in [0, \frac{1}{2}]$. We can then construct a \mathcal{C}' -curve $\gamma_t(s)$ in $R^3 \setminus \{0\}$ that goes from $\mathbf{x}(t)$ to $\mathbf{x}(t+\frac{1}{2})$ by letting cylindrical coordinates vary linearly with respect to s . To be specific, we can assume without loss of generality that the line is the z -axis. In that case, the radial coordinate $r(t) = \sqrt{x^2(t) + y^2(t)}$ is never zero and is a continuously differentiable function of position. We choose $\theta(t)$ as the cylindrical polar coordinate of $\mathbf{x}(t)$ in a continuously differentiable manner for $0 \leq t \leq 1$. (This is possible by piecing together sections of this function over sufficiently small intervals.) We then define $\gamma(s, t) = (x(s, t), y(s, t), z(s, t))$ for $0 \leq s \leq 1, 0 \leq t \leq 1/2$ by

$$\begin{aligned} x(t, u) &= (1-u)r(t) \cos((1-u)\theta(t)) + ur(1-t) \cos(u\theta(1-t)), \\ y(t, u) &= (1-u)r(t) \sin((1-u)\theta(t)) + ur(1-t) \sin(u\theta(1-t)), \\ z(t, u) &= (1-u)z(t) + uz(1-t). \end{aligned}$$

We let the boundary of this cell be $\delta_1 + \delta_2 + \delta_3 + \delta_4$. Here δ_1 is $\gamma(t, 0)$, $0 \leq t \leq 1/2$, which is just $\mathbf{x}(t)$ over the same interval; δ_2 is $\gamma(1/2, u)$, which is the "line segment" from $\mathbf{x}(1/2)$ to $\mathbf{x}(1/2)$, whose range is just a point, and hence counts as 0 when regarded as a 1-chain; δ_3 is $\gamma(1/2-t, 1)$ which is just $\mathbf{x}(t+\frac{1}{2})$, so that $\delta_1 + \delta_3$ represents $\mathbf{x}(t)$ as t goes from 0 to 1. Finally δ_4 is $\gamma(0, u)$, which is the line segment from $\mathbf{x}(1)$ to $\mathbf{x}(0)$, and since \mathbf{x} is a closed curve, these two points are the same. Hence once again δ_4 counts as 0 when regarded as a 1-chain. Thus the boundary of γ is indeed the curve \mathbf{x} .

Exercise 10.27 Let E be an open 3-cell in R^3 , with edges parallel to the coordinate axes. Suppose $(a, b, c) \in E$, $f_t \in \mathcal{C}'(E)$ for $i = 1, 2, 3$,

$$\omega = f_1 dy \wedge dz + f_2 dz \wedge dx + f_3 dx \wedge dy,$$

and assume that $d\omega = 0$ in E . Define

$$\lambda = g_1 dx + g_2 dy$$

where

$$\begin{aligned} g_1(x, y, z) &= \int_c^z f_2(x, y, s) ds - \int_b^y f_3(x, t, c) dt \\ g_2(x, y, z) &= - \int_c^z f_1(x, y, s) ds, \end{aligned}$$

for $(x, y, z) \in E$. Prove that $d\lambda = \omega$ in E .

Evaluate these integrals when $\omega = \zeta$ and thus find the form λ that occurs in part (e) of Exercise 22.

Solution. Since

$$d\lambda = -\frac{\partial g_2}{\partial z} dy \wedge dz + \frac{\partial g_1}{\partial z} dz \wedge dx + \left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} \right) dx \wedge dy,$$

we need only show that

$$\begin{aligned} \frac{\partial g_2}{\partial z} &= -f_1, \\ \frac{\partial g_1}{\partial z} &= f_2, \\ \frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y} &= f_3. \end{aligned}$$

The first two equations are immediate. As for the third, direct computation shows that

$$D_1 g_2(x, y, z) - D_2 g_1(x, y, z) = \int_c^z -(D_1 f_1(x, y, s) + D_2 f_2(x, y, s)) ds + f_3(x, y, c).$$

Now the assumption that $d\omega = 0$ says that

$$D_1 f_1(x, y, s) + D_2 f_2(x, y, s) = -D_3 f_3(x, y, s),$$

Substituting this value into the last expression and evaluating the integral using the fundamental theorem of calculus yields the result $d\lambda = \omega$.

Taking

$$\begin{aligned} f_1(x, y, z) &= \frac{x}{(x^2 + y^2 + z^2)^{3/2}}, \\ f_2(x, y, z) &= \frac{y}{(x^2 + y^2 + z^2)^{3/2}}, \\ f_3(x, y, z) &= \frac{z}{(x^2 + y^2 + z^2)^{3/2}}, \end{aligned}$$

we get

$$\begin{aligned} g_2(x, y, z) &= -\int_c^z \frac{x}{(x^2 + y^2 + s^2)^{3/2}} ds = \\ &= \frac{1}{x^2 + y^2} \left(\frac{cx}{\sqrt{x^2 + y^2 + c^2}} - \frac{zx}{\sqrt{x^2 + y^2 + z^2}} \right), \\ g_1(x, y, z) &= \int_c^z \frac{y}{(x^2 + y^2 + s^2)^{3/2}} ds - \int_b^y \frac{c}{(x^2 + t^2 + c^2)^{3/2}} dt \\ &= \frac{1}{x^2 + y^2} \left(\frac{yz}{\sqrt{x^2 + y^2 + z^2}} - \frac{yc}{\sqrt{x^2 + y^2 + c^2}} \right) \\ &\quad - \frac{1}{x^2 + c^2} \left(\frac{cy}{\sqrt{x^2 + y^2 + c^2}} - \frac{bc}{\sqrt{x^2 + b^2 + c^2}} \right). \end{aligned}$$

It is a routine computation to verify that these functions do indeed provide a primitive for ω .

Exercise 10.28 Fix $b > a > 0$, define

$$\Phi(r, \theta) = (r \cos \theta, r \sin \theta)$$

for $a \leq r \leq b$, $0 \leq \theta \leq 2\pi$. (The range of Φ is an annulus in R^2 .) Put $\omega = x^3 dy$, and compute both

$$\int_{\Phi} d\omega \quad \text{and} \quad \int_{\partial\Phi} \omega$$

to verify that they are equal.

Solution. Since $d\omega = 3x^2 dx \wedge dy$, we have $(d\omega)_{\Phi} = -r dr \wedge d\theta$, and

$$\int_{\Phi} d\omega = - \int_a^b \int_0^{2\pi} 3r^3 \cos^2 \theta d\theta dr = \frac{3\pi}{4}(a^4 - b^4).$$

For the integral over the boundary we have $dy = r \cos \theta d\theta + \sin \theta dr$, and we get

$$\int_0^{2\pi} (a^4 - b^4) \cos^4 \theta d\theta = \frac{3\pi}{4}(a^4 - b^4).$$

Exercise 10.29 Prove the existence of a function α with the properties needed in the proof of Theorem 10.38, and prove that the resulting function F is of class C' . (Both assertions become trivial if E is an open cell or an open ball, since α can then be taken to be a constant. Refer to Theorem 9.42.)

Solution. We are given a convex open set $V \subseteq R^p$ whose projection on R^{p-1} is the convex open set U . We need to show that there is a continuously differentiable function $\alpha : U \rightarrow R$ whose graph is contained in V . If V is a cell or an open ball, there exists a section $x_p = c$ of it (many sections, if it is a cell), whose projection is U , and we can simply define $\alpha(\mathbf{y}) = c$ for all $\mathbf{y} \in U$.

Now write V as a countable union of open balls $V = \bigcup_{i=1}^{\infty} B_i$. Also write V as the union of an increasing sequence of compact sets K_n such that $K_n \subseteq \text{int}(K_{n+1})$.

We claim that, as in Theorem 10.8, there exist continuous functions ψ_i such that the support of ψ_i is contained in the projection of B_i on R^{p-1} , $0 \leq \psi_i(\mathbf{y}) \leq 1$ for all \mathbf{y} , and $\sum_{i=1}^{\infty} \psi_i(\mathbf{y}) = 1$ for all $\mathbf{y} \in U$. Moreover, this sum is *locally finite*, that is, each point \mathbf{y} has a neighborhood $U_{\mathbf{y}}$ such that the set of indices i for which $\psi_i(\mathbf{z}) \neq 0$ for some $\mathbf{z} \in U_{\mathbf{y}}$ is finite.

To construct such functions, for each $\mathbf{x} \in V$, let $i(\mathbf{x})$ be the smallest index r such that $\mathbf{x} \in B_i$. Then, as in the proof of Theorem 10.8, for each $\mathbf{x} \in K_1$, choose open balls $B(\mathbf{x})$ and $W(\mathbf{x})$ centered at \mathbf{x} such that

$$\overline{B(\mathbf{x})} \subset W(\mathbf{x}) \subset \overline{W(\mathbf{x})} \subset B_{i(\mathbf{x})}.$$

Since K_1 is compact, there are points $\mathbf{x}_{11}, \dots, \mathbf{x}_{1N_1}$ such that

$$K_1 \subseteq B(\mathbf{x}_{11}) \cup \dots \cup B(\mathbf{x}_{1N_1}).$$

For later convenience we define $L_1 = K_1$.

Now let $L_2 = K_2 \setminus \bigcup_{j=1}^{N_1} B(\mathbf{x}_{1j})$. For each $\mathbf{x} \in L_2$ there are open balls $B(\mathbf{x})$ and $W(\mathbf{x})$ centered at \mathbf{x} such that

$$\overline{B(\mathbf{x})} \subset W(\mathbf{x}) \subset \overline{W(\mathbf{x})} \subset B_{i(\mathbf{x})} \setminus K_1.$$

Since L_2 is compact, we choose a finite set of points $\mathbf{x}_{21}, \dots, \mathbf{x}_{2N_2}$ such that

$$L_2 \subseteq B(\mathbf{x}_{21}) \cup \dots \cup B(\mathbf{x}_{2N_2}).$$

Notice that $K_2 \subset \bigcup_{k=1}^2 \bigcup_{j=1}^{N_k} B(\mathbf{x}_{kj})$.

Now suppose we have chosen a (possibly empty) collection of open balls $B(\mathbf{x}_{kj})$ and $W(\mathbf{x}_{kj})$, $1 \leq j \leq N_k$, $1 \leq k \leq r$, centered at $\mathbf{x}_{kj} \in L_k = K_k \setminus \bigcup_{i=1}^{k-1} \bigcup_{j=1}^{N_i} B(\mathbf{x}_{ij})$, and such that

$$\overline{B(\mathbf{x}_{kj})} \subset W(\mathbf{x}_{kj}) \subset \overline{W(\mathbf{x}_{kj})} \subset B_{i(\mathbf{x}_{kj})} \setminus K_{k-1},$$

and

$$K_r \setminus \bigcup_{k=1}^{r-1} \bigcup_{j=1}^{N_k} B(\mathbf{x}_{kj}) \subset \bigcup_{j=1}^{N_r} B(\mathbf{x}_{rj}),$$

and

$$K_s \subset \bigcup_{k=1}^s \bigcup_{j=1}^{N_k} B(\mathbf{x}_{kj}),$$

for $1 \leq s \leq r-1$. It then follows from the last two relationships that the last one also holds with $s = r$. By then considering the compact set $L_{r+1} = K_{r+1} \setminus \bigcup_{k=1}^r \bigcup_{j=1}^{N_k} B(\mathbf{x}_{kj})$ and repeating the argument, we can assume that the sets $B(\mathbf{x}_{kj})$ and $W(\mathbf{x}_{kj})$ with these properties have been chosen for all k and all j , $1 \leq k < \infty$, $1 \leq j \leq N_k$. It follows in particular that

$$V = \bigcup_{n=1}^{\infty} K_n \subset \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{N_k} B(\mathbf{x}_{kj}).$$

Now let \tilde{K}_k , $\tilde{B}(\mathbf{x}_{kj})$, and $\tilde{W}(\mathbf{x}_{kj})$ be respectively the projections on R^{p-1} of K_k , $B(\mathbf{x}_{kj})$, and $W(\mathbf{x}_{kj})$, and let

$$\tilde{L}_r = \tilde{K}_r \setminus \bigcup_{k=1}^{r-1} \bigcup_{j=1}^{N_k} \tilde{B}(\mathbf{x}_{kj}).$$

We then choose functions φ_{jk} as smooth as we like such that $\varphi_{jk}(\mathbf{y}) = 1$ on $\tilde{B}(\mathbf{x}_{kj})$ (and hence also on $\overline{\tilde{B}(\mathbf{x}_{kj})}$), $\varphi_{jk}(\mathbf{y}) = 0$ outside $\tilde{W}(\mathbf{x}_{kj})$, and $0 \leq \varphi_{jk}(\mathbf{y}) \leq 1$ on R^{p-1} . Let $\varphi_j(\mathbf{y}) = \varphi_{1j}(\mathbf{y})$ for $1 \leq j \leq N_1$ and $\varphi_j(\mathbf{y}) =$

$\varphi_{k,j-(N_1+\dots+N_{k-1})}(\mathbf{y})$ for $N_1+\dots+N_{k-1} < j \leq N_1+\dots+N_k$, $2 \leq k < \infty$. We define \mathbf{x}_j analogously. Let $\mathbf{x}_j = (\mathbf{y}_j, c_j)$.

We then proceed to define $\psi_1(\mathbf{y}) = \varphi_1(\mathbf{y})$ and

$$\psi_{j+1}(\mathbf{y}) = (1 - \varphi_1(\mathbf{y})) \cdots (1 - \varphi_j(\mathbf{y}))\varphi_{j+1}(\mathbf{y})$$

for $j = 1, 2, \dots$, as in Theorem 10.8. It is obvious that the support of ψ_j is contained in the closure of $\widetilde{W}(\mathbf{x}_j)$ and hence in $\widetilde{B}_{i(\mathbf{x}_j)} \setminus \widetilde{K}_{k-1} \subseteq U \setminus \widetilde{K}_{k-1}$ when $N_1 + \dots + N_{k-1} < j \leq N_1 + \dots + N_k$.

Now by the choice of the sets K_n , if $\mathbf{y} \in U$, there is some n such that $\mathbf{y} \in \widetilde{K}_n \subset \text{int}(\widetilde{K}_{n+1})$, and hence $\psi_j(\mathbf{y}) = 0$ on the open neighborhood $\text{int} \widetilde{K}_{n+1}$ of \mathbf{y} if $j > N_1 + \dots + N_{n+1}$. Therefore the sum and product

$$\sum_{j=1}^{\infty} \psi_j(\mathbf{y}) = 1 - \prod_{i=1}^{\infty} [1 - \varphi_i(\mathbf{y})]$$

are both locally finite at each point. (Local finiteness of the product means all but a finite number of factors equal 1 on a neighborhood of each point.) However, if $\mathbf{y} \in U$, then $\mathbf{y} \in \widetilde{B}(\mathbf{x}_j)$ for some j , and so $\varphi_j(\mathbf{y}) = 1$, from which it then follows that

$$\sum_{j=1}^{\infty} \psi_j(\mathbf{y}) = 1$$

for all $\mathbf{y} \in U$.

Since we have defined c_j so that $\mathbf{x}_j = (\mathbf{y}_j, c_j)$, it follows that the projection of the c_j -section of $B(\mathbf{x}_j)$ on R^{p-1} , which we denote C_j , is the same as the projection of $B(\mathbf{x}_j)$ on this subspace. That is, it is $\widetilde{B}(\mathbf{x}_j)$. We can now let $\alpha(\mathbf{y}) = \sum_{j=1}^{\infty} c_j \psi_j(\mathbf{y})$. For then at each $\mathbf{y} \in U$ there is a finite integer n such that

$$(\mathbf{y}, \alpha(\mathbf{y})) = \psi_1(\mathbf{y})(\mathbf{y}, c_1) + \cdots + \psi_n(\mathbf{y})(\mathbf{y}, c_n).$$

Since $\psi_k(\mathbf{y}) = 0$ if $\mathbf{y} \notin C_k$ and $(\mathbf{y}, c_k) \in B_k \subset V$ if $\mathbf{y} \in C_k$, it follows that $(\mathbf{y}, \alpha(\mathbf{y}))$ is a weighted average of points in V , hence belongs to V for all $\mathbf{y} \in U$.

Exercise 10.30 If \mathbf{N} is the vector given by (135), prove that

$$\det \begin{bmatrix} \alpha_1 & \beta_1 & \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_2 & \beta_2 & \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_3 & \beta_3 & \alpha_2\beta_2 - \alpha_2\beta_1 \end{bmatrix} = |\mathbf{N}|^2.$$

Also, verify Eq. (137).

Solution. The equation in the problem is a straightforward computation, and amounts merely to expanding the determinant along the last column. Likewise Eq. (137), which merely asserts that a cross product is perpendicular to each

of the factors, is routine. The two inner products in the equation can be obtained by replacing the last column of this determinant by either $(\alpha_1, \alpha_2, \alpha_3)$ or $(\beta_1, \beta_2, \beta_3)$. In each case, the result is a determinant with two equal columns, which is therefore zero.

Exercise 10.31 Let $E \subset R^3$ be open, suppose $g \in C''(E)$, $h \in C''(E)$, and consider the vector field

$$\mathbf{F} = g\nabla h.$$

(a) Prove that

$$\nabla \cdot \mathbf{F} = g\nabla^2 h + (\nabla g) \cdot (\nabla h)$$

where $\nabla^2 h = \nabla \cdot (\nabla h) = \sum \partial^2 h / \partial x_i^2$ is the so-called “Laplacian” of h . (b) If Ω is a closed subset of E with positively oriented boundary $\partial\Omega$ (as in Theorem 10.51), prove that

$$\int_{\Omega} [g\nabla^2 h + (\nabla g) \cdot (\nabla h)] dV = \int_{\partial\Omega} g \frac{\partial h}{\partial n} dA$$

where (as is customary) we have written $\partial h / \partial n$ in place of $(\nabla h) \cdot \mathbf{n}$. (Thus $\partial h / \partial n$ is the directional derivative of h in the direction of the outward normal to $\partial\Omega$, the so-called *normal derivative* of h .) Interchange g and h , subtract the resulting formula from the first one, to obtain

$$\int_{\Omega} (g\nabla^2 h - h\nabla^2 g) dV = \int_{\partial\Omega} \left(g \frac{\partial h}{\partial n} - h \frac{\partial g}{\partial n} \right) dA.$$

These two formulas are usually called *Green’s identities*.

(c) Assume that h is *harmonic* in E ; this means that $\nabla^2 h = 0$. Take $g = 1$ and conclude that

$$\int_{\partial\Omega} \frac{\partial h}{\partial n} dA = 0.$$

Take $g = h$, and conclude that $h = 0$ in Ω if $h = 0$ on $\partial\Omega$.

(d) Show that Green’s identities are also valid in R^2 .

Solution. Part (a) is simply the product rule for derivatives.

The main equation in part (b) is simply the divergence theorem applied to \mathbf{F} . Green’s identities then follow by completely routine computation.

(c) Taking $g = 1$ forces $\partial g / \partial n = 0$ and $\nabla^2 g = 0$. Since $\nabla^2 h = 0$ by the assumption that h is harmonic, the result follows. For the other assertion of this part we have to go back to the main equation before taking $g = h$. When we do, we actually get a slightly stronger assertion: $\nabla h = 0$ in Ω , and so h is constant on each component of Ω , if either $h = 0$ or $\partial h / \partial n = 0$ on all of $\partial\Omega$. When $h = 0$ on $\partial\Omega$, obviously the constant value of h must be 0.

(d) The “two-dimensional” divergence theorem is simply Green’s theorem. That is, the assertion that

$$\int_{\Omega} \nabla \cdot \mathbf{F} = \int_{\partial\Omega} \mathbf{k} \times \mathbf{F}$$

follows upon applying Green’s theorem to the one-form $\omega = -F_2 dx + F_1 dy$ corresponding to the vector field $\mathbf{k} \times \mathbf{F} = -F_2 \mathbf{i} + F_1 \mathbf{j}$. Because the dot and cross operations can be interchanged in the scalar triple product, integrating $\mathbf{k} \times \mathbf{F}$ along a curve, that is, taking the product $\mathbf{k} \times \mathbf{F} \cdot \mathbf{r}$, where \mathbf{r} is the tangent to the curve, and then integrating, is the same as integrating $\mathbf{F} \cdot \mathbf{k} \times \mathbf{r}$, which is the normal component of \mathbf{F} . All the same identities now follow.

Exercise 10.32 Fix δ , $0 < \delta < 1$. Let D be the set of all $(\theta, t) \in \mathbb{R}^2$ such that $0 \leq \theta \leq \pi$, $-\delta \leq t \leq \delta$. Let Φ be the 2-surface in \mathbb{R}^3 with parameter domain D given by

$$\begin{aligned} x &= (1 - t \sin \theta) \cos 2\theta \\ y &= (1 - t \sin \theta) \sin 2\theta \\ z &= t \cos \theta \end{aligned}$$

where $(x, y, z) = \Phi(\theta, t)$. note that $\Phi(\pi, t) = \Phi(0, -t)$ and that Φ is one-to-one on the rest of D .

The range $M = \Phi(D)$ is known as a *Möbius band*. It is the simplest example of a nonorientable surface.

Prove the various assertions made in the following description: Put $\mathbf{p}_1 = (0, -\delta)$, $\mathbf{p}_2 = (\pi, -\delta)$, $\mathbf{p}_3 = (\pi, \delta)$, $\mathbf{p}_4 = (0, \delta)$, $\mathbf{p}_5 = \mathbf{p}_1$. Put $\gamma_i = [\mathbf{p}_i, \mathbf{p}_{i-1}]$, $i = 1, 2, \dots, 4$, and put $\Gamma_i = \Phi \circ \gamma_i$. Then

$$\partial\Phi = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4.$$

Put $\mathbf{z} = (1, 0, -\delta)$, $\mathbf{b} = (1, 0, \delta)$. Then

$$\Phi(\mathbf{p}_1) = \Phi(\mathbf{p}_3) = \mathbf{a}, \quad \Phi(\mathbf{p}_2) = \Phi(\mathbf{p}_4) = \mathbf{b},$$

and $\partial\Phi$ can be described as follows.

Γ_1 spirals up from \mathbf{a} to \mathbf{b} ; its projection into the (x, y) -plane has winding number $+1$ around the origin. (See Exercise 23, Chap. 8).

$$\Gamma_2 = [\mathbf{b}, \mathbf{a}].$$

Γ_3 spirals up from \mathbf{a} to \mathbf{b} ; its projection into the (x, y) -plane has winding number -1 around the origin.

$$\Gamma_4 = [\mathbf{b}, \mathbf{a}].$$

$$\text{Thus } \partial\Phi = \Gamma_1 + \Gamma_3 + 2\Gamma_2.$$

If we go from \mathbf{a} to \mathbf{b} along Γ_1 and continue along the “edge” of M until we return to \mathbf{a} , the curve traced out is

$$\Gamma = \Gamma_1 - \Gamma_3,$$

which may also be represented on the parameter interval $[0, 2\pi]$ by the equations

$$\begin{aligned}x &= (1 + \delta \sin \theta) \cos 2\theta, \\y &= (1 + \delta \sin \theta) \sin 2\theta, \\z &= -\delta \cos \theta.\end{aligned}$$

It should be emphasized that $\Gamma \neq \partial\Phi$: Let η be the 1-form discussed in Exercises 21 and 22. Since $d\eta = 0$, Stokes' theorem shows that

$$\int_{\partial\Phi} \eta = 0,$$

But although Γ is the "geometric" boundary of M , we have

$$\int_{\Gamma} \eta = 4\pi.$$

In order to avoid this possible source of confusion, Stokes' formula (Theorem 10.50) is frequently stated only for orientable surfaces Φ .

Solution. The claim about the boundary $\partial\Phi$ follows immediately from the definition of a boundary. The domain D is a cell whose boundary is $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$, so that by definition $\partial\Phi = \Phi(\gamma_1) + \Phi(\gamma_2) + \Phi(\gamma_3) + \Phi(\gamma_4)$.

The claims that $\Phi(\mathbf{p}_1) = \Phi(\mathbf{p}_3) = \mathbf{a}$ and $\Phi(\mathbf{p}_2) = \Phi(\mathbf{p}_4) = \mathbf{b}$ are routine computations.

The description of Γ_1 follows from the fact that γ_1 can be described as the set $(\theta, -\delta)$, $0 \leq \theta \leq \pi$, so that the projection of Γ_1 in the (x, y) -plane is the set of all points $(x(\theta), y(\theta))$, where

$$\begin{aligned}x(\theta) &= (1 + \delta \sin \theta) \cos 2\theta \\y(\theta) &= (1 + \delta \sin \theta) \sin 2\theta.\end{aligned}$$

Regarding the pair $(x(\theta), y(\theta))$ as the complex number $z(\theta) = x(\theta) + iy(\theta) = (1 + \delta \sin \theta)(\cos 2\theta + i \sin 2\theta)$, and using the definition of the winding number, we find this winding number to be

$$n = \frac{1}{2\pi i} \int_0^\pi \frac{z'(\theta)}{z(\theta)} d\theta.$$

Now, $z'(\theta) = 2(1 + \delta \sin \theta)(-\sin 2\theta + i \cos 2\theta) + \delta \cos \theta(\cos 2\theta + i \sin 2\theta)$, so that we get

$$n = \frac{1}{\pi i} \left(\int_0^\pi \frac{-\sin 2\theta + i \cos 2\theta}{\cos 2\theta + i \sin 2\theta} d\theta + \delta \int_0^\pi \frac{\cos \theta}{1 + \delta \sin \theta} d\theta \right).$$

But $-\sin 2\theta + i \cos 2\theta = i(\cos 2\theta + i \sin 2\theta)$, so that the first integral is just πi , and that term contributes $+1$ to the winding number. The second integral is just $\ln(1 + \delta \sin \theta)$, and since this function has the value 0 at both $\theta = 0$ and $\theta = \pi$, it contributes nothing.

As for Γ_2 , since $\theta = \pi$, it is given by $(x(t), y(t), z(t))$, $-\delta \leq t \leq \delta$, where $x(t) = 1$, $y(t) = 0$, $z(t) = -t$. It therefore describes the line segment from **b** to **a** as t goes from $-\delta$ to δ .

The descriptions of Γ_3 and Γ_4 are justified exactly as was just done for Γ_1 and Γ_2 .

As both Γ_1 and Γ_3 spiral upward from **a** to **b**, it is manifest that $\Gamma_1 - \Gamma_3$ represents a spiral that goes from **a** to **b** and back again. It is also easy to see that this spiral does not intersect itself, as the ranges of Γ_1 and Γ_3 meet only in **a** and **b**. For suppose θ and φ are such that $\Gamma_1(\theta) = \Gamma_3(\varphi)$. This means in particular that $-\delta \cos \theta = \delta \cos \varphi$, and so $\theta = \pi - \varphi$. It then follows that $(1 + \delta \sin \theta) \cos 2\theta = (1 - \delta \sin \theta) \cos 2\theta$, so that either $\cos 2\theta = 0$ or $\sin \theta = 0$. Since we also have $(1 + \delta \sin \theta) \sin 2\theta = -(1 - \delta \sin \theta) \sin 2\theta$, the possibility that $\cos 2\theta = 0$ is ruled out, and so $\sin \theta = 0$, i.e., $\theta = 0$ or $\theta = \pi$, meaning the point in common is either **a** or **b**, as asserted.

As for the description of $\Gamma_1 - \Gamma_3$, it is clear that the mapping $T(\theta)$ given by the equations

$$\begin{aligned} x &= (1 + \delta \sin \theta) \cos 2\theta \\ y &= (1 + \delta \sin \theta) \sin 2\theta \\ z &= -\delta \cos \theta \end{aligned}$$

has the property that $T(\theta + \pi)$ is given by the equations

$$\begin{aligned} x &= (1 - \delta \sin \theta) \cos 2\theta \\ y &= (1 - \delta \sin \theta) \sin 2\theta \\ z &= \delta \cos \theta. \end{aligned}$$

Hence it equals describes $\Gamma_1(-\delta, \theta)$ on the interval $[0, \pi]$ and $-\Gamma_3(\delta, \theta)$ (since Γ_3 is given by the latter formulas, but is traversed with θ decreasing from π to 0).

Since $x^2 + y^2 = (1 - t \sin \theta)^2 \geq (1 - \delta)^2 > 0$ on all of M , it follows that η is defined on M . On Γ_1 and Γ_3 we have $\eta = 2 d\theta$, so that

$$\int_{\Gamma} \eta = 4\pi.$$

Chapter 11

The Lebesgue Theory

Exercise 11.1 If $f \geq 0$ and $\int_E f d\mu = 0$, prove that $f(x) = 0$ almost everywhere on E . *Hint:* Let E_n be the subset of E on which $f(x) > 1/n$. Write $A = \cup E_n$. Then $\mu(A) = 0$ if and only if $\mu(E_n) = 0$ for every n .

Solution. The assertion in the hint is immediate. If $\mu(A) = 0$, then $\mu(E_n) = 0$ also, since $E_n \subseteq A$. Conversely, letting $F_n = E_n \setminus \bigcup_{k=1}^{n-1} E_k$, we have $F_n \subset E_n$, $F_m \cap F_n = \emptyset$ if $m \neq n$, and $\cup F_n = \cup E_n = A$. Hence if $\mu(E_n) = 0$, then $\mu(F_n) = 0$ also, and therefore $\mu(A) = 0$ by the countable additivity of μ .

Given the hint, the solution is immediate, since A is the subset of E on which $f(x) > 0$. If $\mu(F_n) > 0$ for any n , then $\int_E f d\mu \geq \int_{F_n} f d\mu \geq \mu(F_n)/n > 0$.

Exercise 11.2 If $\int_A f d\mu = 0$ for every measurable subset A of a measurable set E , then $f(x) = 0$ almost everywhere on E .

Solution. The hypothesis applies in particular if A is the set on which $f(x) > 0$. Since $\chi_A f \geq 0$, the preceding exercise shows that $\mu(A) = 0$. Likewise, taking B as the set on which $-f(x) > 0$, we find that $\mu(B) = 0$. Hence $f(x) = 0$ for almost every x .

Exercise 11.3 If $\{f_n\}$ is a sequence of measurable functions, prove that the set of points x at which $\{f_n(x)\}$ converges is measurable.

Solution. This set can be written as

$$\bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{k=m}^{\infty} \bigcap_{l=m}^{\infty} \{x : |f_k(x) - f_l(x)| < \frac{1}{n}\}.$$

For this set is the set of x such that for every n there exists m such that $|f_k(x) - f_l(x)| < 1/n$ for all $k \geq m, l \geq m$. That is precisely the Cauchy criterion for convergence.

Exercise 11.4 If $f \in \mathcal{L}(\mu)$ on E and g is bounded and measurable on E , then $fg \in \mathcal{L}(\mu)$ on E .

Solution. This follows immediately from the dominated convergence theorem and the fact that $|g(x)| \leq M$ for some constant M . (Take $f_n(x) = g_n(x)f(x)$ for all n , where $g_n(x)$ is a sequence of simple functions converging to $g(x)$ almost everywhere. We can assume $|g_n(x)| \leq M$ and let the dominating function be $M|f(x)|$.)

Exercise 11.5 Put

$$\begin{aligned} g(x) &= \begin{cases} 0 & (0 \leq x \leq \frac{1}{2}), \\ 1 & (\frac{1}{2} < x \leq 1), \end{cases} \\ f_{2k}(x) &= g(x) \quad (0 \leq x \leq 1), \\ f_{2k+1}(x) &= g(1-x) \quad (0 \leq x \leq 1). \end{aligned}$$

Show that

$$\liminf_{n \rightarrow \infty} f_n(x) = 0 \quad (0 \leq x \leq 1),$$

but

$$\int_0^1 f_n(x) dx = \frac{1}{2}.$$

[Compare with (77).]

Solution. Since for each $x \in [0, \frac{1}{2}]$ we have $f_{2k}(x) = 0$ for all k , it follows that the inferior limit at such an x is zero. The same is true for $x \in [\frac{1}{2}, 1]$, since $f_{2k+1}(x) = 0$ for all these x . The value of the integral is immediate, since each $f_n(x)$ is a step function.

The point of this exercise is that strict inequality can easily occur in Fatou's Lemma.

Exercise 11.6 Let

$$f_n(x) = \begin{cases} \frac{1}{n} & (|x| \leq n), \\ 0 & (|x| > n). \end{cases}$$

Then $f_n(x) \rightarrow 0$ uniformly on \mathbb{R}^1 , but

$$\int_{-\infty}^{\infty} f_n dx = 2 \quad (n = 1, 2, 3, \dots).$$

(We write $\int_{-\infty}^{\infty}$ in place of $\int_{\mathbb{R}^1}$.) Thus uniform convergence does not imply dominated convergence in the sense of Theorem 11.32. However, on sets of finite measure, uniformly convergent sequences of bounded functions do satisfy Theorem 11.32.

Solution. The uniform convergence to zero is obvious, since $0 \leq f_n(x) \leq 1/n$ for all x and all n .

Again, since $f_n(x)$ is a step function, the value of the integral is immediate.

Exercise 11.7 Find a necessary and sufficient condition that $f \in \mathcal{R}(\alpha)$ on $[a, b]$. *Hint:* Consider Example 11.6(b) and Theorem 11.33.

A bounded function f belongs to $\mathcal{R}(\alpha)$ on $[a, b]$ if and only if the following two conditions hold:

- (i) f is right-continuous wherever α is not right-continuous and left-continuous wherever α is not left-continuous (that is, one of f and α is right-continuous at each point and one is left-continuous);
- (ii) the set of points where α is continuous and f is not continuous is a set of zero α -variation. That is, this set has μ_α -measure zero, where μ_α is the regular Borel measure generated by the function α , as in Example 11.6(b).

To prove this fact, all we have to do is copy the proof of Theorem 11.33, *mutatis mutandis*, specifically, replacing dx by $d\alpha$ and Δx by $\Delta\alpha$ at every stage. It will follow as a corollary of the proof that if $f \in \mathcal{R}(\alpha)$, then $f \in \mathcal{L}(\mu_\alpha)$ and

$$\int_{[a,b]} f d\mu_\alpha = \mathcal{R} \int_a^b f(x) d\alpha(x).$$

In modifying the proof we need to clear out just one case in order to make the changes run smoothly. To that end, we note that if f and α both have a one-sided discontinuity from the same side and at the same point, it is impossible for f to belong to $\mathcal{R}(\alpha)$. Indeed, suppose p is a common right-sided discontinuity of both f and α . For any partition P we have $x_k \leq p < x_{k+1}$ for some index k , and then

$$U(P, f, \alpha) - L(P, f, \alpha) \geq o \cdot (\alpha(p+) - \alpha(p)) > 0,$$

where o is the limit of $\sup_{p \leq x < p+h} f(x) - \inf_{p \leq x < p+h} f(x)$ as $h \downarrow 0$. (The function $f(x)$ is right-continuous at p if and only if $o = 0$.)

Note that if f and α are discontinuous from opposite sides at a point, it is quite possible that $f \in \mathcal{R}(\alpha)$. For example, let $f(x) = \chi_{[0, 1/2]}(x)$ and $\alpha(x) = \chi_{[1/2, \infty)}(x)$. Then for any partition P of $[0, 1]$ containing $1/2$, we have $x_k = 1/2$ for some k , and

$$U(P, f, \alpha) - L(P, f, \alpha) = 1 - 1 = 0.$$

(It is for this reason that I define $\mathcal{R}(\alpha)$ differently in my courses. I require that for each $\varepsilon > 0$ there must exist $\delta > 0$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$ for all partitions P such that $\max_{1 \leq k \leq n} (x_k - x_{k-1}) < \delta$. When this is done, conditions

(i) and (ii) are no longer sufficient for f to belong to $\mathcal{R}(\alpha)$, and the theory is somewhat simpler. Except for special considerations at discontinuities of α , however, the results are the same in both theories.)

We now suppose that condition (i) holds and prove the necessity of condition (ii). To avoid having to single out the endpoints in what follows, we simply extend α outside the interval $[a, b]$ by specifying $\alpha(x) = \alpha(b)$ for $x > b$ and $\alpha(x) = \alpha(a)$ for $x < a$.

Suppose that f is in $\mathcal{R}(\alpha)$. Let $\{P_k\}$ be a sequence of partitions such that P_{k+1} is a refinement of P_k , the distance between adjacent points of P_k is less than $\frac{1}{k}$, P_k contains all points x at which $\alpha(x+) - \alpha(x-) > \frac{1}{k}$, and

$$U(P_k, f, \alpha) \rightarrow \mathcal{R} \int f d\alpha, \quad L(P_k, f, \alpha) \rightarrow \mathcal{R} \int f d\alpha.$$

We note that every discontinuity of α belongs to some partition P_k . Assume P_k consists of the points $a = x_{k,0} < x_{k,1} < \cdots < x_{k,n_k} = b$.

As in the proof of Theorem 11.33, we define $U_k(x) = M_i$ and $L_k(x) = m_i$ for $x_{k,i-1} < x \leq x_{k,i}$, $1 \leq i \leq n_k$. For definiteness we define $U_k(a) = M_1$ and $L_k(a) = m_1$. Then by definition of the upper and lower sum, the definition of $\mu_\alpha((a, b))$, and the definition of the integral of a simple function,

$$\begin{aligned} \int_{[a,b]} U_k d\mu_\alpha &= U(P_k, f, \alpha) + M_1(\alpha(a+) - \alpha(a)), \\ \int_{[a,b]} L_k d\mu_\alpha &= L(P_k, f, \alpha) + m_1(\alpha(a+) - \alpha(a)). \end{aligned}$$

By condition (i), either $M_1 - m_1 \rightarrow 0$ as $k \rightarrow \infty$ or $\alpha(a+) = \alpha(a)$. It then follows that the monotonic sequences L_k and U_k have limits L and U that are measurable, and either

$$\int_{[a,b]} L d\mu_\alpha = \mathcal{R} \int f d\alpha, \quad \int_{[a,b]} U d\mu_\alpha = \mathcal{R} \int f d\alpha,$$

(when $\alpha(a+) = \alpha(a)$) or

$$\begin{aligned} \int_{[a,b]} L d\mu_\alpha &= \mathcal{R} \int f d\alpha + f(a)(\alpha(a+) - \alpha(a)), \\ \int_{[a,b]} U d\mu_\alpha &= \mathcal{R} \int f d\alpha + f(a)(\alpha(a+) - \alpha(a)), \end{aligned}$$

(when $\alpha(a+) > \alpha(a)$).

If these two integrals are the same, it follows that $U(x) = L(x)$ almost everywhere with respect to the measure μ_α . If x is not a point of any partition P_k and $U(x) = L(x)$, then f is continuous at x . As for points of the partition, they are either points of discontinuity of α or points x such that $\mu_\alpha(\{x\}) = 0$. Since there are only countably many points in all the partitions, the partition points x for which $\mu_\alpha(\{x\}) = 0$ form a set of measure zero. Thus the set of discontinuities of $f(x)$ can be written as the union $A \cup B$, where A consists of points where f is continuous from only one side and α is discontinuous from

that side (these points are all among the points of partition), and B consists of the points of discontinuity of f where α is continuous. We have just shown that B is of zero α -variation, as claimed.

Conversely, if f satisfies these two conditions, we note that $U(x) - L(x)$ at all points where $f(x)$ is continuous. Hence if the discontinuities of $f(x)$ other than one-sided discontinuities at points where α is continuous from the side on which f is discontinuous form a set of zero α -variation, then $U(P_k, f, \alpha) - L(P_k, f, \alpha) \rightarrow 0$.

Exercise 11.8 If $f \in \mathcal{R}$ on $[a, b]$ and if $F(x) = \int_a^x f(t) dt$, prove that $F'(x) = f(x)$ almost everywhere on $[a, b]$.

Solution. We know by Theorem 6.20 that $F'(x) = f(x)$ at every point where $f(x)$ is continuous. Theorem 11.33 shows that if f is Riemann-integrable on $[a, b]$, then it is continuous almost everywhere. Hence the result follows.

Exercise 11.9 Prove that the function F given by (96) is continuous on $[a, b]$.

Solution. The function F is the one in the preceding exercise. Its continuity follows from the dominated convergence theorem, taking $|f|$ as the “dominating” function and letting $f_n = \chi_{[a, x_n]} f$ where $\{x_n\}$ is any sequence of numbers converging to x , so that f_n converges pointwise to $\chi_{[a, x]} f$ except possibly at the point x , which is a set of measure 0. The dominated convergence theorem then guarantees that $F(x_n) \rightarrow F(x)$. Since the sequence $\{x_n\}$ is arbitrary, it follows that F is continuous at x , as in Theorem 4.2.

Exercise 11.10 If $\mu(X) < +\infty$ and $f \in \mathcal{L}^2(\mu)$ on X , prove that $f \in \mathcal{L}(\mu)$ on X . If

$$\mu(X) = +\infty,$$

this is false. For instance, if

$$f(x) = \frac{1}{1 + |x|},$$

then $f \in \mathcal{L}^2$ on R^1 , but $f \notin \mathcal{L}$ on R^1 .

Solution. This follows from Theorems 11.27 and 11.29, if we let $A = \{x : |f(x)| \leq 1\}$ and $B = \{x : |f(x)| > 1\}$. We can then write

$$|f| \leq \chi_A + \chi_B \cdot |f|^2.$$

and χ_A is integrable by Theorem 11.23(a).

As for the counterexample, we have

$$|f|^2 \leq \chi_{[-1,1]} + \chi_{[1,\infty)} \cdot \frac{1}{x^2},$$

which implies that $f \in \mathcal{L}^2$, and

$$f(x) \geq \chi_{[0,n]}(x) \frac{1}{1+x},$$

so that

$$\int f \, dx \geq \int_0^n \frac{1}{1+x} \, dx = \ln(1+n) \rightarrow \infty.$$

Hence $f \notin \mathcal{L}$.

Exercise 11.11 If $f, g \in \mathcal{L}(\mu)$ on X , define the distance between f and g by

$$\int_X |f - g| \, d\mu.$$

Prove that $\mathcal{L}(\mu)$ is a complete metric space.

Solution. We have to regard functions equal almost everywhere as the same function. Given that, it does follow that if $d(f, g) = 0$, then $f = g$. The fact that $d(f, g) = d(g, f)$ is immediate from the definition and the triangle inequality follows from simply integrating the triangle inequality for the values of the functions. Hence \mathcal{L} is a metric space.

To prove that it is complete, we merely repeat the reasoning of Theorem 11.42, replacing \mathcal{L}^2 by \mathcal{L} and taking the function $g(x)$ to be identically equal to 1. When this is done, every step in the proof of Theorem 11.42 follows for \mathcal{L} .

Exercise 11.12 Suppose

- (a) $|f(x, y)| \leq 1$ if $0 \leq x \leq 1, 0 \leq y \leq 1$,
- (b) for fixed x , $f(x, y)$ is a continuous function of y ,
- (c) for fixed y , $f(x, y)$ is a continuous function of x .

Put

$$g(x) = \int_0^1 f(x, y) \, dy \quad (0 \leq x \leq 1).$$

Is g continuous?

Solution. Yes, $g(x)$ is continuous. Let $x_n \rightarrow x$. Then by (c), $f(x_n, y) \rightarrow f(x, y)$ for each $y \in [0, 1]$, in particular for almost every y . Since $|f(x_n, y)| \leq 1$ for all x_n and y by assumption (a), and the set $[0, 1]$ has finite measure, it follows from the dominated convergence theorem that $g(x_n) \rightarrow g(x)$.

Note that property (b) was used only to guarantee that $g(x)$ is actually defined. Thus the word *continuous* could be replaced by *integrable* in this condition.

Exercise 11.13 Consider the functions

$$f_n(x) = \sin nx \quad (n = 1, 2, 3, \dots, -\pi \leq x \leq \pi)$$

as points of \mathcal{L}^2 . Prove that the set of these points is closed and bounded, but not compact.

Solution. We compute by brute force that

$$\|f_m - f_n\|^2 = \begin{cases} 0, & \text{if } m = n, \\ 2\pi, & \text{if } m \neq n. \end{cases}$$

Further, it is easy to see that $\|f_n\|^2 = \pi$. Hence the set $\{f_n\}$ is bounded and has no limit points. (The $\sqrt{\pi/2}$ -neighborhood of any point contains at most one point of this set.) Having no limit points, it contains all of its limit points and is therefore closed. Being infinite, if it were compact, it *would* have a limit point. Therefore it is not compact.

Exercise 11.14 Prove that a complex function f is measurable if and only if $f^{-1}(V)$ is measurable for every open set V in the plane.

Solution. By definition $f = u + iv$, where u and v are real-valued, is measurable if and only if u and v are.

Suppose f is measurable (that is, u and v are measurable). Let V be any open set in the plane and $(x, y) \in V$. Then there exists $\delta > 0$ such that the square $S(x, y) = (x - \delta, x + \delta) \times (y - \delta, y + \delta)$ is contained in V . The union of these open squares is all of V , and there is a countable set of points $(x_n, y_n) \in V$ such that $\bigcup_{n=1}^{\infty} S(x_n, y_n) = V$. (This is proved by appealing to Exercise 23 of Chapter 2.) But then

$$f^{-1}(V) = \bigcup_{n=1}^{\infty} f^{-1}(S(x_n, y_n)) = \bigcup_{n=1}^{\infty} u^{-1}(x_n - \delta, x_n + \delta) \cap v^{-1}(y_n - \delta, y_n + \delta).$$

It follows that $f^{-1}(V)$ is measurable.

Conversely if $f^{-1}(V)$ is measurable for every open set in the plane, then in particular this set is measurable if $V = (a, b) \times R^1$ (where $f^{-1}(V) = u^{-1}((a, b))$) or $V = R^1 \times (a, b)$ (where $f^{-1}(V) = v^{-1}((a, b))$), and hence both u and v are measurable. By definition, that means that f is measurable.

Exercise 11.15 Let \mathcal{R} be the ring of all elementary subsets of $(0, 1]$. If $0 < a \leq b \leq 1$, define

$$\phi([a, b]) = \phi([a, b)) = \phi((a, b]) = \phi((a, b)) = b - a,$$

but define

$$\phi((0, b)) = \phi((0, b]) = 1 + b$$

if $0 < b \leq 1$. Show that this gives an additive set function ϕ on \mathcal{R} , which is not regular and which cannot be extended to a countably additive set function on a σ -ring.

Solution. In brief, since an elementary set A is a finite disjoint union of intervals, $\phi(A)$ is the sum of the lengths of those intervals if 0 is not the endpoint of any interval in A and 1 larger than the sum of the lengths of the intervals if 0 is one of the endpoints. In particular $\phi(A) < 1$ if A is a closed set, since 0 cannot be the endpoint of any closed set that is a finite union of intervals in $(0, 1]$.

(This alternate definition is independent of the particular way in which the set A is represented as a finite disjoint union of intervals, since if $A = \bigcup_{i=1}^m I_i = \bigcup_{j=1}^n J_j$, where each of the collections $\{I_i\}$ and $\{J_j\}$ is a set of pairwise disjoint intervals, one can easily verify that

$$|I_i| = \sum_{j=1}^n |I_i \cap J_j|, \quad |J_j| = \sum_{i=1}^m |I_i \cap J_j|,$$

so that $\sum_{i=1}^m |I_i| = \sum_{j=1}^n |J_j| = \sum_{i,j} |I_i \cap J_j|$. Here $|I|$ is the length of the interval I .)

If two elementary sets A and B are disjoint, at most one of them can have the point 0 as the endpoint of one of its intervals. Then $\phi(A \cup B)$ is the sum of the lengths of the intervals in $A \cup B$ if neither set contains an interval having 0 as the endpoint, and 1 larger than this sum if one of them does contain an interval with 0 as endpoint. In either case $\phi(A \cup B) = \phi(A) + \phi(B)$ when $A \cap B = \emptyset$. Thus the function ϕ is additive.

The function ϕ is not regular, however, since there is no closed subset of $(0, c]$ that can approximate $(0, c]$ if $c < 1$. For $\phi((0, c]) = 1 + c$, but $\phi(A) \leq 1$ if A is closed.

The function ϕ also cannot be extended to a countably additive set function on a σ -ring, since

$$(0, \frac{1}{2}] = \bigcup_{n=1}^{\infty} (\frac{1}{2^{n+1}}, \frac{1}{2^n}],$$

and

$$\phi((0, \frac{1}{2}]) = \frac{3}{2}, \quad \sum_{n=1}^{\infty} \phi((\frac{1}{2^{n+1}}, \frac{1}{2^n}]) = \frac{1}{2}.$$

Exercise 11.16 Suppose $\{n_k\}$ is an increasing sequence of positive integers and E is the set of all $x \in (-\pi, \pi)$ at which $\{\sin n_k x\}$ converges. Prove that $m(E) = 0$. *Hint:* For every $A \subset E$,

$$\int_A \sin n_k x \, dx = 0,$$

and

$$2 \int_A (\sin n_k x)^2 dx = \int_A (1 - \cos 2n_k x) dx \rightarrow m(A) \text{ as } k \rightarrow \infty.$$

Solution. The two statements in the hint follow from the Riemann-Lebesgue lemma (or from Bessel's inequality applied to the Fourier series of χ_A , if you wish). Let $f(x)$ be the limit of $\sin n_k x$ on the set E . Then, since termwise integration is justified by the dominated convergence theorem, we have

$$\int_A [(f(x))^2 - \frac{1}{2}] dx = 0,$$

for all A . Hence, by Exercise 2 above, $f(x) = \pm \frac{1}{\sqrt{2}}$ almost everywhere on E . If we let A be the set of points of E at which $f(x) = \frac{1}{\sqrt{2}}$, we find that $\int_A f(x) dx = 0$, and so by Exercise 1, $f(x) = 0$ almost everywhere on A . Since in fact $f(x) \neq 0$ on A , it follows that A has measure 0. Similarly the set where $f(x) = -\frac{1}{\sqrt{2}}$ has measure 0.

Exercise 11.17 Suppose $E \subset (-\pi, \pi)$, $m(E) > 0$, $\delta > 0$. Use the Bessel inequality to prove that there are at most finitely many integers n such that $\sin nx \geq \delta$ for all $x \in E$.

Solution. For any integer with this property we have

$$\int_E \sin nx dx \geq \delta \mu(E),$$

and the Bessel inequality implies that this inequality can hold for only a finite number of n . (The integral is the imaginary part of the Fourier coefficient of the \mathcal{L}^2 -function χ_E .)

Exercise 11.18 Suppose $f \in \mathcal{L}^2(\mu)$, $g \in L^2(\mu)$. Prove that

$$\left| \int f \bar{g} d\mu \right|^2 = \int |f|^2 d\mu \int |g|^2 d\mu$$

if and only if there is a constant c such that $g(x) = cf(x)$ almost everywhere. (Compare Theorem 11.35.)

Solution. There is a slight mistake in the statement of the problem, since equality certainly holds if $f(x)$ is identically zero, whether $g(x)$ equals zero or not. We must either assume that $f(x)$ is not identically zero, or allow the possibility that $f(x) = cg(x)$.

Equality can hold if $g(x) = 0$ almost everywhere, and in that case $c = 0$ in the relation $g(x) = cf(x)$. Hence assume now that $\int |g|^2 d\mu > 0$. The inequality

$$0 \leq \int (|f| + \lambda|g|)^2 d\mu,$$

which holds for real values of λ , is equivalent to the inequality

$$-2\lambda \int |fg| d\mu \leq \int |f|^2 d\mu + \lambda^2 \int |g|^2 d\mu.$$

In this inequality take $\lambda = -\sqrt{\frac{\int |f|^2 d\mu}{\int |g|^2 d\mu}}$. The result is

$$2 \frac{\int |fg| d\mu \sqrt{\int |f|^2 d\mu}}{\sqrt{\int |g|^2 d\mu}} \leq 2 \int |f|^2 d\mu,$$

which is equivalent to

$$\left[\int |fg| d\mu \right]^2 \leq \int |f|^2 d\mu \int |g|^2 d\mu.$$

Hence the equality in the problem can hold only if equality holds in this last inequality, which, since it implies that

$$\int (|f| + \lambda|g|)^2 d\mu = 0,$$

implies that $|f| = -\lambda|g|$ almost everywhere. In particular f vanishes almost everywhere that g vanishes. In addition, the equality in the hypothesis of the problem requires that

$$\left| \int f\bar{g} d\mu \right| = \int |fg| d\mu.$$

If both sides of this last equality are zero, then at almost every point either $f(x) = 0$ or $g(x) = 0$. Since $|f| = -\lambda|g|$, it then follows that in fact either both functions vanish identically, a case we have already discussed, or $\lambda = 0$, in which case only f vanishes identically. In either case we do have the kind of linear dependence specified in the amended statement of the problem.

Hence assume that neither side of this equality is zero. Let ω be the complex number

$$\omega = \frac{\overline{\int f\bar{g} d\mu}}{|\int f\bar{g} d\mu|},$$

so that $|\omega| = 1$. We note that

$$\int \omega f\bar{g} d\mu = \omega \int f\bar{g} d\mu = \left| \int f\bar{g} d\mu \right| \leq \int |f\bar{g}| d\mu = \int |\omega f\bar{g}| d\mu.$$

This means that the real parts of the two integrals on the extremes here are equal, and the imaginary parts of both are zero. Taking just the real parts, since $\operatorname{Re}(\omega f \bar{g}) \leq |\omega f \bar{g}|$, this implies that the real part of $\omega f \bar{g}$ is equal to $|f g| = -\lambda g \bar{g}$ almost everywhere, and therefore that the imaginary part is zero almost everywhere. But then, almost everywhere where g does not vanish, we can cancel \bar{g} from the equality, getting $f = -\lambda \bar{\omega} g$ wherever g does not vanish. Since this equality also holds almost everywhere where g does vanish, we are done.